INSTITUT FÜR INFORMATIK

UND PRAKTISCHE MATHEMATIK

The Monogenic Signal

Michael Felsberg and Gerald Sommer

Bericht Nr. 2009
September 2000

PAX OPTIMA RERUM

CHRISTIAN-ALBRECHTS-UNIVERSITÄT

KIEL

Institut für Informatik und Praktische Mathematik der
Christian-Albrechts-Universität zu Kiel
Olshausenstr. 40
D – 24098 Kiel

The Monogenic Signal

Michael Felsberg and Gerald Sommer

Bericht Nr. 2009
September 2000

e-mail: mfe@ks.informatik.uni-kiel.de

This work has been supported by German National Merit Foundation and
by DFG Graduiertenkolleg No. 357 (M. Felsberg) and by DFG Grant
So-320-2-2 (G. Sommer).

Abstract

This paper introduces a new two-dimensional generalization of the analytic signal. This novel approach is based on the Riesz transform which is used instead of the Hilbert transform. The combination of a 2D signal with its Riesz transform yields a sophisticated 2D analytic signal, the monogenic signal. The approach is analytically derived from irrotational and sourceless vector fields. An appropriate representation with local amplitude and local phase is presented which preserves the split of identity. This is one of the central properties of the 1D analytic signal that decomposes a signal into structural and energetic information. Furthermore, other properties of the analytic signal concerning symmetry, energy, allpass transfer function, and orthogonality are also preserved. As a central topic of this paper, a theorem about the relation between the 1D analytic signal and the 2D monogenic signal is established using the Radon transform. A possible application of this theorem is sketched and references to other applications are given. A geometric interpretation of the phase of the monogenic signal is discussed and comparisons to other approaches for a 2D analytic signal are presented.

Contents

1 Introduction

The analytic signal is one of the most capable approaches in one-dimensional signal processing. The fundamental property of the analytic signal is the *split of identity*. This means that in polar representation the modulus of the complex signal is identified with a local quantitative measure of a signal, called *local amplitude*, and the argument of the complex signal is identified with a local measure for the qualitative information of a signal, called *local phase*.

Local amplitude and local phase fulfill the properties of *invariance* and *equivariance* [11]. That means that the local phase is invariant wrt. to the local energy of the signal but changes if the local structure varies. The local amplitude is invariant wrt. the local structure but represents the local energy.

Energy and structure are independent informations contained in a signal. The polar representation of the analytic signal is like an *orthogonal decomposition* of this information. We will use the terms *structural information* and *energetic information* in the following. This terminology also gives hints for designing methods for automatic signal analysis. The main information that characterizes the signal is carried by the phase [22].

According to the enhanced representation, the analytic signal is used in plenty of applications: for coding information (phase and frequency modulation), for radar applications, for the processing of seismic data [20], speech recognition, airfoil design [26] etc.

A sophisticated generalization of the analytic signal to two dimensions should keep the idea of the orthogonal decomposition of the information. Hence, it should have a representation which is invariant and equivariant wrt. structural information and energetic information. The problem is now that a one-dimensional measure like the local phase cannot encode 2D structure because it has not enough degrees of freedom.

So the question arises how to encode 2D local structural information. In his thesis [4], Bülow chose an algebraic approach in order to increase the expressiveness of the local phase. In his approach, the local energy is in general not constant if the orientation of the signal is changed, i.e. it is not isotropic. Hence, the invariance/equivariance property is not perfectly fulfilled.

The idea which is applied in this paper is the following. We keep with a one-dimensional phase but add an *orientation information*. This yields an approach that takes the locally strongest *intrinsically one-dimensional* [18] structure and encodes it in the classical 1D phase. The orientation is encoded in a new component which we call according to local phase and local amplitude the *local orientation*. Since orientation is a geometric property, we will call this information *geometric information*.

For intrinsically 2D signals, the properties of our new generalization which we will call *monogenic signal* will be discussed in the context of theorem 3 in section 5. The monogenic signal is also somehow related to the structure tensor (e.g. [11]) but it is *linear*. Actually, we invented it starting from the structure tensor. Therefore, in the first published result [9] we used the term ’structure multivector’.

2 Preliminaries

In this section, we give the mathematical framework for the following sections. Originally, we invented the monogenic signal using geometric algebra (see e.g. [13]) and Clifford analysis (e.g. [3]). The formulation in geometric algebra is preferable because some notational problems are avoided and the derivation is straightforward (see [7]). Since geometric algebra does not belong to the usual mathematical knowledge of a signal theorist, we tried to formulate our approach in vector notation. The only exceptions are some formulae where we made use of the algebra of quaternions.

Throughout this paper, we use the following conventions and notations:

- The considered (real) *signals* are functions $f : I\!R^n \to I\!R$ which are supposed to be ’nice’, i.e. they are continuous derivable and in $I\!L_2(I\!R^n)$, such that all transforms mentioned below do exist.

- *Vectors* in $I\!R^n$ are represented by boldface letters $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ ($\top$ indicates the transpose) and their inner product is denoted by $\langle \cdot, \cdot \rangle$. In 2D, the orthogonal vector of $\boldsymbol{x} = (x_1, x_2)$ is given by $\boldsymbol{x}^\perp = (x_2, -x_1)$. In 3D, $\boldsymbol{x} \times \boldsymbol{y}$ indicates the cross product.

