Solving parity games in quasi-polynomial time – the modal μ way

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January 2018

Solving parity games in quasi-polynomial time – by playing with registers

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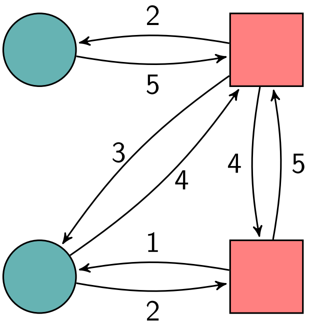
Parity Games



Complexity of parity games

2 5 4
8
6
7
3 1

Priority assignement

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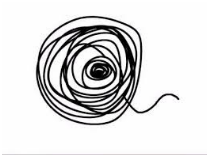

Structural complexity of graph
tree-width, Kelly-width, entanglement...

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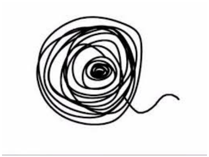

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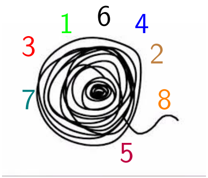

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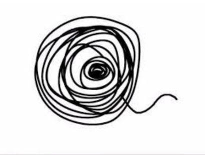

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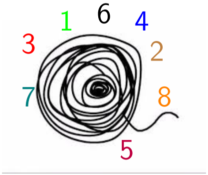

Introducing... *register-index*

Contributions

Definition (Register-index)

A measure capturing the complexity of priority assignments.

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Solving parity games of fixed register-index is in PTIME.

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Register-index is $O(\log n)$ *in the size of the graph.*

Corollary

Quasi-polynomial algorithm for solving parity games.

Theorem

Parity games of bounded register-index have bounded descriptive complexity.

Parity Game

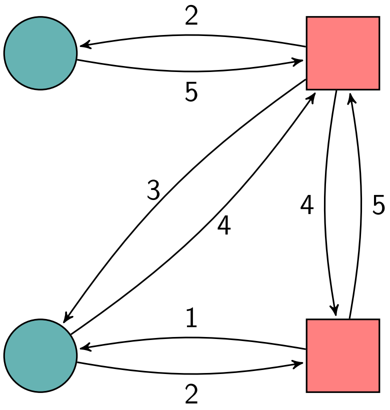


Figure: A parity game

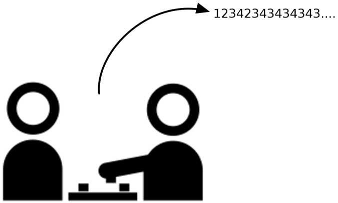
- Even *vs.* Odd
- Circles belong to Odd; Boxes to Even.
- Priorities from $\{0, ..., 5\}$
- Even wins if the **highest** priority seen infinitely often is even.

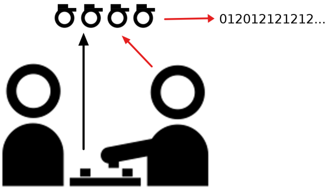
k-register games

Definition (Big picture)

- Parity game,
- Registers record highest priority seen since last reset,
- Even uses registers to produce an output sequence.

Winning condition on the sequence of outputs.





k-register games

Definition

- Registers contain max priority since last reset.
- Registers are ranked by last reset time.

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- New register content: 0

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- New register rank: 1

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- $+1$ to the rank of all registers of rank $< r$.

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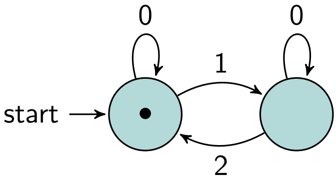
- New register rank: 1
- $+1$ to the rank of all registers of rank $< r$.

Even wins if the maximal output occurring infinitely often is Even.

