TGI WS 2016/17 Christmas Sheet

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First, some definitions.

A certain enchanted forest is inhabited by talking birds. The set of these birds is denoted by $\mathcal{V}$. For all mathematical objects A, when we say "Let A be a bird", then $A \in \mathcal{V}$ is understood. Given any birds A and B in $\mathcal{V}$, if you call out the name of B to A, then A will respond by calling out the name of some bird to you; this bird we designate by $AB \in \mathcal{V}$. Thus AB is the bird named by A upon hearing the name of B.

Definition 1 Let A and B be birds. If we call out B's name to A, A's answer is AB. For abbreviation, we use "A's answer to B is AB".

Furthermore, when listing the members of a finite set, we list each member exactly once. That is, in "$\mathcal{V} = \{A, B, C\}$", $|\mathcal{V}| = 3$ is understood.

Lemma 1 Let A and B be birds. Then $AB = BA$ is not necessarily true.

Proof. It suffices to set $\mathcal{V} := \{A, B\}$ and to set $AB := B$ as well as $BA := A$. □

Lemma 2 Let A, B, C be birds. Then $(AB)C = A(BC)$ is not necessarily true.

Proof. Set $\mathcal{V} := \{A, B, C\}$, $AA := B$, $AB := C$, $BC := A$, $CC := A$.
Then $(AB)C = CC = A \neq B = AA = A(BC)$. □

Definition 2 Suppose $M \in \mathcal{V}$. Call M a *mockingbird* iff $Mx = xx$ holds for all $x \in \mathcal{V}$.

Definition 3 Let A, B, C be birds (not necessarily distinct). Call C a composition of A with B iff for all birds x, we have $Cx = A(Bx)$, i. e. C's response to x is the same as A's response to B's response to x. Furthermore, define

$$C_1 :\Longleftrightarrow \quad \forall A, B \in \mathcal{V} \;\; \exists C \in \mathcal{V} \;\; \forall x \in \mathcal{V} \quad Cx = A(Bx) \quad \text{(composition condition)}$$

and

$$C_2 :\Longleftrightarrow \text{ There exists at least one mockingbird } M \in \mathcal{V} \quad \text{(mockingbird condition)} \quad .$$

Exercise 1

Let $A, B \in \mathcal{V}$. Say that A is *fond* of B if $AB = B$.

Claim If C_1 and C_2 hold, then every bird $A \in \mathcal{V}$ is fond of at least one bird $B \in \mathcal{V}$, where $A = B$ may happen, i. e. there might be a bird that is fond of itself.

Proof Suppose that C_1 and C_2 hold. Then, by C_2, let M be a mockingbird.

Considering C_1,

$$\forall A, B \in \mathcal{V} \quad \exists C \in \mathcal{V} \quad \forall x \in \mathcal{V} \quad Cx = A(Bx) \quad ,$$

substituting M for B yields

$$\forall A \in \mathcal{V} \quad \exists C \in \mathcal{V} \quad \forall x \in \mathcal{V} \quad Cx = A(Mx) \quad .$$

Let $A \in \mathcal{V}$. Then there exists $C \in \mathcal{V}$ such that for all $x \in \mathcal{V}$, $Cx = A(Mx)$ holds. Let C be such. By the definition of mockingbird, $Mx = xx$ for all $x \in \mathcal{V}$; therefore we have

$$\forall x \in \mathcal{V} : Cx = A(xx) \quad .$$

Now, let $x \in \mathcal{V}$. Then $Cx = A(xx)$; in particular, if $x = C$, we get

$$CC = A(CC) \quad ,$$

i. e. A's response upon hearing CC is CC. Therefore, A is fond of CC.

Therefore, every bird is fond of CC.

Therefore, every bird is fond of at least one bird. $\square$

Exercise 2

An *egocentric* bird $x \in \mathcal{V}$ answers the call x with xx.

Claim If C_1 and C_2 hold, then there is always at least one egocentric bird.

Proof Suppose that C_1 and C_2 apply to our forest. In particular, this means we can appeal to Ex. 1, because that exercise states that if C_1 and C_2 hold, we know that every bird in the forest is fond of at least one bird.

Because C_2 holds, the forest is not empty.

We first consider the case that the forest contains exactly one bird. By C_2, the only bird in the forest is a mockingbird.

By Ex. 1, we know that the mockingbird is fond of at least one bird, but the only bird the mockingbird can be fond of is the mockingbird itself, because only the mockingbird exists.

Now consider the case that there are at least two birds in the forest. One of those is a mockingird due to C_2, and let A be another bird distinct from the mockingbird.

By Ex. 1, we know that every bird is fond of at least one bird.

If the mockingbird is fond of itself, we are done.

If the mockingbird is fond of A, we have $MA = A$ by definition of "fond of". We also have $MA = AA$ because M is a mockingbird. Now $A = AA$ is immediate. Therefore A is fond of itself. $\square$

Figure 1: Is this bird egocentric? We won't know until we ask it. Unfortunately, it has a complicated name: `http://www.lto.de/fileadmin/user_upload/kakadu_papagei_620.jpg`

Exercise 3

Two birds A and B are said to *agree on* a bird x iff their responses to x are the same, i. e. iff $Ax = Bx$ holds. A is *agreeable* iff, for any bird B, A agrees with B on at least one bird.

In symbols:

$$A \text{ is agreeable} \iff \quad \forall B \in \mathcal{V} \;\; \exists x \in \mathcal{V} \quad Ax = Bx \quad .$$

Claim If C_1 holds and there is at least one agreeable bird, then every bird is fond of at least one bird.

Proof Suppose that C_1 holds and that there is at least one agreeable bird.

Let A be an agreeable bird, and let B be any bird. By C_1, we know that there exists a bird C such that for all birds x we have $Cx = B(Ax)$. Let C be such.