- The nD *Fourier transform* of a signal $f(\boldsymbol{x})$ is denoted

$$\widehat{f}(\boldsymbol{u}) = \int_{I\!R^n} f(\boldsymbol{x}) \exp(-i2\pi\langle \boldsymbol{x}, \boldsymbol{u}\rangle)\, d\boldsymbol{x} \ .$$

- The algebra of *quaternions* $I\!H$ is spanned by $\{1, i, j, k\}$ and the product is defined by $i^2 = j^2 = -1$ and $ij = -ji = k$. Linear combinations $x_1 i + x_2 j$ are identified with vectors in $I\!R^2$ by the matrix product $(i, j)\boldsymbol{x}$ ($\boldsymbol{x} = (x_1, x_2)^\top$). In contrast to the common embedding, vectors in $I\!R^3$ are identified with quaternions in the span of $\{1, i, j\}$ by $(i, j, 1)\boldsymbol{x} = x_3 + x_1 i + x_2 j$. The conjugate of a quaternion $q = q_1 + q_2 i + q_3 j + q_4 k$ is given by $\bar{q} = q_1 - q_2 i - q_3 j - q_4 k$. Therefore, the norm of q reads $\|q\| = \sqrt{q\bar{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$.

- The *Hilbert transform* is defined by the transfer function $\widehat{h}(u) = i\,\mathrm{sign}(u)$. The transformed signal is denoted $f_H(x)$.

- The *analytic signal* is defined by $f_A(x) = f(x) - i f_H(x)$.

The definitions of the Hilbert transform and the analytic signal are taken from [14, 1]. For a more detailed introduction of quaternions, see e.g. [15].

3 Motivation

As a motivation for the following sections, we will recall some properties of the Hilbert transform and the analytic signal. Furthermore, we will present a derivation of the Hilbert transform from two-dimensional vector fields.

The Hilbert transform has some important properties which are worth to be preserved in its two-dimensional generalization:

- It is anti-symmetric, which means $\widehat{h}(-u) = -\widehat{h}(u)$. This also includes that its *energy* is symmetric.
- It suppresses the DC component ($\widehat{h}(0) = 0$).
- Its energy is equal to one for all non-zero frequencies ($|\widehat{h}(u)| = 1 \ \forall u \neq 0$).

Accordingly, the analytic signal has the following properties:

- Its energy is two times the energy of the original signal (if the DC component is neglected) because f and f_H are orthogonal.
- It is complex and the analytic signal of a symmetric signal is hermitian.
- Considered in polar coordinates, the analytic signal performs a split of identity. This means that the *local amplitude* (the modulus of the complex signal) is a quantitative measure of structure and the *local phase* is a measure for the qualitative information of structure. Therefore, the analytic signal can be considered as an orthogonal decomposition into structural and energetic information.

The Hilbert transform and the Fourier transform are part of harmonic analysis. A harmonic function f is a solution of the Laplace equation $\Delta f = \langle \nabla, \nabla \rangle f = 0$ where $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})^\top$. On the other hand, the real part and the imaginary part of an analytic function are harmonic functions (e.g. [17]). Furthermore, analytic functions can be identified with gradient fields of harmonic potentials (see below). These relations are well known for 2D vector fields. As we will show, it is possible to derive the Hilbert transform from a gradient field of a harmonic potential.

Gradient fields of harmonic potentials can also be designed in higher dimensions. Though there is no link between 3D vector field theory and the (complex) analytic function, there is indeed a function theory that combines higher dimensional field theory and analysis: the Clifford analysis (e.g. [3]). In section 4, we will use 3D vector fields (and therefore, implicitly Clifford analysis) to derive a generalized Hilbert transform, the Riesz transform. This way to derive the Riesz transform is taken from [21].

At first, we will derive the Hilbert transform as a motivation. The starting point is a two-dimensional vector field $\boldsymbol{g}(\boldsymbol{x})$ which is irrotational and sourceless in the half-space $x_2 < 0$:

$$\operatorname{rot} \boldsymbol{g}(\boldsymbol{x}) = \langle \nabla, \boldsymbol{g}(\boldsymbol{x})^{\perp} \rangle = 0 \quad \text{and} \tag{1}$$

$$\operatorname{div} \boldsymbol{g}(\boldsymbol{x}) = \langle \nabla, \boldsymbol{g}(\boldsymbol{x}) \rangle = 0 \tag{2}$$

with $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}))^\top$ and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^\top$. If we identify $I\!R^2$ with the complex plane according to $z = x_2 + ix_1$ and embed $\boldsymbol{g}$ according to $g_C = g_2 - ig_1$, these equations are just the Cauchy-Riemann (CR) equations. Complex functions that fulfill the CR equations are called *analytic*[1] functions.