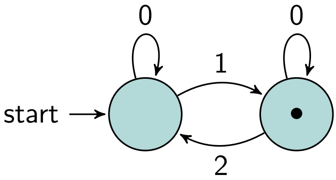
Example



$r_1 : 0$

output:

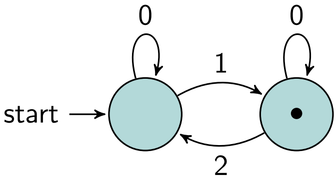
Example



$r_1 : 1$

output:

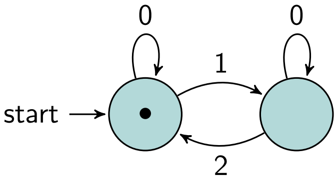
Example



r_1 : ~~1~~ 0

output: 3

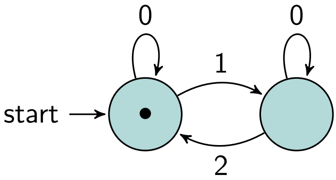
Example



$r_1 : \not{1}\not{0}\ 2$

output: 3

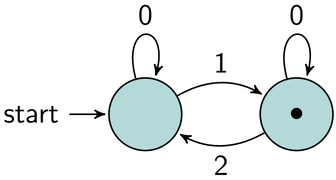
Example



r_1 : ~~102~~ 0

output: 32

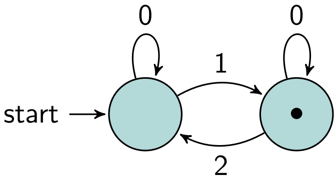
Example



r_1 : ~~1020~~ 1

output: 32

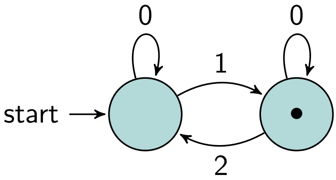
Example



r_1 : $\not{1}\not{0}\not{2}\not{0}\not{1}$ 0

output: 323

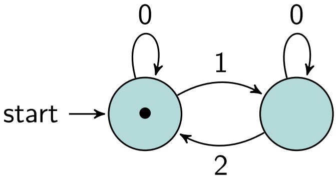
Example



r_1 : ~~1~~~~0~~~~2~~~~0~~~~1~~ 0...

output: 323232323...

Example

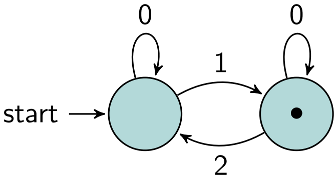


$r_2 : 0$

$r_1 : 0$

output:

Example

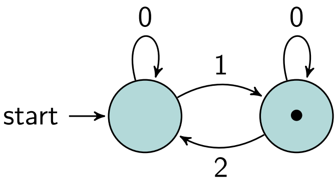


r_2 : 1

r_1 : 1

output:

Example

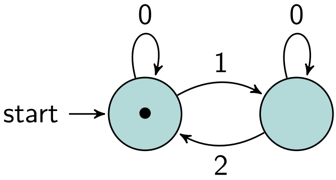


r_2 : 1

r_1 : ~~1~~ 0

output: 3

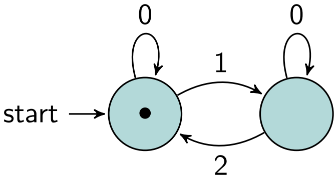
Example



r_2 : ~~1~~2
r_1 : ~~1~~~~0~~ 2

output: 3

Example

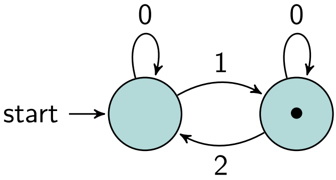


r_2 : ~~1~~~~2~~ 2

r_1 : ~~1~~~~0~~~~2~~ 0

output: 34

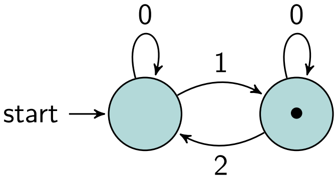
Example



r_2 : ~~1~~~~2~~ 2
r_1 : ~~1~~~~0~~~~2~~~~0~~ 1

output: 34

Example

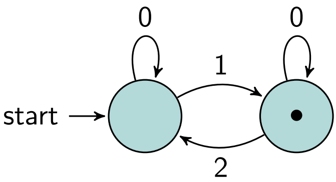


$r_2 : \not{1}\not{2}\ 2$

$r_1 : \not{1}\not{0}\not{2}\not{0}\not{1}\ 0$

output: 343

Example



r_2 : ~~1~~~~2~~ 2...

r_1 : ~~1020~~~~1~~ 0...

output: 34343434343...

