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An Elementary Introduction to Lie Groups

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Chapter 1

Introduction

These notes grew out from some informal lectures which the first author gave in the seminar of Prof. Sommer (Inst. für Informatik, Cognitive Systeme, Christian Albrechts Universität Kiel). The lectures were meant for non-mathematicians which had from their working background a special interest in these things. The content of these lectures was not fixed from the beginning but they were strongly influenced by the discussions with the audiance in the seminar.

During a week in spring 2003, the first author was invited by Prof. Rita Vincenti for an Erasmus Visit to the University of Perugia. Using the material of Kiel, he gave five lectures on an elementary introduction to Lie groups. A listener of these lectures, the second author, took notes, worked it over and made a latex file from it. Some time later, both authors met a week at Kiel University, and brought the notes to the final form.

Though the lectures are now in a written form, it is not strictly a scientific text but it is still informal. The aim of the notes is to give students or interested people from other fields a very first glimps to some elementary notions of Lie groups and Lie algebras and their connection. The examples of the text may possibly help to start reading some book on Lie groups.

Chapter 2

Lie Groups and Transformation Groups

Definition 1 *A Lie group G is a group G on which there are introduced real coordinates such that the product and the inverse are differentiable maps with respect to these coordinates.*

Remark 1 Differentiable means real analytic (i.e. representable by power series). Therefore these functions are "very nice functions", like polynomials, cosine, sine, exponential.

Example 1 $GL(2,\mathbb{R})$, the group of regular 2×2 matrices over $\mathbb{R}$.
We write each regular matrix $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ as a 4-tupel $(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$, then the multiplication and the inverse have form, respectively:

$$(x_1,x_2,x_3,x_4)(y_1,y_2,y_3,y_4) = (x_1y_1+x_2y_3, x_1y_2+x_2y_4, x_3y_1+x_4y_3, x_3y_2+x_4y_4),$$

$$(x_1,x_2,x_3,x_4)^{-1} = (\frac{x_4}{x_1x_4-x_2x_3}, \frac{-x_2}{x_1x_4-x_2x_3}, \frac{-x_3}{x_1x_4-x_2x_3}, \frac{x_1}{x_1x_4-x_2x_3}).$$

The results of both maps have components which are polynomials or quotients of polynomials with denominator $\neq 0$, so these maps are differentiable. By definition $GL(2,\mathbb{R})$ is a Lie group. Since the 4-tuples which correspond to the regular matrices form an open subset of $\mathbb{R}^4$, we have here natural coordinates for the Lie group.

Example 2 Take the group of all affine maps of $\mathbb{R}$:

$$x \longmapsto ax+t,\ a=e^s,\ s\in\mathbb{R},\ t\in\mathbb{R}.$$

Then the multiplication of two maps $x\longmapsto e^{s_1}x+t_1$,$x\longmapsto e^{s_2}x+t_2$ is

$$x\longmapsto e^{s_1}x+t_1\longmapsto e^{s_2}(e^{s_1}x+t_1)+t_2 = e^{s_1+s_2}x+e^{s_2}t_1+t_2.$$

In this way we get a group $G=\{(s,t)|s,t\in\mathbb{R}\}$ where the multiplication and the inverse are respectively:

$$(s_1,t_1)(s_2,t_2) = (s_1+s_2, e^{s_2}t_1+t_2),\ (s,t)^{-1} = (-s,-e^{-s}t)).$$

The components on the right sides are differentiable in s_1,t_1,s_2,t_2 or s,t, respectively, so G is a Lie group which we call the Lie group L_2.

Example 3 The group of Euclidean motions of the plane has the maps:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}, \ \varphi, m, n \in \mathbb{R}.$$

The composition of two maps with parameters (φ_1, m_1, n_1) and (φ_2, m_2, n_2)), respectively, is:

$$(\varphi_1, m_1, n_1)(\varphi_2, m_2, n_2)$$
$$= (\varphi_1 + \varphi_2, \cos\varphi_2 m_1 - \sin\varphi_2 n_1 + m_2, \sin\varphi_2 + m_1 + \cos\varphi_2 n_1 + n_2)$$

and the inverse of a map is:

$$(\varphi, m, n)^{-1} = (-\varphi, -\cos(-\varphi)m + \sin(-\varphi)n, -\sin(-\varphi)m - \cos(-\varphi)n).$$

The three components on the right sides are differentiable, so we have a Lie group

$$\tilde{E} = \{(\varphi, m, n) | \varphi, m, n \in \mathbb{R}\}.$$

Introducing coordinates is no problem here, since we take the coordinates from $\mathbb{R}^3$.

Remark 2 Note that one usually considers the group

$$E = \{(\varphi, m, n) | \varphi \in \mathbb{R} \text{ mod } 2\pi, \ m, n \in \mathbb{R}\}.$$

Here the parameter φ runs through $\mathbb{R}$ mod 2π, therefore the underlying space is not $\mathbb{R}^3$ and the two Lie groups $\tilde{E}$ and E are not isomorphic.

Example 4 *The Lie transformation group $(GL(2, \mathbb{R}), \mathbb{R}^2)$.*
Each element of $GL(2, \mathbb{R})$ is a 2×2 matrice and defines a linear map on $\mathbb{R}^2$. Therefore, we have a map $\varphi : G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

which is called the action or operation of G on $\mathbb{R}^2$.

If we write the elements of $GL(2, \mathbb{R})$ again as (x_1, x_2, x_3, x_4) and the points of $\mathbb{R}^2$ as (y_1, y_2), then we have the map:

$$((x_1, x_2, x_3, x_4), (y_1, y_2)) \longmapsto (x_1 y_1 + x_2 y_2, x_3 y_1 + x_4 y_2)$$

where the two components on the right side are polynomials in x_1, x_2, x_3, y_1, y_2, so differentiable functions. The Lie group $GL(2, \mathbb{R})$ "acts" in a differentiable way on $\mathbb{R}^2$.
The word action means: if we write the map $G \times X \mapsto X$ as $(g, x) \longmapsto g(x)$ then the following axioms hold:
(a) $g_2(g_1(x)) = (g_2 g_1)(x)$ and
(b) $e(x) = x$ for all $x \in X$, where e is the identity element of G
In Example 4 this is clear from the rules for matrix multiplication. All this is summarized in the following

Definition 2 *A Lie transformation group (G, X) consists of a Lie group G and a space X (with real coordinates on X) and an action $G \times X \longrightarrow X$ which is differentiable with respect to the coordinates on G and X.*

Example 5 The Lie transformation group $(L_2, \mathbb{R})$.
The elements of L_2 were defined as affine maps $x \longmapsto e^s x + t,\ s, t \in \mathbb{R}$ on the real line $\mathbb{R}$. So we have the action $((s,t), x) \longmapsto e^s x + t$, which is differentiable.

The same group can act in different ways and on different spaces.

Example 6 The Lie transformation group $(L_2, \mathbb{R}^2)$.
We can write L_2 as a subgroup of $GL(2, \mathbb{R})$:

$$\begin{pmatrix} 1 & \\ t_2 & e^{s_2} \end{pmatrix} \begin{pmatrix} 1 & \\ t_1 & e^{s_1} \end{pmatrix} = \begin{pmatrix} 1 & \\ t_2 + e^{s_1} t_1 & e^{s_2 + s_1} \end{pmatrix}.$$

Then we have the linear action of L_2 on $\mathbb{R}^2$:

$$\left(\begin{pmatrix} 1 & \\ t & e^s \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \longmapsto \begin{pmatrix} 1 & \\ t & e^s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ tx + e^s y \end{pmatrix},$$

or formally:

$$((s,t), (x,y)) \longmapsto (x, tx + e^s y).$$

The notion of a Lie transformation group has three ingredients: the Lie group G, the space X and the action of G on X. Note that the group G may act on the space X in different ways:

Example 7 Take $G = (\mathbb{R}, +)$ and $X = \mathbb{R}$. Then we define two actions of G on X.

$$a)\ (t, x) \longmapsto t + x \qquad b)\ (t, x) \longmapsto e^t x.$$

The first action gives the translation group and the second action is the group of homotheties. These two actions are not isomorphic, since the first is transitive (each point can be mapped to each other point by a suitable map) and the second has a fixed point (all maps leave the origin invariant).

Example 8 In Example 4 we saw the natural action of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$ but there is also an action on $\mathbb{R}^3$:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \longmapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

One says: $GL(2, \mathbb{R})$ is represented by 3×3 matrices (see the field of mathematics which is called Representation Theory).

Definition 3 *The n-dimensional vector group $\mathbb{R}^n$ is the group $(\mathbb{R}^n, +)$ with usual (componentwise) addition.*

Definition 4 *$T^n = \mathbb{R}^n/\mathbb{Z}^n$ is called the n-dimensional torus group.*

Note that for $n = 1$ this is the rotation group $SO_2(\mathbb{R})$.

Theorem 1 *Each commutative connected Lie group is a vector group or a torus group or a product of these.*

Theorem 2 *Each topological group where the underlying topology is a topological manifold, is a Lie group.*

Remark 3 The first theorem is not a deep theorem but follows from the elementary notions of Lie theory. On the contrary the second is a very deep theorem: the solution of 5^{th} Hilbert's problem.

How to introduce coordinates in G?
a) best situation: the set G is an $\mathbb{R}^n$, then we have already coordinates.
b) next best : G is an open subset of an $\mathbb{R}^n$, then we can take the coordinates from $\mathbb{R}^n$.
c) taking quotients, for instance : $\mathbb{R} \longmapsto \mathbb{R}/\mathbb{Z}2\pi$.
d) general case: cover G by coordinate charts which overlap differentiably, then G is a differentiable manifold. This is explicitly explained in books on differentiable manifolds.

Chapter 3

The Lie Algebra of a Lie Group

The Lie algebra of a Lie group can be calculated via *one-parameter subgroups*. Instead of giving the exact definition, we show some examples.

Example 9 One-parameter subgroups of $GL(2,\mathbb{R})$:

$$\left\{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} | \varphi \in \mathbb{R}\right\} \text{ rotation group,}$$
$$\left\{\begin{pmatrix} e^t & \\ & e^t \end{pmatrix} | t \in \mathbb{R}\right\} \text{ group of homotheties,}$$
$$\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in \mathbb{R}\right\} \text{ group of shears with respect to } x\text{-axis,}$$
$$\left\{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} | c \in \mathbb{R}\right\} \text{ group of shears with respect to } y\text{-axis,}$$
$$\left\{\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} | t \in \mathbb{R}\right\} \text{ hyperbolic subgroup,}$$
$$\left\{\begin{pmatrix} e^t & \\ & 1 \end{pmatrix} | t \in \mathbb{R}\right\}, \quad \left\{\begin{pmatrix} 1 & \\ & e^t \end{pmatrix} | t \in \mathbb{R}\right\} \text{ homotheties in } x\text{- and } y\text{-direction.}$$

Let $E = \{g(t)|t \in \mathbb{R}\}$ be a one-parameter subgroup. By differentiating it with respect to t and evaluate at $t = 0$ we obtain a matrix (this matrix is not necessarily regular, but only an endomorphism). In the example of one-parameter subgroups given above differentiating leads to the following endomorphisms:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}.$$

To define the Lie algebra of a Lie group G we make this procedure for all one-parameter subgroups of G. This gives a set of matrices $\mathcal{L}(G)$. This set turns out to be a real vector space. On this vector space we define an operation $[\ ,\ ]:\ \mathcal{L}(G) \times \mathcal{L}(G) \longrightarrow \mathcal{L}(G)$ by $(A, B) \longmapsto [A, B] := BA - AB$.
This operation is called the commutator of A and B and $(\mathcal{L}(G), [\ ,\ ])$ is called the Lie

algebra of the Lie group G.