As a consequence of (1), there is a real function φ such that $\boldsymbol{g}$ is the gradient of φ. For the following considerations, we switch to the frequency domain wrt. x_1, which means that we apply the 1D Fourier transform. From (2) it follows that

$$i2\pi u_1 \widehat{g}_1(u_1, x_2) + \frac{\partial}{\partial x_2} \widehat{g}_2(u_1, x_2) = 0$$

and substituting $\boldsymbol{g} = \nabla\varphi$ yields

$$-4\pi^2 u_1^2 \widehat{\varphi}(u_1, x_2) + \frac{\partial^2}{\partial x_2^2} \widehat{\varphi}(u_1, x_2) = 0 \ \ .$$

This differential equation is solved in the half-space $x_2 < 0$ by $\widehat{\varphi}(u_1, x_2) = C(u_1) \exp(2\pi |u_1| x_2)$ where $C(u_1)$ is a function which is independent of x_2. Therefore, the components of the gradient field read

$$\begin{aligned} \widehat{g}_1(u_1, x_2) &= i2\pi u_1 \widehat{\varphi}(u_1, x_2) \qquad \text{and} \\ \widehat{g}_2(u_1, x_2) &= 2\pi |u_1| \widehat{\varphi}(u_1, x_2) \ \ . \end{aligned}$$

According to the fact that $\boldsymbol{g}$ can be considered as an analytic function g_C, g_1 is the harmonic-conjugate of g_2 and vice versa [17]. Note that $\boldsymbol{g}$ also fulfills (1) for this choice of g_1 and g_2.

[1] In mathematical terms, these functions are called holomorphic. Analytic means, that there is a local power series expansion about each point, which is a complete characterization of holomorphic functions [17].

Now, we consider the continuous extension of $\boldsymbol{g}$ for $x_2 \to -0$ and we obtain that $\widehat{g}_1(u_1, x_2 = 0) = \widehat{h}(u_1)\widehat{g}_2(u_1, x_2 = 0)$, such that g_C is consistent with the definition of the analytic signal ($f(x) = g_2(x, 0)$ and $f_H(x) = g_1(x, 0)$).

In mathematical terms, eq. (2) with a given f on the line $x_2 = 0$ constitutes a Dirichlet problem (the line $x_2 = 0$ is the boundary of the open subspace $x_2 < 0$). The Hilbert transform then corresponds to a switching between the two different partial derivatives on the line $x_2 = 0$ (see fig. 1).

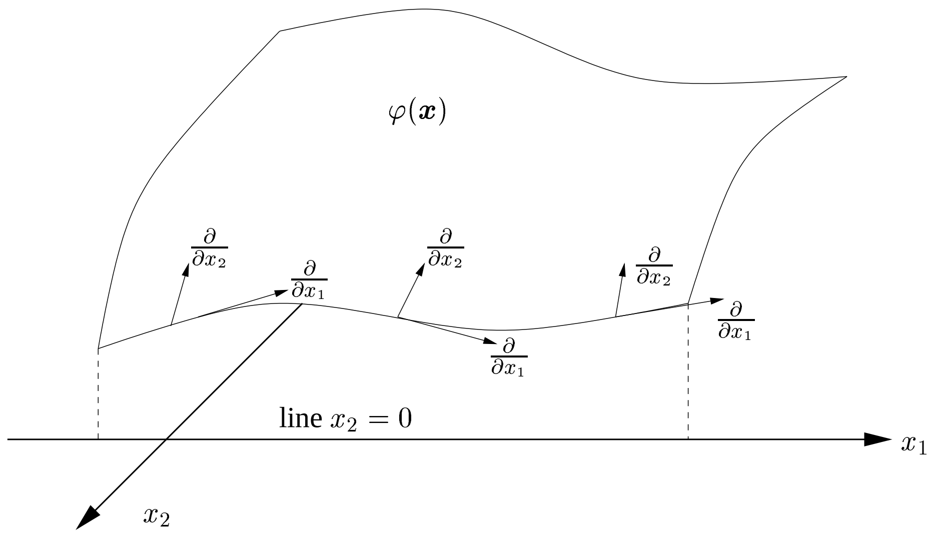

Figure 1: Dirichlet problem of second kind: $\varphi(\boldsymbol{x})$ fulfills the Laplace equation in the open domain $x_2 < 0$ and a partial derivative on the line $x_2 = 0$ is given. The other partial derivative is obtained by the Hilbert transform.

As far as we know, the following approaches for generalizing the Hilbert transform to higher dimensions can be found in the literature (for a more extensive discussion, see [4]).

- Partial Hilbert transform: The Hilbert transform is performed wrt. a half-space which is chosen by introducing a *preference direction* [11]: $\widehat{h}(\boldsymbol{u}) = i \operatorname{sign}(\langle \boldsymbol{u}, \boldsymbol{n} \rangle)$. The main drawback is the missing isotropy of the transform.

- Total Hilbert transform. The Hilbert transform is performed wrt. both axes: $\widehat{h}(\boldsymbol{u}) = -\operatorname{sign}(u_1)\operatorname{sign}(u_2)$ (see [12]). This approach is not a valid generalization of the Hilbert transform, since it does not perform a phase shift of $\pi/2$.

- A combination of partial Hilbert transforms and the total Hilbert transform [12]: $\widehat{h}(\boldsymbol{u}) = i(\mathrm{sign}(u_1) + \mathrm{sign}(u_2) + \mathrm{sign}(u_1)\,\mathrm{sign}(u_2))$. This approach is neither complete nor isotropic.

- Combined partial and total Hilbert transforms in the quaternionic Fourier domain: Instead of using the complex Fourier transform, the quaternionic Fourier transform [6] is used. The result is discussed in detail in [4]. As already pointed out in the introduction, this approach is not isotropic, either.