Register-index

Definition

A parity game has register-index k if its winner also wins the k register-index.

- The number of priorities is an upper bound on register-index.

Algorithmic complexity

Theorem

Solving parity games of k-bounded register-index is in P.

Proof.

Given parity game G with priorities from I, solve this parity game with $2k+1$ priorities:

□

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- States: $(p, \bar{x}, i)$ where:
  - $p \in G$: current position in G,
  - $\bar{x} \in I^k$: recorded priorities,
  - $i \in \{0, 1\}$: time to summon detectives or turn in parity game.
- Edges represent moves in G and detective summons.
- Edges inherited from G have priority 1,
- Edge representing summon has priority of the output.

□

Descriptive complexity

Theorem

- There is a L_μ formula $\mathbf{Win}_k^I$ which holds in a parity game G with priorities I if and only if Even wins the k-register game.
- The alternation-depth of this formulas depends on k, not I.

Recap

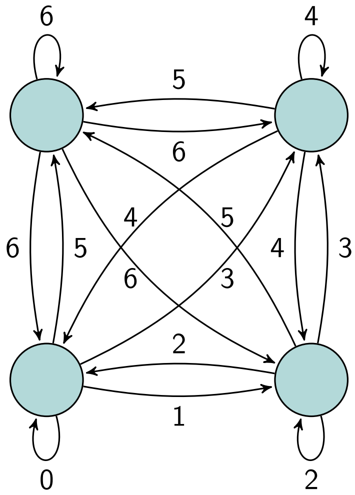
So far:

- Low register-index $\implies$ low complexity.

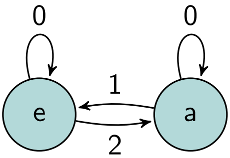
Next:

- Examples
- Register-index is $O(\log n)$ in the size n of the parity game.

Examples of constant register-index: 1



Examples of high register-index: 2



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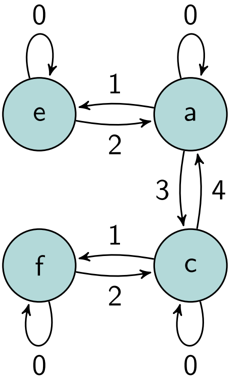
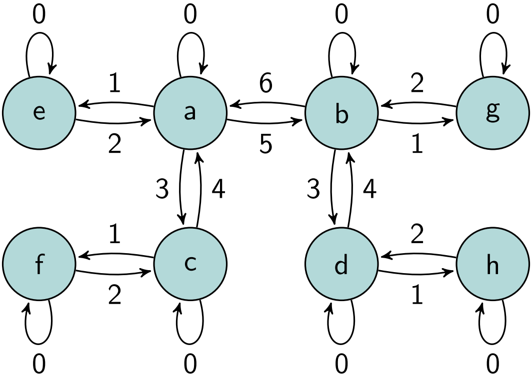
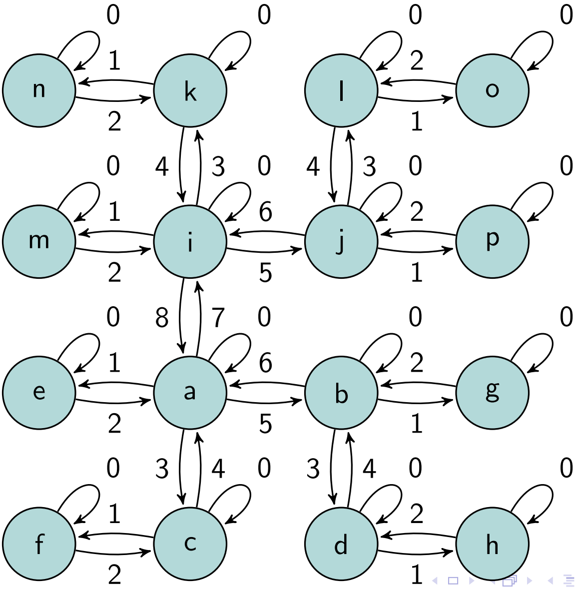