Example 10 The Lie algebra of the Lie group $L_2 \leqslant GL(2,\mathbb{R})$:
Here we see two outstanding one-parameter subgroups defined by the parameter s and the parameter t. The process of differentiating gives

$$\begin{pmatrix} e^{-s} & \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 & \\ & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix},$$

and the commutator of the resulting endomorphisms is

$$\left[\begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & \\ & 0 \end{pmatrix}\begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & \\ & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Abbreviate:

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we have $[e,f] = e$, and the commutators for linear combinations of e and f can be calculated by linear extension:
$[x_1e + x_2f, y_1e + y_2f] = x_1y[e,e] + x_2y_1[f,e] + x_1y_2[e,f] + x_2y_2[f,f] =$
$= 0 - x_2y_1[e,f] + x_1y_2[e,f] + 0 = (x_1y_2 - x_2y_1)e.$
Therefore the Lie group L_2 has the 2-dimensional real Lie algebra

$$\mathcal{L}(L_2) = \ell_2 = \{< e, f >_{\mathbb{R}}, \ [e,f] = e\}.$$

For the commutator of two matrices, $[A,B] = BA - AB$, the following rules hold:

Proposition 1 *a)* $[A,A] = 0$*, for all* A
b) $[A,B] = -[B,A]$*, for all* A,B
c) $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$*, for all* A,B,C*. (Jacobi identity)*

Proof : a) $[A,A] = AA - AA = 0$,
b) $[A,B] = BA - AB = -(AB - BA) = -[B,A]$,
c)$[[A,B],C] + [[B,C],A] + [[C,A],B] = C(BA - AB) - (BA - AB)C + A(CB - BC) - (CB - BC)A + B(AC - CA) - (AC - CA)B = CBA - CAB - BAC + ABC + ACB - ABC - CBA + BCA + BAC - BCA - ACB + CAB = 0$.

Example 11 The Lie algebra of the Lie group $SO(3,\mathbb{R})$ (orthogonal group of $\mathbb{R}^3$).
We choose the groups of rotations with respect x,y,z axis:

$$\left\{\begin{pmatrix} 1 & & \\ & \cos\varphi & -\sin\varphi \\ & \sin\varphi & \cos\varphi \end{pmatrix}, \varphi \in \mathbb{R}\right\} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} := e$$

$$\left\{\begin{pmatrix} \cos\varphi & & -\sin\varphi \\ & 1 & \\ \sin\varphi & & \cos\varphi \end{pmatrix}, \varphi \in \mathbb{R}\right\} \longmapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} := f,$$

$$\left\{\begin{pmatrix} \cos\varphi & & -\sin\varphi \\ \sin\varphi & \cos\varphi & \\ & & 1 \end{pmatrix}, \varphi \in \mathbb{R}\right\} \longmapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} := g.$$

Let us calculate the commutators:

$$[e,f] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = g.$$

Similar we get $[f,g] = e$, $[g,e] = f$. Therefore, the Lie algebra of $SO(3,\mathbb{R})$ is

$$\mathcal{L}(SO(3,\mathbb{R})) = \{< e, f, g >_{\mathbb{R}},\ [e,f] = g,\ [f,g] = e,\ [g,e] = f\}.$$

Note that $[\ ,\ ]$ is only given for the basis elements. If we have two general elements $x = (x_1, x_2, x_3) = x_1e + x_2f + x_3g$ and $y = (y_1, y_2, y_3) = y_1e + y_2f + y_3g$, then we can calculate $[x,y]$ by linear extension:

$[x_1e + x_2f + x_3g, y_1e + y_2f + y_3g] = x_1y_1[e,e] + x_2y_1[f,e] + x_3y_1[g,e] + x_1y_2[e,f] + x_2y_2[f,f] + x_3y_2[g,f] + x_1y_3[e,g] + x_2y_3[f,g] + x_3y_3[g,g] =$
$= (x_1y_2 - x_2y_1)g + (x_3y_1 - x_1y_3)f + (x_2y_3 - x_3y_2)e = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$

Comparing the result with the vector product on $\mathbb{R}^3$:

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1,$$

we see: the Lie algebra of $SO(3,\mathbb{R}))$ is exactly $\mathbb{R}^3$ with the vector product.

Example 12 *The Lie algebra of the Euclidian motion group of the plane,*

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ s & \cos\varphi & -\sin\varphi \\ t & \sin\varphi & \cos\varphi \end{pmatrix}, s, t, \varphi \in \mathbb{R}\ mod\ 2\pi\right\},$$

can be generated by the following one-parameter groups: translations in x-direction, translations in y-direction and rotations around the origin.

$$\left\{\begin{pmatrix} 1 & & \\ s & 1 & \\ 0 & & 1 \end{pmatrix}, s \in \mathbb{R}\right\} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = e,$$

$$\left\{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ t & & 1 \end{pmatrix}, t \in \mathbb{R}\right\} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = f,$$

$$\left\{ \begin{pmatrix} 1 & & \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \varphi \in \mathbb{R} \right\} \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = r.$$

Commutators:

$$[e,f] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

$$[e,r] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = f,$$

$$[f,r] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -e.$$

Therefore the Lie algebra of the Euclidian motion group of plane is

$$\mathbb{R}^3 = < e, f, r >_{\mathbb{R}}, [e,f] = 0, [e,r] = f, [f,r] = -e.$$

Definition 5 *A Lie algebra is a vector space V (over field K) with a map $V \times V \longrightarrow V$, called the commutator $[,]$, such that:*
a) $[x,x] = 0$, for all $x \in V$, $[x,y] = -[y,x]$, for all $x, y \in V$
b) $[h_1x_1 + h_2x_2, y] = h_1[x_1,y] + h_2[x_2,y]$, $[x, h_1y_1 + h_2y_2] = h_1[x,y_1] + h_2[x,y_2]$, for all $x_1, x_2, y_1, y_2, x, y \in V$, for all $h_1, h_2 \in K$
c) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$. for all $x, y, z \in V$ (Jacobi identity)

Remark 4 *Each Lie group G gives rise to a Lie algebra $\mathcal{L}(G)$, and isomorphic Lie groups have isomorphic Lie algebras.*

There is another way to calculate the Lie algebra of a Lie group, namely from a Lie transformation group (G, X). Each one-parameter subgroup of G defines a one-parameter transformation group on X (a flow) and by derivation of this flow we get a vector field on X:
Let us explain this in dimension 2.
Let $\gamma : t \longmapsto (\alpha = x(t), \beta = y(t)) : \mathbb{R} \longrightarrow X$ be a differentiable flow. Then for each

differentiable function $f : X \longrightarrow \mathbb{R}$ we calculate the derivation of f with respect to t evaluated at $t = 0$. By chainrules we get

$$\frac{\partial f}{\partial t}_{|t=0} = \frac{\partial f}{\partial x}\frac{\partial \alpha}{\partial t}_{|t=0} + \frac{\partial f}{\partial y}\frac{\partial \beta}{\partial t}_{|t=0} = (\frac{\partial \alpha}{\partial t}_{|t=0}\frac{\partial f}{\partial x} + \frac{\partial \beta}{\partial t}_{|t=0}\frac{\partial f}{\partial y}),$$

so we get the operator $f \longmapsto Xf$ with $X = \left(\frac{\partial \alpha}{\partial t}_{|t=0}\frac{\partial}{\partial x} + \frac{\partial \beta}{\partial t}_{|t=0}\frac{\partial}{\partial y}\right)$.
Such an operator

$$X = (\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}).$$

with differentiable functions ξ and η is called a vector field and it behaves like a derivation, namely

Remark 5 One easily checks that the rules for a derivation are satisfied :
a) $X(rf + sg) = rX(f) + sX(g)$,
b) $X(fg) = X(f)g + fX(g)$, (product rule)

For two vector fields X and Y one defines the commutator $[X, Y] = XY - YX$, which can be computed formally, using the product rule as will be seen in the following examples. Now we can describe the Lie algebra of a Lie transformation group (G, X).

Definition 6 *All vectors fields which come from one-parameter subgroups of G form a real vector space, and this vector space together with the commutator* $[\ ,\ ]$ *is called the Lie algebra* $(\mathcal{L}(G, X), [\ ,\ ])$ *of the Lie transformation group* (G, X)*.*

Example 13

$$L_2 = \left\{ \begin{pmatrix} e^s & \\ t & 1 \end{pmatrix}, s, t \in \mathbb{R} \right\}$$

acting linearly on $X = \mathbb{R}^2$.

The one-parameter group

$$\begin{pmatrix} e^s & 0 \\ 0 & 1 \end{pmatrix}, s \in \mathbb{R},$$

defines the s-flow

$$(x,y) \longrightarrow \begin{pmatrix} e^s & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^s x \\ y \end{pmatrix}.$$

Here we have $(\alpha(s,x,y) = e^s x$ and $\beta(s,x,y)) = y$ and calculation of the derivation with respect to s (at $s = 0$) gives:

$$\frac{\partial \alpha}{\partial s}_{|s=0} = x, \quad \frac{\partial \beta}{\partial s}_{|s=0} = 0, \quad X = x\frac{\partial}{\partial x}.$$

Similar the flow induced by the parameter t is

$$(x,y) \longrightarrow \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ tx + y \end{pmatrix}$$

Here we have $\alpha(t,x,y) = x$ and $\beta(t,x,y)) = tx + y$ and derivation gives

$$\frac{\partial \alpha}{\partial t}_{|t=0} = 0, \ \frac{\partial \beta}{\partial t} = x, \ Y = x\frac{\partial}{\partial y}.$$

We commutate these two vector fields :

$$[X,Y] = x\frac{\partial}{\partial y}(x\frac{\partial}{\partial x}) - x\frac{\partial}{\partial x}(x\frac{\partial}{\partial y}) =$$

$$x(\frac{\partial x}{\partial y} + x\frac{\partial^2}{\partial y \partial x}) - (x\frac{\partial x}{\partial x}\frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x \partial y}) = -x\frac{\partial}{\partial y} = -Y.$$

Here we have used the product rule for differentation and the second order derivation terms cancel because of the rule

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$$

Remark 6 In practice one can omit second order terms at once: if such a term arises, one can leave it away since the negative partner surely will follow and cancel it.

Set $Y = e$, $X = f$, then

$$[e,f] = [X,Y] = -[X,Y] = -(-Y) = Y = e,$$

so we have an isomorphism to the Lie algebra ℓ_2 considered earlier.