Hence, we can summarize that the common drawback of all approaches is the missing isotropy. This is also reflected by the wide-spread opinion in the signal processing community that no odd 2D filter with constant energy (up to the origin) exists (e.g. [16]). We will show that this conjecture is wrong if vector-valued filters are considered.

4 The Monogenic Signal

Appropriate to the previous section, we start with a 3D vector field which is irrotational and sourceless in the half-space $x_3 < 0$, i.e. it is *monogenic* for $x_3 < 0$ [19]. We have

$$\begin{aligned} \mathrm{rot}\, \boldsymbol{g}(\boldsymbol{x}) &= \nabla \times \boldsymbol{g}(\boldsymbol{x}) = 0 \quad \text{and} \qquad (3) \\ \mathrm{div}\, \boldsymbol{g}(\boldsymbol{x}) &= \langle \nabla, \boldsymbol{g}(\boldsymbol{x}) \rangle = 0\ . \qquad (4) \end{aligned}$$

Again, we conclude from (3) that there exists a potential φ such that $\boldsymbol{g}$ is its gradient field. Furthermore, we switch to the frequency domain wrt. x_1 and x_2, which means that we apply the 2D Fourier transform. Then we get from (4) the differential equation

$$\frac{\partial^2}{\partial x_3^2} \widehat{\varphi}(u_1, u_2, x_3) = 4\pi^2 (u_1^2 + u_2^2) \widehat{\varphi}(u_1, u_2, x_3)$$

which is solved in the half-space $x_3 < 0$ by

$$\widehat{\varphi}(u_1, u_2, x_3) = C(u_1, u_2) \exp\left(2\pi \sqrt{u_1^2 + u_2^2}\, x_3 \right) \ ,$$

where $C(u_1, u_2)$ is a function which is independent of x_3. Consequently, we obtain for the components of the gradient (see also [21]):

$$\begin{aligned} \widehat{g}_1(u_1, u_2, x_3) &= i2\pi u_1 \widehat{\varphi}(u_1, u_2, x_3) \qquad (5) \\ \widehat{g}_2(u_1, u_2, x_3) &= i2\pi u_2 \widehat{\varphi}(u_1, u_2, x_3) \qquad (6) \\ \widehat{g}_3(u_1, u_2, x_3) &= 2\pi \sqrt{u_1^2 + u_2^2}\, \widehat{\varphi}(u_1, u_2, x_3)\ . \qquad (7) \end{aligned}$$

Note that for this choice of $g_1,\ g_2$ and g_3 the equations (3) are fulfilled.

Finally, we get the formulae

$$\widehat{g}_1(u_1,u_2,x_3) \;=\; i\frac{u_1}{\sqrt{u_1^2+u_2^2}}\widehat{g}_3(u_1,u_2,x_3) \tag{8}$$

$$\widehat{g}_2(u_1,u_2,x_3) \;=\; i\frac{u_2}{\sqrt{u_1^2+u_2^2}}\widehat{g}_3(u_1,u_2,x_3) \tag{9}$$

which are also valid for the continuous extension $x_3 \to -0$. In that case, eqs. (8) and (9) constitute a generalization of the Hilbert transform which is known as the *Riesz transform*[2] in mathematical literature [25] (up to a minus sign which is dependent on whether taking the half-space $x_3 > 0$ or $x_3 < 0$).

From a field-theoretic point of view, the Riesz transform allows us to switch directly between the partial derivatives of a sourceless 3D potential. In mathematical terms, this means that we can change the boundary conditions of a Dirichlet problem of second kind from any partial derivative to another one.

If we embed $\mathbb{R}^3$ into the subspace of $\mathbb{H}$ spanned by $\{1,i,j\}$ (also called *paravectors* [23]) according to $q = x_3 + x_1 i + x_2 j$ and $g_Q = g_3 - g_1 i - g_2 j$, the equations (4) and (3) are equivalent to the generalized Cauchy-Riemann equations from Clifford analysis [3]. Functions that fulfill these equations are called (left) *monogenic* functions[3]. Therefore, we introduce the following terminology:

Definition 1 (monogenic signal) *The quaternion valued 2D signal*

$$\widehat{f}_M(\boldsymbol{u}) = \widehat{f}(\boldsymbol{u}) - \frac{(i,j)\boldsymbol{u}}{\|\boldsymbol{u}\|}\widehat{f}(\boldsymbol{u})i \tag{10}$$

is called the monogenic signal.

Note that we have changed the notation slightly in order to get a compact expression. Indeed, we used $(i,j)\boldsymbol{u}$ to embed $\mathbb{R}^2$ in $\mathbb{H}$ and we took the is from (8) and (9) to the right side. This enables us to write the transfer functions of the Riesz transform into one expression

$$\widehat{h}(\boldsymbol{u}) = \frac{(i,j)\boldsymbol{u}}{\|\boldsymbol{u}\|} \tag{11}$$

such that the monogenic signal is obtained by

$$f_M(\boldsymbol{x}) = f(\boldsymbol{x}) - f_R(\boldsymbol{x})i = f(\boldsymbol{x}) - h(\boldsymbol{x}) * f(\boldsymbol{x})i \ . \tag{12}$$

[2]At this place, we want to thank T. Bülow for alluding to the existence of the Riesz transform and for giving us the references [25, 21] which enabled us to identify the transform (8) in [9] with it.