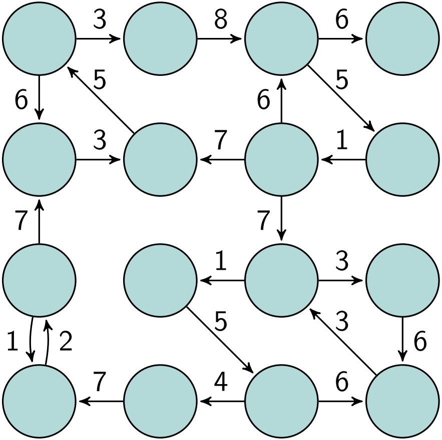
Examples of high register-index: 4



Parity games of growing register-index: 5



Logarithmic bound on register index



- Game G with max priority $q = 8$

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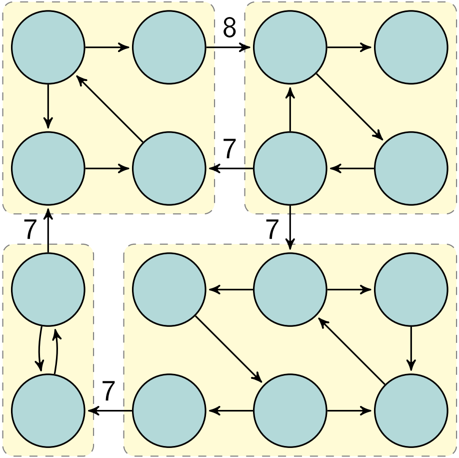
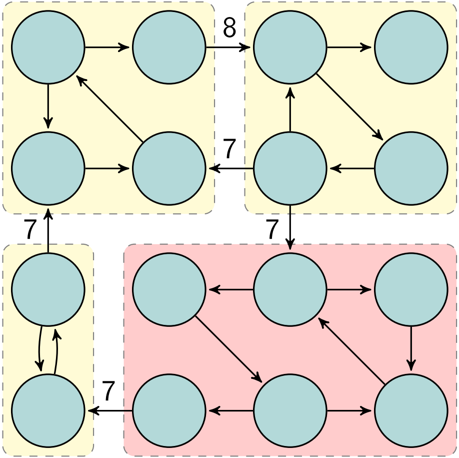


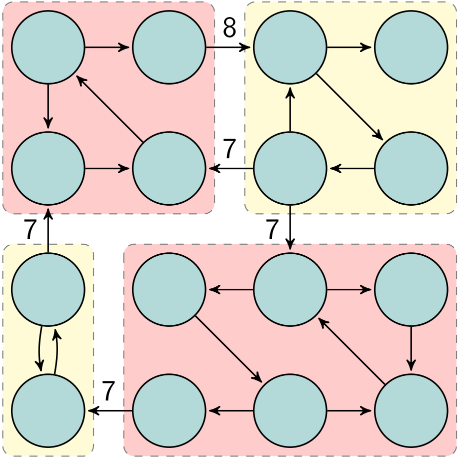
- Game G with max priority $q = 8$
- Subgames G_i with max priority $q - 2 = 6$
- Register indices r_i
- $r_{\max}$

Logarithmic bound on register index



Case: Unique $r_{\text{max}} > 1$
Then $r = r_{\text{max}}$.

Logarithmic bound on register index



Case: 2 or more $r_{\max}$
Then $r = r_{\max} + 1$.

Theorem

The register-index of a parity game of size n is $O(\log n)$.

Corollary

Parity games are solvable in quasi-polynomial time.

Proof.

Solve the k-register game on G instead of G, with $k = 1 + \log n$.
Size: $O(kn^{k+1})$
Priorities: $2k + 1$
Complexity: $2^{O((\log n)^3)}$ □

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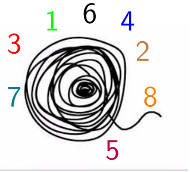
Proof.

Model check a modal μ formula of size $O(kn^k)$ with alternation depth k, with $k = 1 + \log n$. □

Summary

- **Register-index** measures the complexity of the priority assignment.
- Solving parity games of bounded register-index is in PTIME.
- Register-index is $O(\log n)$ in the size of parity games.
- Alternative quasi-polynomial algorithm.
- Modal μ-account.



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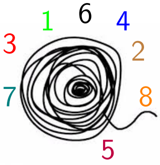

Open questions:

- Practical parametrised algorithm?
- What about node ownership?