Example 14 *Take the affine action of L_2 on $X = \mathbb{R}$:*

$$x \longmapsto e^s x + t.$$

The s-flow is $x \longmapsto e^s x, \ \alpha(s,x) = e^s x$, and the derivation leads to

$$\frac{\partial \alpha}{\partial s}_{s=0} = e^s x_{s=0} = x, \ Y = x\frac{\partial}{\partial x}$$

. The t-flow (translations) is $x \longmapsto x + t, \ \alpha(t,x) = x + t$, and gives

$$\frac{\partial \alpha}{\partial t}_{t=0} = 1_{t=0} = 1, \ X = \frac{\partial}{\partial x}.$$

We commute these two vector fields X and Y:

$$[X,Y] = x\frac{\partial}{\partial x}(\frac{\partial}{\partial y}) - \frac{\partial}{\partial x}(x\frac{\partial}{\partial x}) = x\frac{\partial^2}{\partial x \partial x} - (x\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y}) = -X.$$

So $X \mapsto e$, $Y \mapsto -f$ is an isomorphism to the Lie algebra ℓ_2.

Remark 7 Among the various actions, a Lie group may have, there is one special: the action G on itself from the right. Derivation of the one-parameter transformation groups lead to the left invariant vector fields on G. These left invariant vector fields are theoretically well suited to define the Lie algebra of a Lie group.

Example 15 Given two vector fields on $\mathbb{R}^n$:

$$X = \sum_{i=1}^{n} \xi_i(x_1, x_2, ..., x_n)\frac{\partial}{\partial x_i} \quad Y = \sum_{i=1}^{n} \eta_i(x_1, x_2, ..., x_n)\frac{\partial}{\partial x_i},$$

where ξ_i,and η_i are differentiable. Then

$$[X, Y] = XY - YX = \sum_{i=1}^{n}[\sum_{j=1}^{n}(\xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j})]\frac{\partial}{\partial x_i}$$

(this is a simple direct calculation).

Chapter 4

Structure of Lie Algebras

After switching from Lie groups to Lie algebras ($G \longrightarrow \mathcal{L}(G)$), we are in an algebraic situation which allows better calculations (but note that we have lost information). In this section we study Lie algebras purely algebraically without thinking of Lie groups. Let A be a finite dimensional Lie algebra over the field K, where we assume $K = \mathbb{C}$ or $K = \mathbb{R}$, only.

Definition 7 *(Structural constants) Let $e_1, ..., e_n$ be a basis for A, then A can described by defining $[e_i, e_j]$, for all pairs of basis elements:*
$[e_i, e_j] = c_{ij}^1 e_1 + c_{ij}^2 e_2 + ... + c_{ij}^n e_n = \sum_{k=1}^n c_{ij}^k e_k$. *The commutator for an arbitrary pair of elements then follows by linear extension. Since $[e_i, e_j] = -[e_j, e_i]$, the elements with $i \leq j$ suffice.*
The elements $c_{ij}^k \in K$ are called the structural constants.

Remark 8 The c_{ij} must be well chosen such that the Jacobi identity holds.

Example 16 $\ell_2 =< e_1, e_2 | [e_1, e_2] = e_1 >$, explicitly
$[e_1, e_2] = e_1 = 1e_1 + 0e_2 : c_{12}^1 = 1, c_{12}^2 = 0$
$[e_1, e_1] = 0 = 0e_1 + 0e_2 : c_{11}^1 = 0, c_{11}^2 = 0$
$[e_2, e_1] = -e_1 = -1e_1 + 0e_2 : c_{21}^1 = -1, c_{21}^2 = 0$
$[e_2, e_2] = 0 = 0e_1 + 0e_2 : c_{22}^1 = 0, c_{22}^2 = 0.$

Definition 8 *(Complexifying) Let A be a real Lie algebra , given by the structural constants $c_{ij}^k \in \mathbb{R}$. Since $\mathbb{R} \leq \mathbb{C}$ we can think of $c_{ij}^h \in \mathbb{C}$. Then $< e_1, ...e_n >$, $c_{ij}^h \in \mathbb{C}$ is a complex Lie algebra, called the complexification B of A, and A is called a real form of the complex algebra B.*

Definition 9 *(Reallifying) Given a complex algebra A, $c_{ij}^k \in \mathbb{C}$, then by splitting each $c_{ij} = a_{ij}^k + ib_{ij}^k$, into two real numbers, one obtains a real Lie algebra (of double dimension) called the reallification of A.*

Example 17 We give an example for this procedure for groups: Since multiplication with the complex number $c = a + ib$ can be written as application of the 2×2 matrix

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, we can describe the elements $\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in GL(2,\mathbb{C})$ as real 4×4-matrices :
$\begin{pmatrix} a_1 & -b_1 & a_2 & -b_2 \\ b_1 & a_1 & b_2 & a_2 \\ a_3 & -b_3 & a_4 & -b_4 \\ b_3 & a_3 & b_4 & a_4 \end{pmatrix} \in GL(4,\mathbb{R})$.

Way of classifying real Lie algebras:

a) One first classifies Lie algebras over $\mathbb{C}$ (this is much simpler since $\mathbb{C}$ is algebraically closed).
b) Then one looks for all real forms and for all reallifications of these complex Lie algebras.

Example 18 *The real Lie algebras $so(3,\mathbb{R})$ and $sl_2(\mathbb{R}) = \mathcal{L}(SL(2,\mathbb{R}))$ are not isomorphic. But their complexifications are isomorphic Lie algebras over $\mathbb{C}$, namely $SL(2,\mathbb{C})$. So $so(3,\mathbb{R})$ and $sl_2(\mathbb{R})$ are two different real forms of the complex Lie algebra $sl_2(\mathbb{C})$.*

a) To show this, we first calculate the Lie algebra of
$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |ad - bc = 1 \right\}$.
We use the following one-parameter subgroups:

$$\left\{ \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} |t \in \mathbb{R} \right\} \longmapsto \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} := X,$$
$$\left\{ \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} |t \in \mathbb{R} \right\} \longmapsto \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} := Y$$
$$\left\{ \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} |t \in \mathbb{R} \right\} \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} := H.$$

Calculation of the commutators of the three endomorphisms X, Y, and H leads to the Lie algebra
$sl_2(\mathbb{R}) =< X, Y, H >$, $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$.

b) The Lie algebra $so(3,\mathbb{R})$ is not isomorphic to $sl_2(\mathbb{R})$.
In fact, in $sl_2(\mathbb{R})$ there are two elements H, X with $[H, X] = 2X$. But in $so(3,\mathbb{R})$ which is $\mathbb{R}^3$ with the vector product as the commutator, for two elements H, X the commutator is $H \times X \perp < H, X >$.

c) In oder to compare the complexifications of $so_3(\mathbb{R})$ and $sl_2(\mathbb{R})$ we use another representation for $sl_2(\mathbb{R})$:
Make the substitution $E = -\frac{X-Y}{2}$, $F = -\frac{H}{2}$, $G = -\frac{X+Y}{2}$, then
$[E, F] = G$, $[F, G] = -E$, $[G, E] = F$.
This is nearly the same as $so_3(\mathbb{R}) =< e, f, g >$, $[e, f] = g$, $[f, g] = e$, $[g, e] = F$, the only difference is the minus in the middle term.

d) Over $\mathbb{C}$ there is an isomorphism of Lie algebras from $< e, f, g >$ to $< E, F, G >$.

Proof: We define $\varphi :< e, f, g > \rightarrow < E, F, G >$ by
$\varphi(e) = E$, $\varphi(f) = iF$, $\varphi(g) = iG$, then
$\varphi([e, f]) = \varphi(g) = iG = i[E, F] = [E, iF] = [\varphi(e), \varphi(f)]$,
$\varphi([f, g]) = \varphi(e) = E = -[F, G] = i^2[F, G] = [iF, iG] = [\varphi(f), \varphi(g)]$,
$\varphi([g, e]) = \varphi(f) = iF = i[G, E] = [iG, E] = [\varphi(g), \varphi(e)]$.

Definition 10 *Let A be a Lie algebra, then $U \leq A$ is a Lie subalgebra if U is a vector subspace of A and $[u_1, u_2] \in U$ for all $u_1, u_2 \in U$.*

Next we want to have the notion of the quotient of a Lie algebra. In group theory it is not enough to have a subgroup U of the group G in order to define the quotient group G/U. We have to take special subgroups, namely normal subgroups N of G. Similar, for Lie algebras, one has to take special Lie subalgebras which we now define.

Definition 11 *$N \leq A$ is an ideal, if N is a Lie subalgebra of A and $[n, a] \in N$, for all $n \in N$ and all $a \in A$.*

Proposition 2 *If $N \leq A$ is an ideal, then we can construct the quotient algebra A/N.*

Proof: For vector space coset classes $a + N$ we want to set $[a + N, b + N] = [a, b] + N$. We have to check that this definition does not depend on the special representants a, b we have used. Let a_1, a_2, b_1, b_2 be elements with $a_1 - a_2 \in N$, $b_1 - b_2 \in N$, then
$[a_1, b_1] - [a_2, b_2] = [a_1, b_1] - [a_1, b_2] + [a_1, b_2] - [a_2, b_2] = [a_1, b_1 - b_2] + [a_1 - a_2, b_2] \in N + N = N$.
Here we have used $a_1 - a_2$,$b_1 - b_2 \in N$, and therefore $[a_1, b_1 - b_2], [a_1 - a_2, b_2] \in N$ since N is an ideal.

Remark 9 *One has the notion of a homomorphism $\varphi : A \longrightarrow B$ between Lie algebras. This is a linear map which is compatible with the commutator. As for other structures there is the Isomorphism theorem: if $\varphi : A \longrightarrow B$ is a surjective homomorphism then $A/_{Kernel(\varphi)} \cong B$. So, the ideals are exactly the kernels of homomorphisms.*

Definition 12 *The Lie algebra A is called simple if it has no ideals (besides A or 0). The Lie algebra A is called semi-simple if it is the product of simple Lie algebras.*

Example 19 *The Lie algebra $so_3(\mathbb{R})$ is simple.*

Proof: The Lie algebra $so_3(\mathbb{R})$ has no 2-dimensional subalgebras (and therefore no 2-dimensionale ideals): if $< x, y >$ is a two dimensional subspace, then $[x, y] = x \times y \perp < x, y >$, and therefore $[x, y] \notin < x, y >$.
Each one dimensional subspace is a (commutative) subalgebra, but again, it is not an ideal: if $< x >$ is a one-dimensional subspace, we take $y \notin < x >$. Then $[x, y] \perp < x, y >$ and therefore $[x, y] \notin < x >$.

Definition 13 *Let A be a Lie algebra und A' be the subspace of the vector space A which is generated by the set of all commutators $[a, b], a, b \in A$. Then A' turns out to be a subalgebra of A*

which is even an ideal of A. It is called the commutator algebra of A. The Lie algebra A is called solvable if the series $A \geq A' \geq A'' \geq ... \geq A^{(n)} = 0$ reaches 0 after a finite numbers of steps.

Theorem 3 *Fundamental theorem of Levi:*
Each Lie algebra A has a uniquely determined maximal solvable ideal R, called the radical, and the quotient A/R is semi-simple.