[3]Originally, monogenic was another, somehow archaic term for holomorphic [17]. People from Clifford analysis reused it for expressing the multidimensional character.

This is the same formula as in the 1D case with the exception that the i has to be on the right of $h(\boldsymbol{x})$ because $I\!H$ is a skew field (non-commutative). The signal $f_R(\boldsymbol{x})$ denotes the Riesz transform of $f(\boldsymbol{x})$ embedded into $I\!H$. The explicit spatial representation of $h(\boldsymbol{x})$ is obtained in the following way: a well known 2D Fourier correspondence is $\widehat{\|\boldsymbol{x}\|^{-1}} = \|\boldsymbol{u}\|^{-1}$ which can be obtained using the Hankel transform [2]. Applying the derivative theorem of the 2D Fourier transform, we obtain

$$h(\boldsymbol{x}) = -\frac{1}{2\pi}\frac{(1,k)\boldsymbol{x}}{\|\boldsymbol{x}\|^3} \tag{13}$$

which is also given in [25]. If we think of the Riesz transform as a function in $I\!R^3$ with $x_3 = 0$ again, we have an interesting relation to the field theory: consider for example the gravitation field. The potential of a point mass in the origin is proportional to $\|\boldsymbol{x}\|^{-1}$ (Newton potential). The resulting field is therefore proportional to $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$. This analogy yields a new interpretation of 2D signals (images): instead of interpreting an image as a surfaces in 3D space, better consider it as a mass distribution in the plane $x_3 = 0$. This mass distribution yields a potential and the gradient of this potential in the plane $x_3 = 0$ is the Riesz transform of the mass distribution. In field theoretic terminology we replace the flow through the plane $x_3 = 0$ by a field of sources.

Now, since we have defined the monogenic signal as a generalized analytic signal, we can start to check whether the properties of the later are fulfilled. First, we take a look at some properties of the Riesz transform:

- It is anti-symmetric since $\widehat{h}(-\boldsymbol{u}) = -\widehat{h}(\boldsymbol{u})$ implies that $h(-\boldsymbol{x}) = -h(\boldsymbol{x})$. Note in this context that symmetry in 2D can be wrt. to a point or wrt. a line. Choosing the symmetry is the fundamental decision for designing the generalization of the Hilbert transform (in 1D, there is only one symmetry). Obviously, the Riesz transform corresponds to the point-symmetry, whereas the approach in [4] corresponds to a line-symmetry wrt. the coordinate axes.

- It suppresses the DC component. We have a singularity at $\boldsymbol{u} = 0$. If we remove it by continuously extending the two components of the Riesz transform along the lines $u_1 = 0$ (eq. (8)) and $u_2 = 0$ (eq. (9)), we immediately get $\widehat{h}(0) = 0$.

- The energy is of value one for all none-zero frequencies, i.e. $\|\widehat{h}(\boldsymbol{u})\| = 1$ $\forall \boldsymbol{u} \neq 0$. This follows directly from the definitions of $\widehat{h}(\boldsymbol{u})$ and the norm.

These properties can be vividly verified in fig. 2.

According to the properties of the Riesz transform and in comparison to the analytic signal, the monogenic signal fulfills the following two statements:

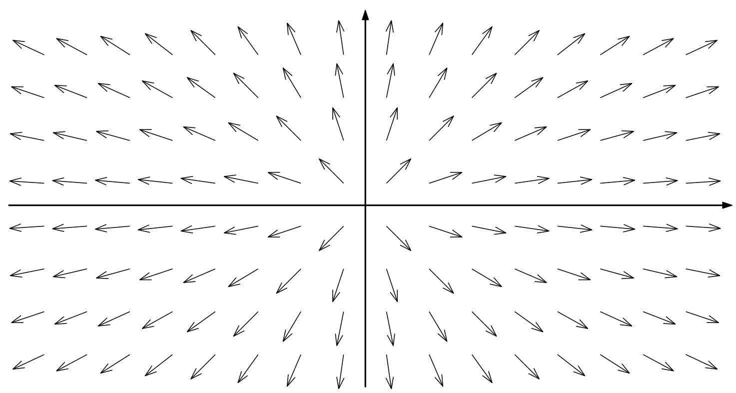

Figure 2: Transfer function of the Riesz transform displayed as a vector field.

- Its energy is two times the energy of the original signal if the DC component is neglected (proof see below).

- It is quaternion valued and the monogenic signal of a point symmetric signal ($f(-\boldsymbol{x}) = f(\boldsymbol{x})$) is quaternionic-hermitian (i.e. $f_M(-\boldsymbol{x}) = \bar{f}_M(\boldsymbol{x})$). This follows directly from the symmetry relations of the Fourier transform.

The third property of the analytic signal (split of identity in polar coordinates) is also fulfilled but this will be discussed in detail in section 5.