We know that A is agreeable, i. e. for all birds B', we have

$$\exists x \in \mathcal{V} \quad Ax = B'x \quad . \tag{1}$$

As equation (1) holds for all birds B', it is true for C in particular. Therefore, there exists a bird y such that $Ay = Cy$. Let y be such. We then have $Ay = Cy$, which means $Ay = B(Ay)$ by C_1, in other words B is fond of Ay.

We have shown that for any agreeable bird A and for any bird B, there is a bird y such that B is fond of Ay.

Therefore, every bird is fond of at least one bird. □

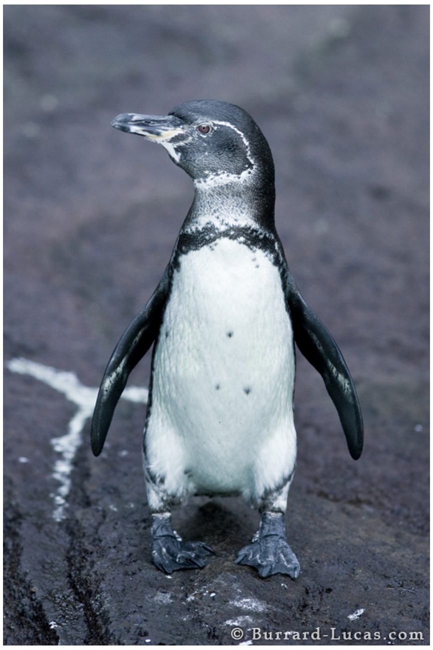

Figure 2:
This bird clearly *is* egocentric. Just look how it sticks its nose up!

http://img.burrard-lucas.com/galapagos/full/galapagos_penguin.jpg

Exercise 4 – a.k.a. *How to think like Nikita Danilenko*[1]

Let A, B, C be birds such that C is the composition of A and B.

Claim If C_1 holds and C is agreeable, then A is also agreeable.

Proof

We will illustrate our reasoning using the diagram shown to the right.

$$\begin{tikzcd}
Cx \arrow[d, "\gamma"] & Dx \arrow[l, "\beta"'] \\
A(Bx) & E(Bx) \arrow[l, "\Omega"'] \arrow[u, "\alpha"']
\end{tikzcd}$$

Recall that C_1 states

$$\forall A, B \in \mathcal{V} \quad \exists C \in \mathcal{V} \quad \forall x \in \mathcal{V} \quad Cx = A(Bx)$$

and that "C is agreeable" is shorthand for

$$\forall D \in \mathcal{V} \quad \exists x \in \mathcal{V} \quad Cx = Dx \quad .$$

Now suppose that C_1 holds and C is agreeable.
We wish to show that A is agreeable, that is, for every bird E there is a bird y such that $Ey = Ay$.

Let E be any bird.

By C_1, we know that there exists a bird D' that composes E with B. Let D be such a bird (cf. α).

We know that C is agreeable, i. e. for any bird D', there is a bird x' such that $Cx' = D'x'$. In particular, there is a bird x such that $Cx = Dx$ (cf. β). Let x be such. Finally, we may write $Cx = A(Bx)$ because we are given that C composes A with B (cf. γ). So by tracing α, β, γ, in this order, we conclude $E(Bx) = Dx = Cx = A(Bx)$ (cf. Ω).

Therefore, for any bird E, we have found a bird y – namely, Bx – such that A's and E's answer upon hearing y are the same, i. e. A is agreeable. $\square$

So where does the idea of doing it this way come from?

We know that $Cx = A(Bx)$, for any bird x. Therefore, if we can somehow infer another bird's answer to Bx as well, we are done. So composing with B seems appropriate. Now the composition condition C_1 comes in handy, because it allows us to specify the bird D as the composition of E with B, and the rest falls into place – we get $Cx = Dx$ from the fact that C is agreeable.

[1]With apologies to Mr Danilenko, who taught me the value of (not necessarily commutative) diagrams, and to Kevin Houston, whose book *How to Think Like a Mathematician* I just might get around to reading, one of these days... http://www1.maths.leeds.ac.uk/~khouston/pdf/htwm.pdf

What happens if the mockingbird M hears its own name?

If M is egocentric, then we know that M responds with M upon hearing M. But otherwise? We'll have to ask someone else, I guess…

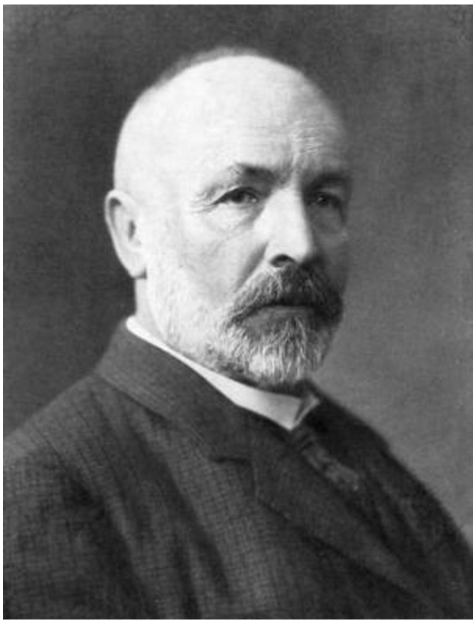

Figure 3:
Above: A mockingbird – no doubt about that.

`http://cdn.litlepups.net/2016/03/04/white-cute-bird.jpg`

To the left: Georg Cantor. Instead of *mockingbird*, he would use the term *diagonal bird*, for sure…

`https://i.kinja-img.com/gawker-media/image/upload/s--YAAYCGzg--/c_scale,f_auto,fl_progressive,q_80,w_800/18dykvtfjl4dvjpg.jpg`