Example 20 *The Euclidean Group $\mathbb{R}^3 \ltimes SO_3(\mathbb{R})$.*
The 3-dimensional translation group corresponds to the radical, the subgroup fixing the origin $0 \in \mathbb{R}^3$ corresponds to the semi-simple quotient. Here the quotient is even simple, the group $SO_3(\mathbb{R})$.

In order to classify Lie algebras one has by Levi to study simple Lie algebras (well done), solvable Lie algebras (technical and difficult) and then apply extension theory. This means the following: Given an algebra N and an algebra Q, determine all algebras A which have an ideal N such that $A/N \cong Q$.

Theorem 4 *All simple Lie algebras over $\mathbb{C}$ are classified:*
There are four families A_n, B_n, C_n, D_n and five sporadic algebras G_2, F_4, E_6, E_7, E_8. The families are the Lie algebras of the following Lie groups: The special linear groups $SL_{n+1}(\mathbb{C})$, the orthogonal groups $SO_{2n+1}(\mathbb{C})$, the symplectic groups $SP_{2n}(\mathbb{C})$ and the orthogonal groups $SO_{2n}(\mathbb{C})$, respectively.

Theorem 5 *All simple Lie algebras over $\mathbb{R}$ are classified.*
This was done by finding all real forms of all complex simple Lie algebras (and adding all reallifications of complex simple Lie algebras).

After this rough overlook to the theory of Lie algebras we now give some examples of very small or some special Lie algebras. We think of real Lie algebras though most propositions are more general.

Proposition 3 *In dim 1 there is exactly one Lie algebra, which is commutative.*

Proof : Let $\mathbb{R}^1 =< e >$ be generated by the element e. By the axioms of Lie algebra we have $[e, e] = 0$. For an arbitrary pair of elements $ae, be, a, b \in \mathbb{R}$ it follows $[ae, be] = ab[e, e] = 0$, so the algebra is commutative.

Proposition 4 *In dimension 2 there are exactly two Lie algebras, the commutative algebra $(\mathbb{R}^2, [\ ,\ ] = 0)$ and the algebra $l_2(\mathbb{R})$.*

Proof : If the Lie algebra is not commutative then there exist elements $g \neq h$ with $[g, h] = e \neq 0$. Then g and h are linear independent, and for an arbitrary pair of elements in $\mathbb{R}^2$ it follows $[ag+bh, cg+dh] = ac[g, g]+bc[h, g]+ad[g, h]+bd[h, h] = (ad-bc)e$. Now take another basis $e, f \notin e\mathbb{R}$, then $[e, f] = re, r \in \mathbb{R}$. We substitute $\bar{f} = \frac{1}{r}f$ and get $[e, \bar{f}] = [e, \frac{1}{r}f] = \frac{1}{r}[e, f] = \frac{1}{r}re = e$. This is the Lie algebra $l_2(\mathbb{R})$.

Proposition 5 *Each 3-dimensional real Lie algebra is either simple or solvable.*

Proof : This follows from Levi's theorem because in dimension 1 and dimension 2 the algebras are not simple.

Proposition 6 *(without proof) There are - up to isomorphism - exactly two 3-dimensional simple real Lie algebras: $so(3, \mathbb{R})$ and $sl_2(\mathbb{R})$.*

Proposition 7 *Each 3-dimensional solvable Lie algebra over $\mathbb{R}$ has a 2-dimensional commutative ideal ($\mathbb{R}^2, [\ ,\ ] = 0$).*

Remark 10 *These Lie algebras correspond to 3-dimensional affine groups G of the real plane $\mathbb{R}^2$. Such a group is the extension of the 2-dimensional translation group (corresponding to the 2-dimensional ideal) by a 1-dimensional subgroup of $GL(2, \mathbb{R})$, the stabilizer of G on some point. A well known example for this situation is the Euclidean motion group of the plane.*

Definition 14 *For a Lie algebra A define $A^1 =< [A, A] >$, $A^2 =< [A, A^1] >$, $A^3 =< [A, A^2] >$,... . Then A is called nilpotent if the series $A \geq A^1 \geq A^2 \geq ... \geq 0$ reaches 0 after a finite number of steps.*

Remark: Compare the definition with the definition of solvability. Then we see that each nilpotent Lie algebra is solvable (but in general not vice versa).

Example 21 *The 3-dimensional Lie algebra "NIL".*
Define $NIL =< e, f, g >$, $[e, f] = g$. This means $[f, e] = -g$ and all other commutators are zero.
Here $A^1 =< g >$, $A^2 =< [A, < g >] >= 0$, therefore NIL is a 3-dimensional nilpotent real Lie algebra.

It turns out that NIL is the only nilpotent Lie in dimension 3. This important Lie algebra is the Lie algebra of the so-called Heisenberg group, which we will now define.

Definition 15 *The Heisenberg group consists of all maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form*

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix},$$

i.e., it is the semidirect product of the group of translations

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix},$$

and the group of shears with respect to the y-axis:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We write this group as a group of 3×3-matrices

$$\left\{ \begin{pmatrix} 1 & & \\ m & 1 & \\ n & c & 1 \end{pmatrix}, m, n, c \in \mathbb{R} \right\}.$$

and calculate the Lie algebra.
Endomorphisms induced by one-parameter subgroups:

$$\begin{pmatrix} 1 & & \\ m & 1 & \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} = e,$$

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ n & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{pmatrix} = g,$$

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & c & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 1 & 0 \end{pmatrix} = f.$$

Calculation of commutators :

$$[e, f] = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = g$$

and all other commutators are 0. So we have seen that the Lie algebra NIL is the Lie algebra of the Heisenberg group.

Definition 16 *The center of a Lie algebra A is $C = \{x \in A | [x, a] = 0,$ for all $a \in A\}$.*

Remark 11 The center is a commutative ideal of A.

Example 22 *Let us compare the following two Lie algebras:*

$$NIL = \{< e, f, g > | [e, f] = g, ([e, g] = 0, [f, g] = 0)\},$$
$$l_2 \times \mathbb{R} = \{< e, f, g > | [e, f] = e, ([e, g] = 0, [f, g] = 0)\}.$$

The Lie algebra NIL has the one-dimensional commutator algebra $< g >$ and the one-dimensional center $< g >$ which coincide. The Lie algebra $l_2 \times \mathbb{R}$ has the one-dimensional commutator $< e >$ and the one-dimensional center $< g >$. These do not coincide. The Lie algebra $l_2 \times \mathbb{R}$ is solvable but not nilpotent.

Chapter 5

Construction of Lie Groups from Lie Algebras

The transition $G \longrightarrow \mathcal{L}(G)$ from Lie groups to Lie algebras was achieved by derivation of one-parameter subgroups, i.e.: $\{A(t)|t \in \mathbb{R}\} \rightarrow M$. Going back from the Lie algebra to the Lie group is an integration process, namely, one integrates all endomorphisms M of the Lie algebra in order to get all one-parameter subgroups of the Lie group.

Example 23 *Derivation of a one-parameter subgroup:*

$$\left\{ \begin{pmatrix} e^t & \\ & e^t \end{pmatrix} |t \in \mathbb{R} \right\} \longmapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = M.$$

Integration of the endomorphism M:

$$\begin{aligned} \exp\ tM &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + t \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \dots \\ &= \begin{pmatrix} 1 + t + \frac{t^2}{2} + \dots & 0 \\ 0 & 1 + t + \frac{t^2}{2} + \dots \end{pmatrix} = \begin{pmatrix} e^t & \\ & e^t \end{pmatrix} \end{aligned}$$

Therefore, by integrating the endomorphism M, we get back the one-parameter group, we started from. Formally, we write:

$$\exp\ tM = 1 + tM + \frac{t^2}{2} M^2 + \frac{t^3}{3} M^3 \dots$$

Proposition 8 *This series converges for each endomorphism (=matrix) M and defines a one-parameter subgroup:*

$$M \mapsto \{A(t) = \exp\ tM | t \in \mathbb{R}\}.$$

Derivation of this one-parameter subgroup is the endomorphism M from which we started.

Example 24 *Derivation of a one-parameter subgroup:*

$$\left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} |t \in \mathbb{R} \right\} \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = M$$

Calculation of $\exp\ tM$*:*

$$M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, M^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M^5 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = M^9 = M^{13}, ..., M^6 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = M^{10} = M^{14}, ...$$

$$M^7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M^{11} = M^{15}, ..., M^8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M^{12} = M^{16}, ...$$

The series at the position $(1,1)$ *is :*

$$1 + t0 + \frac{t^2}{2}(-1) + \frac{t^3}{3!}0 + \frac{t^4}{4!}1 + ... = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!}... = \cos t.$$

The series at the position $(1,2)$ *is :*

$$0 + t(-1) + \frac{t^2}{2}(0) + \frac{t^3}{3!}1 + ... = -t + \frac{t^3}{3!} - \frac{t^5}{5!}... = -\sin t,$$

Therefore, we get back the one-parameter group

$$\left\{ \begin{pmatrix} cost & -sint \\ sint & cost \end{pmatrix} | t \in \mathbb{R} \right\}.$$

Example 25 *Derivation of the group of shears:*

$$\left\{ \begin{pmatrix} 1 & \\ s & 1 \end{pmatrix} | s \in \mathbb{R} \right\} \longrightarrow \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} = M.$$

Calculation of $\exp$ *for this endomorphism:*

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, M^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, ...$$

$$\exp\ tM = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + t\begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix} + 0 + 0 + ... = \left\{ \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} | t \in \mathbb{R} \right\}.$$

In the situation of a Lie transformation group we applied the differentiation process to a one-parameter transformation group (a so called differential flow) and got as result a vector field. Also in this situation we can get back the flow from the vectorfield by an integration process, i. e., by calculating formally the series $\exp$:

Example 26 *Take the one-parameter transformation group*
$\{(x,y) \longrightarrow (x+t, ye^t) | t \in \mathbb{R}\}$ *in the plane. This is a group of translations in x-direction with simultaneously stretching in y-direction.*
Let $\alpha = x + t$ *and* $\beta = ye^t$ *and calculate the derivation:*
$\frac{\partial \alpha}{\partial t}_{t=0} = 1$, $\frac{\partial \beta}{\partial t}_{t=0} = ye^t_{t=0} = y$, *so* $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Now we show, how we can get back the flow by integrating this vector field X: We set formally:

$$\exp\ tX = \begin{pmatrix} x \\ y \end{pmatrix} + tX \begin{pmatrix} x \\ y \end{pmatrix} + \frac{t^2}{2} X^2 \begin{pmatrix} x \\ y \end{pmatrix} + \ldots$$

Now we calculate

$$tX \begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} \frac{\partial x}{\partial y} + y\frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y} \end{pmatrix} = t \begin{pmatrix} 1 \\ y \end{pmatrix},$$

$$\frac{t^2}{2} X^2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{t^2}{2} X \begin{pmatrix} 1 \\ y \end{pmatrix} = \frac{t^2}{2} \begin{pmatrix} 0 \\ y \end{pmatrix},$$

$$\frac{t^3}{3!} X^3 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{t^3}{3!} \begin{pmatrix} 0 \\ y \end{pmatrix}, \ldots$$

and get

$$\exp\ tX = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 1 \\ y \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 \\ y \end{pmatrix} + \frac{t^3}{3} \begin{pmatrix} 0 \\ y \end{pmatrix} + \ldots = \begin{pmatrix} x + t \\ ye^t \end{pmatrix}.$$

We will, however, consider only Lie groups in the following, and not Lie transformation groups.