The energy property can be proved easily: Suppose that $f(\boldsymbol{x})$ is a DC-free real signal. The transfer function $\widehat{h}(\boldsymbol{u})$ is anti-symmetric and in the span of $\{i, j\}$. Using the symmetry relations of the Fourier transform, we obtain that $h(\boldsymbol{x})$ is also anti-symmetric but in the span of $\{1, k\}$. Consequently, we obtain by using Parseval's theorem and the orthogonality of the tripel $(1, \frac{u_1}{\|\boldsymbol{u}\|}, \frac{u_2}{\|\boldsymbol{u}\|})$:[4]

$$
\begin{aligned}
\int \|f_M(\boldsymbol{x})\| \, d\boldsymbol{x} &= \int \|1 - \widehat{h}(\boldsymbol{u}) i\| \, \|\widehat{f}(\boldsymbol{u})\| \, d\boldsymbol{u} \\
&= \int (1 + \|\widehat{h}(\boldsymbol{u})\|)\| \, \|\widehat{f}(\boldsymbol{u})\| \, d\boldsymbol{x} \\
&= \int |f(\boldsymbol{x})| \, d\boldsymbol{x} + \int \|\widehat{h}(\boldsymbol{u})\| \, \|\widehat{f}(\boldsymbol{u})\| \, d\boldsymbol{u} \\
&= \int |f(\boldsymbol{x})| \, d\boldsymbol{x} + \int \|\widehat{f}(\boldsymbol{u})\| \, d\boldsymbol{u} \\
&= 2 \int |f(\boldsymbol{x})| \, d\boldsymbol{x} \quad .
\end{aligned}
$$

[4] Note that $\|f_M(\boldsymbol{x})\|$ indicates the pointwise norm of $f_M(\boldsymbol{x})$ in contrast to $\|f_M\|$ which is the integral of $\|f_M(\boldsymbol{x})\|$.

Since the energy of an arbitrary signal is only modified by a constant real factor, we can conclude that the amplitude of the monogenic signal is *isotropic* which means that there is no dependence on the orientation of a signal (see also fig. 3). The only restriction we have is that $f(\boldsymbol{x})$ must be DC-free (same as in the case of the 1D analytic signal).

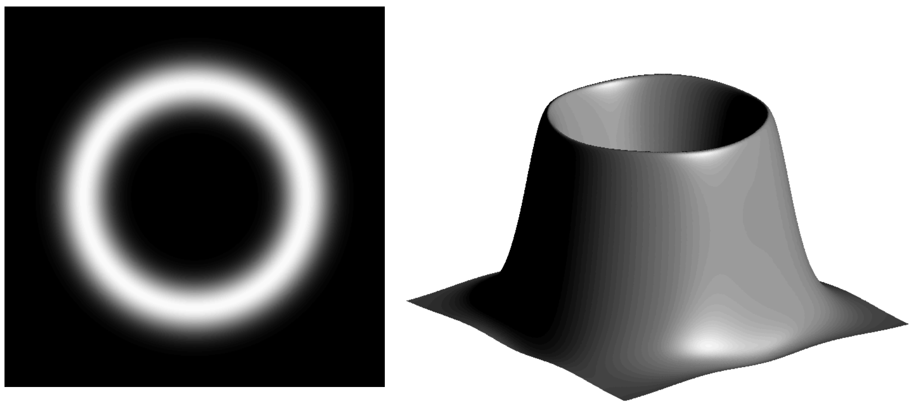

Figure 3: Left image: test-image containing all orientations; right image: the energy of the corresponding monogenic signal is isotropic.

5 The Phase of the Monogenic Signal

The phase of a complex signal a is measure for the rotation of a real signal in the complex plane. In 2D space, the rotation axis is unique, except for the *direction* of rotation. Therefore, the polar representation of a complex number $z = x + iy$ is uniquely defined by $(r, \varphi) = (\sqrt{z\bar{z}}, \mathrm{atan2}(y, x))$, where $\mathrm{atan2}$ can be defined as follows:

$$\mathrm{atan2}(y, x) = \mathrm{sign}(y)\,\mathrm{atan}\left(\frac{|y|}{x}\right) \tag{14}$$

with $\mathrm{atan}(\cdot) \in [0, \pi)$. The factor $\mathrm{sign}(y)$ indicates the direction of rotation. If we use this definition, the negative real numbers are singular because they have an angle of π wrt. positive *and* negative rotations. This is comparable to the complex logarithm which is defined on the complex plane without the negative real axis (recall the definition of the complex logarithm: $\ln(z = r\exp(i\varphi)) = \ln(r) + i\varphi$). The imaginary part of the complex logarithm also represents the argument of a complex number. Both definitions, the one of $\mathrm{atan2}$ and the one of the complex logarithm, can be extended to the whole complex plane by taking the angle modulo 2π. In that case, $\lim_{\varepsilon\to 0} \mathrm{Im}(\ln(-x+i\varepsilon)) = \lim_{\varepsilon\to 0} \mathrm{Im}(\ln(-x-i\varepsilon)) = \pi$ (with $x \in I\!R^+$, also obtained by the Cauchy principal value of $\ln(z)$).