Let us now sketch roughly how one can construct a Lie group from its Lie algebra. Given a Lie algebra A, then a theorem of Ado tells that one can write this Lie algebra as linear Lie algebra. This means it is a set of matrices with the commutator of two of its matrices defined by $[M, N] = MN - NM$. Now we take one after the other all matrices of the Lie algebra and integrate them to one parameter groups, as was just explained. Then we collect all one-parameter groups coming out by this process and take the elements of these one-parameter groups. They will form a Lie group $\tilde{G}$ with $\mathcal{L}(\tilde{G}) = A$.

Example 27 1-*dimensional Lie algebra :*

$$\left\{ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} | t \in \mathbb{R} \right\}.$$

Application of exp *leads to*

$$\left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} | \varphi \in \mathbb{R} \right\},$$

the rotation group with φ running through $\mathbb{R}$ (not $\mathbb{R} \mod 2\pi$).

So, starting with the one-dimensional (real) Lie algebra, we constructed a Lie group for it, namely $\tilde{G} = \widetilde{SO_2(\mathbb{R})} = \mathbb{R}$.

But also

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} | \varphi \in \mathbb{R} \bmod 2\pi \right\}$$

has the one-dimensional Lie algebra as it's Lie algebra. This shows that for a given Lie algebra, there may exist several Lie groups which have this algebra as their Lie algebra. We get $SO_2(\mathbb{R})$ from $\tilde{G}$ by taking the quotient group $\mathbb{R}/\mathbb{Z}2\pi$. $\mathbb{Z}2\pi$ is a discrete normal subgroup of $\mathbb{R}$. We will explain this notion immediately.

We are going to formulate the main theorem on the connection between Lie groups and Lie algebras. For this we need some notions on topological groups. A topological group is a group on which also a topology is defined such that the group operations (multiplication and inverse) are continuous with respect to this topology. Since we have on a Lie group real coordinates, these coordinates also define a topology and therefore each Lie group is also a topological group. We call a Lie group (and a topological group) connected, if the underlying topology is connected.

Definition 17 *A subgroup N of a topological group G is discrete if there exists an open neighborhood of 1 such that this neighborhood intersects N in the neutral element 1 only.*

Example 28 $G = \{(x,y)|x,y \in \mathbb{R}\}$*, addition componentwise.*
$N_1 = \{(x,y)|x,y \in \mathbb{Z}\}$, $N_2 = \{(x,y)|x \in \mathbb{R}, y \in \mathbb{Z}\}$.
Then N_1 is a discrete subgroup of G, but N_2 is not.

Well known is the notion of a connected topological space. We need also:

Definition 18 *A topological space is simply connected, if each loop (beginning at some chosen point and ending at that point) can continuously be deformed to that point.*

Example 29 *Let $X = \mathbb{R}^2 \setminus \{0\}$ be the punctured plane, then a loop which goes around the point $\{0\}$ cannot be deformed to a point, so this space is not simply connected.*

For the following we need only to know that the spaces $\mathbb{R}^n, n \geq 0,$ and the spheres $S^n, n \geq 2,$ are simply connected. Now we are ready to formulate the

MAIN THEOREM:
a) Each real Lie algebra comes from a Lie group i.e., for each Lie algebra $\mathcal{A}$ there exists a Lie group G with $\mathcal{L}(G)$ isomorphic to $\mathcal{A}$.
b) For each Lie algebra $\mathcal{A}$ there is a unique connected, simply connected Lie Group $\tilde{G}$ with $\mathcal{L}(\tilde{G}) = \mathcal{A}$. This group is called the *universal Lie group* belonging to this algebra $\mathcal{A}$.
c) Each other connected Lie group G with $\mathcal{L}(G) = \mathcal{A}$ is the quotient group $\tilde{G}/D$, where D is a discrete normal subgroup of $\tilde{G}$.
d) (Isomorphism) $\tilde{G}/D_1 \cong \tilde{G}/D_2$ if D_1 and D_2 are "equally imbedded" in $\tilde{G}$, i.e., if there is an automorphism $\varphi : \tilde{G} \longrightarrow \tilde{G}$ with $D_1^{\varphi} = D_2$.

We cannot prove this theorem here, but we will give several examples, so that one can see how the theorem works.

Example 30 $A = (\mathbb{R}, [,] = 0)$ *the one-dimensional algebra (wich is commutative). The universal group is* $\tilde{G} = (\mathbb{R}, +)$*. Look for discrete normal subgroups* D*. Such a group has an element* $a \in \mathbb{R}$*,* $a > 0$ *minimal, and therefore* $D = a\mathbb{Z}$*. Up to an automorphism of* $\mathbb{R}$ *we can choose* $a = 1$*, so* $G = \mathbb{R}/\mathbb{Z}$ *is the second (connected) Lie group belonging to the* 1*-dimensional algebra.*

We have proved our very first theorem:

Theorem 6 *There are exactly two connected one-dimensional Lie groups,* $(\mathbb{R}, +)$ *and* $SO(2, \mathbb{R}) = \mathbb{R}/\mathbb{Z}$*.*

Next we start with two-dimensional Lie algebras and we already know that there are two of them: $(\mathbb{R}^2, [,] = 0)$ and ℓ_2.
a) Lie algebra $(\mathbb{R}^2, [,] = 0)$: This Lie algebra has the universal group $(\mathbb{R}^2, +)$, and up to isomorphisms we find two non-trivial discrete normal subgroups: $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 1$. They lead to the following quotient groups: $SO_2 \times SO_2$ (the 2-torus) and $SO_2 \times \mathbb{R}$ (the cylinder group).
b) Lie algebra ℓ_2: We know a Lie group with this Lie algebra, namely L_2. Since L_2 has underlying space $\mathbb{R}^2$, it is connected and simply connected, therefore we have already found the universal group $\tilde{G} = L_2$. As the next step we have to find discrete normal subgroups of L_2. Here we use the following simple but important lemma:

Lemma 1 *Each discrete normal subgroup* D *of a connected group* G *is contained in the center of* G*.*

Proof: We remind: Center $C(G) = \{x \in G | xg = gx \text{ for all } g \in G\}$. Take a fixed $d \in D$ and look at the map

$$\varphi : G \longrightarrow G : g \mapsto g^{-1}dg.$$

Since G is connected and this map is continuous (by the axioms of a topological group), the image is also connected. The image contains $d = e^{-1}de$, and since D is discrete, the image consists only of d. Therefore $g^{-1}dg = d$ for all $g \in G$ and d is in the center of G.

Proposition 9 *The center of* L_2 *is* 1 *(simple direct calculation).*

By this proposition, there is no proper quotient of L_2, in other words, L_2 is the only connected Lie group belonging to the algebra ℓ_2. We summarize:

Theorem 7 *There are exactly four connected Lie groups of dimension* 2*: the commutative Lie groups* $(\mathbb{R}^2, +)$*,* $\mathbb{R} \times SO_2$*,* $SO_2 \times SO_2$ *and the non-commutative Lie group* L_2*.*

As the last example of this section we study the Lie algebra $so_3(\mathbb{R})$. This Lie algebra was defined as the Lie algebra of the orthogonal group $SO_3(\mathbb{R})$, therefore $\mathcal{L}(SO_3(\mathbb{R})) = so_3(\mathbb{R})$ and we know already one Lie group with this algebra. But it turns out that the underlying topological space of $SO_3(\mathbb{R})$ is homeomorphic to the 3-dimensional real projective space and therefore not simply connected. So by the main theorem there must be at least one other connected Lie group $\tilde{G}$ which has $so_3(\mathbb{R})$ as its Lie algebra. In order to define this universal group we use the quaternions.

Remind the definition of the quaternions H: On the vector space $\mathbb{R}^4 =< e_0, e_1, e_2, e_3 >$ define a multiplication for the basis elements:

$$e_0 e_i = e_i e_0 = e_i, e_i^2 = -1, i = 1, 2, 3 \text{ and}$$
$$e_i e_j = e_h, e_j e_i = -e_h, i = 1, 2, 3, j = 2, 3, 1, l = 3, 1, 2.$$

By linear extension we get a multiplication for all pairs of elements of $H = \mathbb{R}^4$, and H with this multiplication and componentwise additition is a non-commutative field (skewfield), called the quaternions.
For each element $x = (x_0, x_1, x_2, x_3) = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$ we define the norm and the conjugate element by

$$|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad \bar{x} = (x_0, -x_1, -x_2, -x_3),$$

then $|xy| = |x||y|$ and $x^{-1} = \frac{\bar{x}}{|x|}$. Put $H^* = H \setminus 0\}$, then H^* is a group under multiplication, called the multiplicative subgroup.

Definition 19 *All quaternions with norm* 1 *form a subgroup of* H^**, called* $Spin(3)$*. The space of* $Spin(3) = \{(x_0, x_1, x_2, x_3)|x_o^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$ *is a* 3*-sphere and therefore simply connected.*

Theorem 8 $Spin(3)$ *is the universal group* $\tilde{G}$ *for the Lie algebra* $so(3, \mathbb{R})$*.Its center is* $\mathbb{Z}_2\{(-1, 0, 0, 0), (1, 0, 0, 0)\}$ *and* $Spin(3)/\mathbb{Z}_2 = SO(3, \mathbb{R})$*. Therefore, there are exactly two connected Lie groups with Lie algebra* $so(3, \mathbb{R})$*, namely* $Spin(3)$ *and* SO_3*.*

Proof : We define a map $\varphi : Spin(3) \longrightarrow SO_3$ as follows: for given $q \in Spin(3)$ set $i_q = (x \longmapsto q^{-1}xq) : H \longrightarrow H$. Then i_q is an orthogonal map of $\mathbb{R}^4$ since $|q^{-1}xq| = |q^{-1}||x||q|$. It leaves the real axis $< e_0 >$ invariant and therefore (by orthogonality) also the orthogonal complement $\mathbb{R}^3 =< e_1, e_2, e_3 >$. Therefore, on $\mathbb{R}^3$, there is induced a map from SO_3. Take this map as the image of q under φ. Now one checks the following: The map φ is surjective and has kernel $\mathbb{Z}_2$. Therefore $\varphi : Spin(3) \longrightarrow SO_3$ is a 2-fold covering, $Spin(3)$ and $SO_3(\mathbb{R})$ are locally isomorphic and have the same Lie algebra. We summarize, using the main theorem: The universal group for the Lie algebra $so_3(\mathbb{R})$ is the group $Spin(3)$. The center of $Spin(3)$ is $\mathbb{Z}_2 = \{(-1, 0, 0, 0), (1, 0, 0, 0)\}$, therefore one has only one quotient, namely $Spin(3)/\mathbb{Z}_2 = SO_3(\mathbb{R})$.

Remark 12 *This description of the elements of* SO_3 *using the quaternions is used in practice, for instance in applied physics, in cristallography, in robotics and even in designing computer plays.*

Without proof we note another description of $Spin(3)$:

Theorem 9 *$Spin(3) \cong SU(2,\mathbb{C})$, where*

$$SU(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1 \right\}.$$

Chapter 6

Classification of all Connected 3-dimensional Lie Groups

Using the main theorem, we will first classify all 3-dimensional Lie algebras, then determine the corresponding universal Lie groups and finally look for discrete normal subgroups in order to take the quotients. So let us first study 3-dimensional Lie algebras.

Proposition 10 *Each 3-dimensional Lie algebra is either solvable or simple.*

This can be seen by using the theorem of Levi: if the Levi decomposition is proper, i.e., the radical has dimension 1 or 2, then the quotient would be a semisimple Lie algebra of dimension 2 or 1. But we know already all Lie algebras of dimension 1 and 2, and they are solvable and not semisimple.
In both cases we can say more:

Proposition 11 *a) If the 3-dimensional Lie algebra A is solvable, then there exists a 2-dimensional commutative ideal N of A. b) If the 3-dimensional Lie algebra is simple, then it is isomorphic to $so_3(\mathbb{R})$ or isomorphic to $sl_2(\mathbb{R})$.*

The first part of the proposition can be proved be a simple direct calculation which uses the Jacobi identity. Also the second part is not a deep result, we only cite it here.

By part a) of the last proposition, each 3-dimensional solvable Lie algebra has the following structure: there is a 2-dimensional commutative ideal $< e, f >$, $[e, f] = 0$, and if we choose a third basis element g of the vector space, we get (by the ideal property) the following commutator relations:

$$[g, e] = ae + bf, \quad [g, f] = ce + df.$$

Therefore, the Lie algebra is determined by a 2×2-matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that this matrix is not uniquely determined by the Lie algebra, but if we make a change of basis, we get another matrix,

$$M^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} M,$$

where $M \in GL_2(\mathbb{R})$ is a regular 2×2-matrix. If we take in the construction the element $kg, k \neq 0,$ instead of g, then the matrix is multiplied by a factor $\neq 0$. This leads to the following

Proposition 12 *The isomorphism classes of solvable* 3*-dimensional Lie algebras correspond bijectively to the equivalence classes of* 2×2*-matrices taken projectively (i.e., up to a factor* $\neq 0$*).*

From linear algebra we know the equivalence classes of endomorphisms of $\mathbb{R}^2$ (Jordan normal form) and taking these modulo a factor $\neq 0$, we get the following

List of possible matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \\ & d \end{pmatrix},\ d \neq 0, 1, \text{"}hyperbolic\text{"}, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix},\ \text{"}parabolic\text{"},$$

$$\begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}, s \geq 0, \text{"}elliptic\text{"}.$$

The first endomorphism leads to the commutative 3-dimensional Lie algebra $(\mathbb{R}^3, [,] = 0)$. Let us treat this case separately. The corresponding universal group is the vector group $(\mathbb{R}^3, +)$. Suitable discrete normal subgroups are $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z} \times 1$, $\mathbb{Z} \times 1 \times 1$, $1 \times 1 \times 1$ and taking quotients, we obtain, respectively, SO_2^3, $SO_2^2 \times \mathbb{R}$, $SO_2 \times \mathbb{R}^2$, $\mathbb{R}^3$. Therefore,

Proposition 13 *There exist exactly the following 3-dimensional commutative connected Lie groups:* SO_2^3, $SO_2^2 \times \mathbb{R}$, $SO_2 \times \mathbb{R}^2$, $\mathbb{R}^3$.

By a similar argument it follows

Proposition 14 *Each commutative connected Lie group is either a vector group* $\mathbb{R}^m$ *or a torus group* $(SO_2)^n$ *or the product of both.*

The next endomorphism $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ leads to the product algebra $\ell_2 \times \mathbb{R}$, where we mean by $\mathbb{R}$ the one-dimensional Lie algebra $(\mathbb{R}, [,] = 0)$. The corresponding universal Lie group is the group $L_2 \times \mathbb{R}$, (here $\mathbb{R}$ denotes the one-dimensional Lie group $(\mathbb{R}, +)$). The center is easily calculated to be the second factor $\mathbb{R}$, and dividing by a discrete normal subgroup $\{0\} \times \mathbb{Z}$ gives

Proposition 15 *There are exactly two connected Lie groups, which have as Lie algebra the algebra $\ell_2 \times \mathbb{R}$, namely $L_2 \times \mathbb{R}$ and $L_2 \times SO_2(\mathbb{R})$.*

Now we come to the general case, to the Lie algebra with a 2-dimensional commutative ideal, which is defined by the 2×2-matrix M. We integrate this matrix

$$\exp\ tM = \begin{pmatrix} a_{22}(t) & a_{23}(t) \\ a_{32}(t) & a_{33}(t) \end{pmatrix}$$

and we build the affine group of the plane which consists of the translation group $\{(x,y) \mapsto (x+m, y+n), m, n \in \mathbb{R}\}$ extended by this one-parameter group.

Proposition 16 *The affine group of the plane*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & a_{22}(t) & a_{23}(t) \\ n & a_{32}(t) & a_{33}(t) \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, m, n, t \in \mathbb{R} \right\}$$

is a Lie group which has as its Lie algebra the one we startet with, i.e., the Lie algebra which was constructed using the endomorphism M. Moreover, it is even the universal Lie group belonging to this Lie algebra.

To prove this, we take the following one-parameter groups: the translation groups $\{(x,y) \mapsto (x+m, y) | m \in \mathbb{R}\}$, $\{(x,y) \mapsto (x, y+n) | n \in \mathbb{R}\}$ in x-direction and y-direction, respectively, and the t-parameter group. We derivate them and compute the commutators. So we get the Lie algebra in question. The fact that the three parameters m, n and t run over all of $\mathbb{R}$ means that we have found the universal group. The elements of the Lie group are the triples (m, n, t) and not the induced maps. So it may happen that different triples $(m_1, n_1, t_1) \neq (m_2, n_2, t_2)$ induce the same map on the plane $\mathbb{R}^2$. This fact will play a role when we will take quotients later on.
We apply this proposition to the other endomorphisms (matrices) of the list.

Proposition 17 *The Lie algebra constructed with* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *has the universal group*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & t & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix},\ m, n, t \in \mathbb{R} \right\}.$$

This is the so-called Heisenberg group, which is a 3-dimensional nilpotent group. Geometrically this group is an extension of the translation group of the plane (parameters m and n) by the group of shears with respect to the y-axis (parameter t).
The Lie algebra constructed with $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ *has the universal group*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & e^t & 0 \\ n & 0 & e^t \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix},\ m, n, t \in \mathbb{R} \right\}.$$

This group, the translation group extended by the group of homotheties is also called the group of dilatations: each line of the plane is mapped to a parallel line.
The Lie algebra constructed with $\begin{pmatrix} 1 & \\ & d \end{pmatrix}$, $d \neq 0, 1$, *has the universal group*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & e^t & 0 \\ n & 0 & e^{dt} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \ m, n, t \in \mathbb{R} \right\}.$$

Here, two parallel classes of lines are invariant, the horizontal lines and the vertical lines. The group is called hyperbolic.
The Lie algebra constructed with $\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$ *has the universal group*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & e^t & 0 \\ n & te^t & e^t \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \ m, n, t \in \mathbb{R} \right\}.$$

Since this group leaves only one parallel class invariant (the verticals), it is called parabolic.
The Lie algebra constructed with $\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$, $s \geq 0$, *has the universal group*

$$\left\{ \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ m & e^{st}\cos t & -e^{st}\sin t \\ n & e^{st}\sin t & e^{st}\cos t \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \ m, n, t \in \mathbb{R} \right\}.$$

These groups leave no parallel class invariant and are therefore called elliptic. There is a structural difference between the cases $s = 0$ *and* $s > 0$*: for* $s = 0$ *we get the universal Euclidean group* $\tilde{E}$*, for* $s > 0$ *we get an extension of the translation group by a spiral group, which depends on the spiral parameter* s.

So far the universal groups which belong to the solvable 3-dimensional Lie algebras. We have given models in form of affine groups on the plane, since in this way one gets a geometric understanding of these groups. Of course, one can also regard the 3×3-matrices as matrix groups only.

In order to get all connected Lie groups which belong to solvable Lie algebras, we have to divide out discrete normal subgroups from the universal groups (if possible). Since such a subgroup lies in the center of $\tilde{G}$, we calculate the centers of the groups. For the Heisenberg group one finds $C = \mathbb{R}$, the t-parameter group. Dividing by $\mathbb{Z} < \mathbb{R}$ we get one quotient described by the triples $\{(m, n, t) | m, n \in \mathbb{Z}, t \in \mathbb{R} \ mod \ \mathbb{Z}\}$.

There is only one other affine group with a center $\neq 0$, the universal Euclidean group $\tilde{E}$. Here the center is the set of triples $(0, 0, \mathbb{Z}2\pi)$, we write shortly $\mathbb{Z}2\pi$. Now we can take quotients $\tilde{E}/N$, where N is a subgroup of $\mathbb{Z}2\pi$. There are two extreme posibilities: we can take $N = \mathbb{Z}2\pi$, that is, we divide by all the center, then we get the Euclidean group E. Or we divide only by $1 < \mathbb{Z}2\pi$, then we get nothing new, but remain with $\tilde{E}$. In between, we may divide by the group $N = k\mathbb{Z}2\pi$ for $k > 1$. This gives a k-fold covering group $E^{(k)}$ of E. We will not explain here the notions of covering spaces and covering groups, but only look at the example $k = 2$: the parameter t describes the angle of the rotation around the origin. For the Euclidean group ($k = 1$) we define a full rotation 2π to be the same as the identity element. If $k = 2$ then we arrive at the identity element only at the parameter $2 \cdot 2\pi$, i. e., we must do two windings. This may give some idea that we have a 2-fold covering.

We have concluded the solvable case and take now the two simple 3-dimensional Lie algebras. One of them, the algebra $so_3(\mathbb{R})$ was looked at in the last section: there are exactly two connected Lie groups, $Spin(3)$ and $SO_3(\mathbb{R})$. The other simple Lie algebra is $sl_2(\mathbb{R})$, where we already know one connected Lie group, the group $SL_2(\mathbb{R})$. Unfortunately, this is not the universal group. It is easy to see at least one other Lie group which has the Lie algebra $sl_2(\mathbb{R})$, the group

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} := \Omega.$$

Without proof we state here that for $sl_2(\mathbb{R})$ there is a similar family of covering groups as for E, namely the family $\Omega^{(k)}, 1 \leq k \leq \infty$. In this family also occurs the group $SL_2(\mathbb{R})$, it is the group $\Omega^{(2)}$. The universal group appears as $\Omega^{(\infty)}$, it has a center isomorphic to $\mathbb{Z}$. This group is not so easy to see and to understand, but it has a nice interpretation as the collineation group of the Moulton planes, see Salzmann et al. [3].

We have now finished the classification of all connected 3-dimensional Lie groups and we collect the result in the following list.