In 3D space, the rotation axis is represented by a unit vector. The straightforward generalization of a 2D angle is then a vector with the length corresponding to the rotation angle and the direction corresponding to the rotation axis. This vector is called *rotation vector*. Consequently, we define a new arctangent function as follows:

Definition 2 (3D arctangent) *Let $\boldsymbol{x} \in I\!R^3$ be a non-zero 3D vector. The rotation vector that corresponds to the rotation of $(0, 0, \|\boldsymbol{x}\|)^\top$ into $\boldsymbol{x}$ is obtained by the 3D arctangent:*

$$\mathrm{atan3}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \boldsymbol{x}_0 \;\; \mathrm{atan}\left(\frac{\|(0,0,1)^\top \times \boldsymbol{x}_0\|}{\langle (0,0,1)^\top, \boldsymbol{x}_0 \rangle} \right) \tag{15}$$

where $\boldsymbol{x}_0 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$.

Again, there is a singularity if $\boldsymbol{x} = (0, 0, -\|\boldsymbol{x}\|)^\top$. But now it becomes obvious why this is really a singularity: the magnitude of the rotation is again well defined by π, but the rotation axis is arbitrary in the 2D subspace orthogonal to $\boldsymbol{x}$. Therefore, *any* rotation vector in that subspace is a correct solution! The only possibility to extend the definition of (15) is to define a 'modulo' on the 2D sub-space such that all vectors of length π are identical.

If we have a smooth vector field and we want to use (15) as a definition of the *phase* of the vector field (as we will do in the next paragraph), there is another possibility to extend the definition: if we consider the values of (15) in an open ball with radius ε around the singular point and by letting ε tend to zero, we get a well defined rotation vector because of the smoothness of the vector field. This continuous extension of the orientation is used in [8] for a stable orientation estimation.

Using (15), we are able to define the phase of the monogenic signal (abbreviation: monogenic phase):

Definition 3 (local phase of the monogenic signal) *The local phase of the monogenic signal $f_M(\boldsymbol{x})$ is defined by*

$$\boldsymbol{\varphi}(\boldsymbol{x}) = \mathrm{atan3}(\boldsymbol{f}_M(\boldsymbol{x})) \;\; , \tag{16}$$

where $\boldsymbol{f}_M$ is the vector field such that $f_M = (i, j, 1)\boldsymbol{f}_M$.

The rotation vector field $\boldsymbol{\varphi}(\boldsymbol{x})$ represents the rotation of the real valued signal $\|f_M(\boldsymbol{x})\|$ into the quaternionic valued signal $f_M(\boldsymbol{x})$. Note that the real component is the third component of the 3D vector (see also (15)).

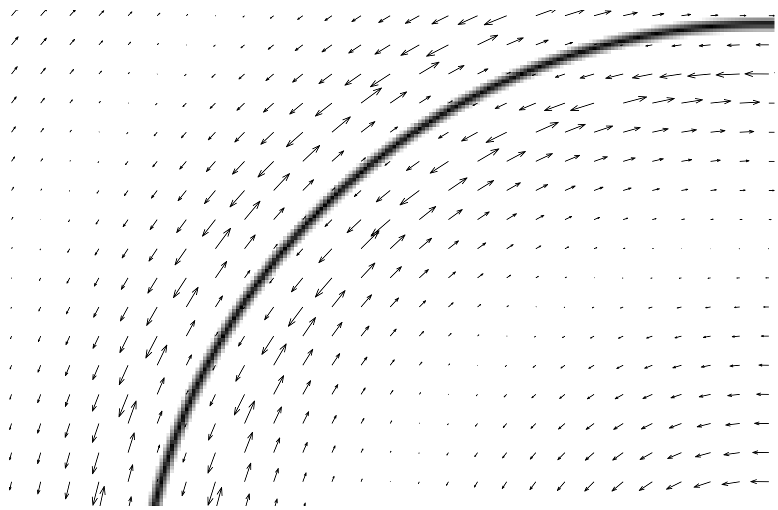

Figure 4: Monogenic phase of a simple image. The phase is represented by the vector field and the image (arc) is represented by the grayscale pattern. Note also the phase-wrapping.

The rotation vector $\boldsymbol{\varphi}$ lies always in the plane spanned by i and j (see fig. 4). Considering the vectors in a local neighborhood, there is some kind of *wrapping* of the vectors: if a vector in a certain direction would exceed the amplitude π, it is replaced by the vector minus 2π times the unit vector in that direction. In other words, it is replaced by a vector in the opposite direction with amplitude 2π minus the 'correct' amplitude. This is the same effect as the phase-wrapping in 1D.

In section 4, we already used the norm of the quaternions for calculating the energy of a monogenic signal. Indeed, the norm is used for defining the *local amplitude* of $f_M(\boldsymbol{x})$ by

$$\|f_M(\boldsymbol{x})\| = \sqrt{f^2(\boldsymbol{x}) - f_R^2(\boldsymbol{x})} \ . \tag{17}$$

The nice thing about these definitions of local phase (16) and local amplitude (17) is that the monogenic signal can be calculated from these two functions:

Theorem 1 (polar representation of the monogenic signal)
Let $\boldsymbol{\varphi}(\boldsymbol{x})$ be the local phase and let $\|f_M(\boldsymbol{x})\|$ be the local amplitude of the monogenic signal $f_M(\boldsymbol{x})$. Then the latter can be reconstructed by

$$f_M(\boldsymbol{x}) = \|f_M(\boldsymbol{x})\| \exp(\boldsymbol{\varphi}^{\perp}(\boldsymbol{x})) \ , \tag{18}$$

where $\exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!}, \ q \in I\!H$.