List of all 3-dimensional connected Lie groups

a) The commutative groups: $\mathbb{R}^3,\ \mathbb{R}^2 \times SO_2,\ \mathbb{R} \times (SO_2)^2$ and $(SO_2)^3$,
b) the groups $L_2 \times \mathbb{R}$ and $L_2 \times SO_2$,
c) the Heisenberg group (or group of shears)

$$G = \left\{ \begin{pmatrix} 1 & & \\ u & 1 & \\ v & t & 1 \end{pmatrix},\ u, v, t \in \mathbb{R} \right\}$$

and its quotient

$$G/\mathbb{Z} = \left\{ \begin{pmatrix} 1 & & \\ u & 1 & \\ v & t & 1 \end{pmatrix},\ u, t \in \mathbb{R}, v \in \mathbb{R} \text{ mod } \mathbb{Z} \right\},$$

d) the group of dilatations :

$$\left\{ \begin{pmatrix} 1 & & \\ u & e^t & \\ v & & e^t \end{pmatrix},\ u, v, t \in \mathbb{R} \right\},$$

e) the parabolic group :

$$\left\{ \begin{pmatrix} 1 & & \\ u & e^t & \\ v & te^t & e^t \end{pmatrix},\ u, v, t \in \mathbb{R} \right\},$$

f) the hyperbolic groups:

$$\left\{ \begin{pmatrix} 1 & & \\ u & e^t & \\ v & & e^{dt} \end{pmatrix},\ u, v, t \in \mathbb{R} \right\}, d \neq 0, 1, |d| \leq 1,$$

g) the spiral groups :

$$\left\{ \begin{pmatrix} 1 & & \\ u & e^{st}\cos t & -e^{st}\sin t \\ v & e^{st}\sin t & e^{st}\cos t \end{pmatrix}, u, v, t \in \mathbb{R} \right\},\ s > 0,$$

h) the covering groups $E^{(k)}$, $1 \leq k \leq \infty$, of the Euclidian group E,
i) the groups $Spin(3)$ and SO_3,
j) the covering groups $\Omega^{(k)}$, $1 \leq k \leq \infty$, of the group $\Omega = PSL_2(\mathbb{R})$.

Chapter 7

Transitive Lie Transformation Groups

We come back to the notion of a transformation group, also called permutation group or action of a group on a set, or shortly a G-action or a G-set. There are various names for the same notion. In the last section we have seen actions of the 3-dimensional Lie groups an the space $\mathbb{R}^2$. Let us begin very general with groups G and sets X without any topological or differentiable structure on it.

Suppose we have a set X and two bijections $\varphi_1, \varphi_2 : X \longrightarrow X$ of this set, then usually one defines a product of φ_1 and φ_2 by first applying the map φ_1 and then applying φ_2. The result is again a bijection of X. Now, if there is given a group G whose elements are bijections of X, and the multiplication in G is defined in this way, then one has a *group of permutations* or as is called a *permutation group (G,X)*.

Example 31 *Let $X = \{1, 2, 3, ..., n\}$ be the finite set with n elements and let S_n be the group of all bijections of X, then we have a permutation group (S_n, X), called the symmetric group on n elements. We recall that $|S_n| = n!$.*

Example 32 *Let $(E, \mathbb{R}^2)$ be the Euclidean motion group. Then each Euclidian motion $\varphi \in E$ is a bijection of $\mathbb{R}^2$ and multiplication is the composition of maps.*

Now we formulate explicitly the definition (in a slight more special form than is usually done).

Definition 20 *A transformation group (or permutation group) (G, X) is a group G where all elements of this group are permutations of a given set X and group multiplication is the composition of maps. One also says, G "acts" or "operates" on X.*

For $x \in X$ and $g \in G$ one either writes $g(x)$ for the image of x under g or one uses the notation x^g. In the first case one gets as the product (i. e., the composition) of g_1 and g_2 the element $g_2(g_1(x))$. One is accustomed to that when one works with matrices which are applied from left. In the second case we get $x^{g_1 g_2}$, here we have g_1 and g_2 in the correct order. We will use this way of writing when we proof general theorems.
Note that the same group G may act as a transformation group on different sets, even G may act on the same set X in different ways, as we will see.

Definition 21 *The transformation group (G, X) is transitive (or G acts transitively on X), if for each pair of points $x_1, x_2 \in X$ there is some $g \in G$ with $g(x_1) = x_2$.*

Example 33 *a) The Euclidean group E acts transitively on $\mathbb{R}^2$: The group E has as a subgroup the group $(\mathbb{R}^2, +)$ of all translations, and it is clear that each point can be mapped to each other point by a translation.*
b) The action of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$ is not transitive: The origin is fixed by all group elements, therefore the point 0 cannot be mapped to a point $\neq 0$.
c) The transformation group $(GL(2, \mathbb{R}), \mathbb{R}^2 \setminus \{0\})$ is transitive. To prove this, let be given two points $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \neq \{0\}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \neq \{0\}$. We have to find a regular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

This follows from the theorem on systems of linear equations, since the matrix is regular. Note that the space $\mathbb{R}^2 \setminus \{0\}$ is homeomorphic to a cylinder, so we have a transitive action of $GL(2, \mathbb{R})$ on the cylinder.

In order to find transitive actions of a group, we use the following

Construction principle for transitive actions:

Take a group G, a subgroup $H \leq G$ and define the following set $X : X = \{Hg, g \in G\} = G/H$ is the set of coset classes of H. Then we may define an action of G on $X = G/H$ in the following way: for each $g \in G$ we define a bijection of the set G/H by

$$Hg_0 \longmapsto (Hg_0)^g := (Hg_0)g = H(g_0 g).$$

It is easy to see that this is a bijection and we check the transitivity: given two coset classes Hg_1 and Hg_2. Take $g = g_1^{-1} g_2$, then $(Hg_1)^g = (Hg_1)(g_1^{-1} g_2) = H(g_1 g_1^{-1}) g_2 = Hg_2$. This construction principle is very important, since it does not only give some transitive actions, but each transitive action can be described in this way:

Theorem 10 *Each transitive transformation group can be written as a coset action $((G, G/H), H \leq G$, for suitable G and $H \leq G$.*

Proof Let (G, X) be the given transitive transformation group, then we have already the group G and only must find a suitable subgroup $H \leq G$. We choose a fixed point $x_0 \in X$. Since the tranformation group is transitive, all points are equivalent, so is does not matter, which point we take. Then we define

$$H = \{g \in G, g(x_0) = x_0\},$$

the set of all group elements, which fix x_0. This is easily seen to be a subgroup of G, called the stabilizer of G on x_0 (or the isotropy group). Now one can show, that the given transformation group (G, X) and the coset action $(G, G/H)$ are "the same". Before we do that, we must say what is the same. To make this precise, we define:

Definition 22 *Two transformation groups (G_1, X_1) and (G_2, X_2) are isomorphic (as transformation groups) if there exists a bijection $f : X_1 \longmapsto X_2$ and an isomorphism of groups $\alpha : G_1 \longmapsto G_2$ such that*

$$(x^g)^f = (x^f)^{g^\alpha} \text{ for all } x \in X_1 \text{ and all } g \in G_1.$$

In words: you may first apply a group element $g \in G_1$ to a point $x \in X_1$ (in the first transformation group) and then go over with f to X_2. This is the same as first mapping $x \in X_1$ with f to X_2 and then applying (in the second transformation group) the group element g^α.

Now we can complete the proof of $(G, G/H) \cong (G, X)$: we take as group isomorphism $\alpha : G \longmapsto G$ the identity map and define the following map

$$f : X_1 \longmapsto X_2 : Hk \mapsto (x_0)^k.$$

Remind that x_0 is the special point in X with which we defined the stabilizer H, and $Hk, k \in G$ is a given coset in G/H. As an example we take the coset H which consists of the subgroup H itself. It may be written as $Hk, k \in H$ and is mapped to $(x_0)^k = x_0$ by the definition of the stabilizer. One can check that f is well defined, i. e., the image depends only on the coset class Hk and not on the special k which is used to describe it. Now we are ready to check the isomorphism condition:
a) We begin with an element Hk of the first point set and apply the group element $g \in G$: $(Hk)^g = H(kg)$. Now we map to the second point set with help of f: $H(kg)^f = (x_0)^{kg}$.
b) We first map Hk to the second point set: $Hk^f = (x_0)^k$ and then apply in the second transformation group the group element $g^\alpha = g : ((x_0)^k)^g = (x_0)^{kg}$. So, in both ways we get the same result and the pair α, f defines an isomorphism of transformation groups. One also says that α and f intertwine.

In the same spirit the following can be proved:

Theorem 11 *Two transitive transformation groups $(G_1, G_1/H_1), H_1 \leq G_1$, and $(G_2, G_2/H_2), H_2 \leq G_2$, are isomorphic if and only if the subgroups H_1 and H_2 are similarly embedded in G_1 and G_2, respectively. This means the following: there exists an isomorphism of groups $\alpha : G_1 \longmapsto G_2$ with $(H_1)^\alpha = H_2$.*

We skip the proof here and recommend it as an exercise to the reader. We prefer to give some more examples.

Example 34 *Let W be the group of motions of the cube, more explicitly, the subgroup of the orthogonal group $SO_3(\mathbb{R})$ which maps the cube onto itself. This is a group of order 24, and we will show that this group has different transitive actions.*
a) the action on the set of 6 faces:
Take as set X the set of 6 faces of the cube. Then it is clear that W acts transitively on X. The stabilizer H on one face has order 4 (is isomorphic to the cyclic group $\mathbb{Z}_4$) and we get the coset action $(W, W/H)$ with $|W| = 24$, $|H| = 4$. Therefore the set $X = W/H$ has $24/4 = 6$ elements, corresponding to the six faces.
b) the action of W on the set of 8 vertices:
Here the stabilizer H of W on one vertex is the cyclic group $\mathbb{Z}_3$ of order 3. So we get the coset

action $(W, W/H)$ and $X = W/H$ has order $24/3 = 8$ corresponding to the 8 vertices.
c) We have also a transitive action of W on the set of 12 edges of the cube. For this action the stabilizer H on one element is the group $\mathbb{Z}_2$ of order 2, so the set $X = W/H$ has order $24/2 = 12$ which is the number of edges of the cube.

The group W has order 24, and we know one special group of order 24, the symmetric group S_4. It turns out that the groups W and S_n are isomorphic (as groups). More than that: S_4 has a natural action on 4 points (where the elements of the group are all permutations), and also in the model of the cube one can find 4 objects, such that W operates on these objects in the same way as S_4 acts on 4 elements:

Proposition 18 *Let D be the set of 4 spatial diagonals of the cube, then $(W, D) \cong (S_4, \{1, 2, 3, 4\})$.*

This is a nice exercise for the reader. Doing this he can get a bit more familiar with these elementary notions of permutation groups.

Now let us consider Lie transformation groups (G, X): here the group G is a Lie group and X is not only a set but has real coordinates on it (is a so-called differentiable manifold) and the operation of G on X is differentiable. We will not go into details here, but only note that X is in particular a topological manifold and the action is continuous. Again we will assume that the action is transitive.

We begin with an example which shows that the same Lie group can act in different ways as a transitive transformation group.

Example 35 *Let E be the Euclidean motion group, then*
a) E acts transitively on the plane $\mathbb{R}^2$. This is the standard affine action on points which we already know. But there is another transitive action,
b) E acts transitively on the space of all lines, so we have a transitive action of E on the Moebius strip.

To proof b), let K and L be two lines in the plane. Then using a suitable rotation ρ we can map K into a line K' which is parallel to L. After this we apply a suitable translation τ which maps K' to L. The product of ρ and τ is then a map in E which brings K to L. Now we explain how the set of all lines can be viewed as a topological space. Let us first consider oriented lines (each line has two directions, we choose one of them to define an oriented line). The oriented lines can be described by two parameters: the first parameter is the direction $\varphi \in [0, 2\pi)$. The second parameter is the distance d of the origin from the line. Since we consider oriented lines, we have $d \in \mathbb{R}$. In this way we get the space $S^1 \times \mathbb{R}$ for the set of orientied lines, and this is a cylinder. In the second step we identify each oriented line with the opposite oriented line, and this identification leads to the Moebius strip.

Proposition 19 *The two Lie transformation groups of the last example are not isomorphic.*

Proof : In case of an isomorphism there would exist a topological bijection from $\mathbb{R}^2$ to the Moebius strip. But $\mathbb{R}^2$ and the Moebius strip are not homeomorphic. Another way to prove the proposition is to use the coset actions. The point action is $(E, E/E_p)$, where E_p is the stabilizer of a point. The second action (the line action) is the coset action $(E, E/E_L)$, where E_L is the subgroup fixing a line. Now we mention that the group $(\mathbb{R}^2, +)$ of translations is a characteristic subgroup of E (since it is the commutator subgroup). But the group E_L is a subgroup of the translation group and E_p is complementary to the translation group. Therefore, there cannot exist an isomorphism $\alpha : E \longmapsto E$ with $E_L^\alpha = E_p$.

In the study of transitive Lie transformation groups the following questions may occur: a) for a given connected Lie group G determine all transitive actions, it can have. b) for a given space X determine all transitive actions of connected Lie groups on this space, c) for manifolds of a fixed dimension (say 1 or 2) what transitive Lie group actions are possible on them? Usually one makes some more assumptions, for instance that the group is compact or that the space has special properties.
Since the action (G, X) is transitive, we can write it as a coset action $(G, G/H)$, where H is the stabilizer on a point. G is a Lie group and it can be shown, that H is a closed subgroup and also Lie. So for both groups we can calculate the Lie algebra: $\mathcal{L}(G) = \mathcal{A},\ \mathcal{L}(H) = \mathcal{B}$, and $\mathcal{B} \leq \mathcal{A}$ is a subalgebra.

Definition 23 *A pair of Lie algebras $\mathcal{A} \geq \mathcal{B}$ consists of a Lie algebra $\mathcal{A}$ and a subalgebra $\mathcal{B}$. Two pairs $\mathcal{A}_1 \geq \mathcal{B}_1$ and $\mathcal{A}_2 \geq \mathcal{B}_2$ are called isomorphic, if there exists a Lie algebra isomorphism $\varphi : \mathcal{A}_1 \longmapsto \mathcal{A}_2$ with $\mathcal{B}_1^\varphi = \mathcal{B}_2$.*

The following proposition follows directly from the definitions.

Proposition 20 *Each transitive action $(G, G/H)$ of a Lie group G defines a Lie algebra pair $\mathcal{L}(G) \geq \mathcal{L}(H)$. Isomorphic transitive actions define isomorphic Lie algebra pairs.*

Note, however, that non-isomorphic Lie transformations groups may have isomorphic Lie algebra pairs. But in this case the two Lie transformation groups are at least "locally isomorphic".

Way of classifying transitive Lie transformation groups

Similar to the classification of Lie groups via Lie algebras, one can classify transitive Lie group actions using Lie algebra pairs.
a) The purely algebraic part: Classify all Lie algebra pairs. We note here that one is only interested in so-called effective Lie algebra pairs $\mathcal{A} \geq \mathcal{B}$, where $\mathcal{B}$ does not contain a non-trivial ideal of $\mathcal{A}$.
b) The universal transformation group. Again, there is a unique universal transformation group for the Lie algebra pair, where universal means: the group and the space are connected and simply connected.
c) Take suitable quotients. Here one has to use the notion of quotient of a transformation

group. One takes the quotient of the acting group G and, simultaneously, a compatible quotient of the space X in the sense of topology.

We cite

Theorem 12 *(Brouwer 1909) All transitive actions of connected Lie groups on one-dimensional manifolds are classified:*
a) actions on the line $\mathbb{R}$: the translation group $(\mathbb{R},\mathbb{R})$, the affine group $(L_2,\mathbb{R})$ and the universal covering $(PSL_2(\mathbb{R}),P_1(\mathbb{R}))^{(\infty)}$ of the projective group (which acts on the projective line $P_1(\mathbb{R})$).
b) actions on the circle S^1: the rotation group $(SO_2(\mathbb{R}),S^1)$ and the finite coverings $(PSL_2(\mathbb{R}),P_2(\mathbb{R}))^{(k)}, 1 \leq k < \infty$.

Theorem 13 *(Lie 1893 and Mostow 1950) All transitive actions of connected Lie groups an 2-dimensional manifolds (surfaces) are classified.*

Remarks: Lie (1893) gave a classification by vector fields. This is the classification of local isomorphism types, or, if one translates it into Lie algebra language, this is the classification of Lie algebra pairs. Mostow (1950) did the globalizing, i.e., he really classified all Lie transformation groups belonging to these local types. This wonderful and fundamental paper is worked out in a lecture by D. Betten: Transitive Wirkungen auf Flächen, Kiel 1977.

We append the list of vector fields. In the list, the following abbreviations are used: $p = \frac{\partial}{\partial x}$ and $q = \frac{\partial}{\partial y}$. Let us further explain what "Richtungsgruppe" means: Locally the action is like an affine action, i. e., if one fixes a point, the stabilizer behaves like a subgroup of $GL_2(\mathbb{R})$. This group induces an action on the space of lines through 0, and this is the "Richtungsgruppe". For the Euclidian motion group it has dimension 1, so we must look for the Euclidian motion group in section II: we find it in no II 12 with $\gamma = 0$ (for $\gamma > 0$ we get the spiral groups). The full affine group of the plane has 3-dimensional direction group and must be in part IV: it is no IV 2.

List of vector fields

0-dimensional Richtungsgruppe:

$I1$ p, q
$I2$ $p, q + xp$
$I3$ $p, q, xp + yq$

1-dimensional Richtungsgruppe:

$II1$ $p, q, xp + (x + y)q$
$II2$ $p, q + xp, 2xq + x^2p$
$II3$ $p, w(x)q, w'(x)q, ..., w^{(r-1)}q$, $r > 1$, where w solves the differential equation: $w^{(r)} = c_0w^{(r-1)} + ... + c_{r-1}w$

$II4$ $p, q, xq, x^2q, xp + yq, x^2p + 2xyq$
$II5$ $p, q, xp + yq, xq, ..., x^sq$
$II6$ p, q, xp, x^2p
$II7$ $p, q, (c + 1)xp + (c - 1)yq$
$II8$ $p + y(xp + yq), q + x(xp + yq), xp - yq$
$II9$ p, xp, x^2p, q, yq, y^2q
$II10$ p, xp, x^2p, q, yq
$II11$ p, xp, q, yq
$II12$ $p, q, yp - xq + \gamma(xp + yq)$
$II13$ $p, q, xp + yq, yp - xq$
$II14$ $p, q, xp + yq, yp - xq, (x^2 - y^2)p + 2xyq, 2xyp + (y^2 - x^2)q$
$II15$ $p + x(xp + yq), q + y(xp + yq), yp - xq$
$II16$ $p - x(xp + yq), q - y(xp + yq), yp - xq$

2**-dimensional Richtungsgruppe :**

$III1$ $p, q, xq, 2xp + yq, x(xp + yq)$
$III2$ $p, q, xq, ..., x^sq, 2xp + syq, x(xp + syq)$, $s > 2$
$III3$ $p, q, xq, ..., x^sq, xp + ryq$, $r \neq 1$, $s > 0$
$III4$ $p, q, xq, ..., x^sq, xp + (s + 1)yq + x^{s+1}q$, $s > 0$
$III5$ $p, wq, w'q, ..., w^{(r-1)}q, yq$, $r > 1$ where w solves the differential equation
$w^r = c_0w^{(r-1)} + ... + c_{r-1}w$
$III6$ $p, q, xp, xq + \frac{1}{2}x^2p$
$III7$ $p, q, xp, yq, xq, x^2q, ..., x^sq$
$III8$ $p, q, xp, yq, xq, x^2q, ..., x^sq, x(xp + syq)$, $s > 0$

3**-dimensional Richtungsgruppe :**

$IV1$ $p, q, yq, xq, xp - yq$
$IV2$ p, q, xp, yp, xq, yq
$IV3$ $p, q, xp, yp, xq, yq, x(xp + yq), y(xp + yq)$

In the following list we give only some books, from which the first author learned at his time or which he could use in courses. We point out to the original books by Lie, especially [1], also note the nice biography [2]. We strongly recommend appendix 9 of the book [3], where a comprehensive survey on Lie Theory - with literature - is given.

Bibliography

[1] **Lie, S.** Theorie der Transformationsgruppen, dritter und letzter Abschnitt, Teubner Leipzig 1893

[2] **Stubhaug, A.** Es war die Kühnheit meiner Gedanken, Der Mathematiker Sophus Lie, Springer 2003.

[3] **Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R., Stroppel, M.,** Compact projective planes, de Gruyter 1995.

[4] **Brouwer, L. E. J.,** Die Theorie der endlichen kontinuierlichen Gruppen, unabhängig von den Axiomen von Lie, Math. Ann. 67, 246-267, 1909.

[5] **Mostow, G. D.,** The extensibility of local Lie groups of transformations and groups on surfaces, Ann. of Math. 52, 606-636, 1950.

[6] **Cohn, P. M.,** Lie Groups, Cambridge Univ. Press 1957.

[7] **Helgason, S.,** Differential Geometry and Symmetric Spaces, Academic Press 1962

[8] **Jacobson, N.,** Lie Algebras, Interscience Publ. 1962

[9] **Tits, J.,** Liesche Gruppen und Algebren, Vorlesung Bonn 1963/64, compare also Springer Verlag,

[10] **Tondeur, P.,** Introduction to Lie Groups and Transformation Groups, Springer Lecture Notes, 1965.

[11] **Tits, J.,** Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen, Springer Lecture Notes, 1967

[12] **Hausner, M. and Schwartz, J.,T.** Lie Groups; Lie Algebras, Gordon and Breach 1968.

[13] **Sagle, A., A. and Walde, R.,E.** Introduction to Lie Groups and Lie Algebras, Academic Press 1973.

[14] **Betten, D.,** Transitive Wirkungen auf Flächen, Vorlesung Kiel 1977.

[15] **Varadarajan, V. S.** Lie Groups, Lie Algebras, and their Representations, Prentice Hall, Springer 1984.

[16] **Hilgert, J. und Neeb, K.-H.,** Lie-Gruppen und Lie-Algebren, Vieweg 1991.