Proof:
Straightforward calculation yields ($\varphi = \|\boldsymbol{\varphi}\|$, $A = \|f_M(\boldsymbol{x})\|$ and $\boldsymbol{\varphi}_0 = \frac{\boldsymbol{\varphi}}{\|\boldsymbol{\varphi}\|} = -\sin(\theta)i + \cos(\theta)j$):

$$\begin{aligned}
& A\exp(\boldsymbol{\varphi}^{\perp}(\boldsymbol{x})) \\
= \;& A(\cos(\varphi) + \boldsymbol{\varphi}_0^{\perp}\sin(\varphi)) \\
= \;& A(\cos(\varphi) + \cos(\theta)\sin(\varphi)i + \sin(\theta)\sin(\varphi)j) \\
= \;& (i,j,1)\begin{bmatrix} \cos\theta\cos\varphi & -\sin\theta & \cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta & \sin\theta\sin\varphi \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \\
= \;& (i,j,1)R_3(\theta)R_2(\varphi)(0,0,A)^{\top}\ ,
\end{aligned}$$

where $R_3(\theta)$ indicates the rotation matrix for a rotation by θ about the third coordinate axis and $R_2(\varphi)$ for a rotation by φ about the second coordinate axis, see also fig. 5. ■

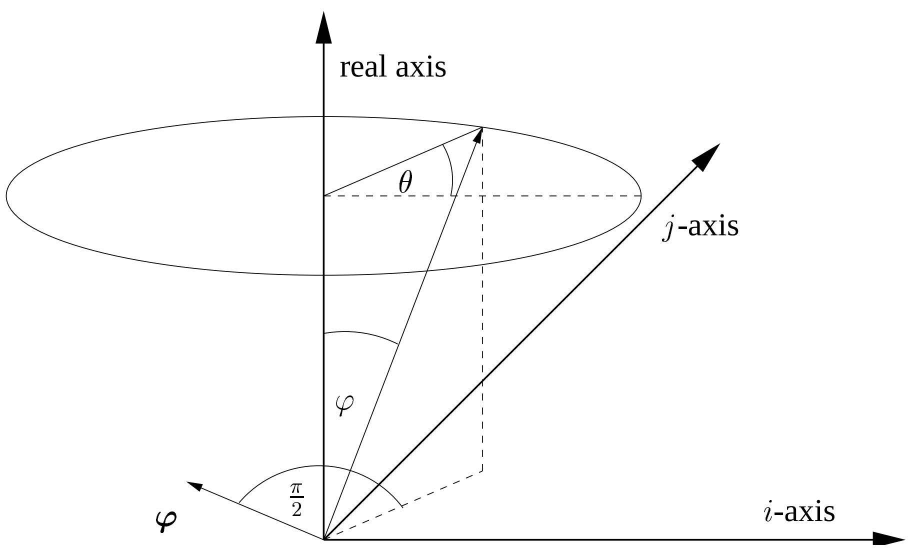

Figure 5: Rotation vector and rotations about the second axis (j-axis) by φ and about the third axis (real axis) by θ.

Note that $\ln(\|f_M(\boldsymbol{x})\|) + \boldsymbol{\varphi}(\boldsymbol{x})$ can be considered as the logarithm of the monogenic signal.

In the previous section, we omitted the property 'split of identity'. Now, having a definition of the monogenic phase, we recognize that amplitude and phase are again orthogonal. The local amplitude includes energetic information and the phase includes structural information. In contrast to the 1D case, the phase also includes *geometric information*[5], now. This problem is discussed at the end of the next section.

6 The Radon Transform and the Monogenic Signal

Up to now, we considered the 1D analytic signal and the 2D monogenic signal as two approaches which are only related by the fact that the latter is the generalization of the first one wrt. the dimension of the domain. In signal theory, there is a well known relation between 1D and 2D signals: the Radon transform [24]. The Radon transform maps a 2D signal onto a family of 1D signals with an orientation parameter. It is defined as follows:

Definition 4 (Radon transform) *The Radon transform of the 2D signal $f(\boldsymbol{x})$ is obtained by*

$$F(t,\theta) = \int_{I\!R^2} f(\boldsymbol{x})\delta_0(\langle \boldsymbol{x}, \boldsymbol{n}_\theta\rangle - t)\, d\boldsymbol{x} \ , \tag{19}$$

where $\delta_0(\cdot)$ is the Dirac-delta and $\boldsymbol{n}_\theta = (\cos\theta, \sin\theta)^\top$ with $\theta \in [0,\pi)$, i.e. an orientation[6].

Geometrically, the Radon transform projects (orthogonally) the 2D signal onto a line with orientation θ. Doing this for all orientations yields a family of 1D functions with the parameter θ. The Radon transform is invertible and there are some important theorems about signals in the Radon domain [2].

One of the most important theorems which we also need in this section is the Fourier slice theorem (e.g. [14]). We give the proof here because it is quite short and gives more insight which will help to understand the rest of this section.

Theorem 2 (slice theorem) *The 1D Fourier transform of the Radon transform with angle θ is identical to the slice of the 2D Fourier transform in orientation θ:*

$$\widehat{F}(u,\theta) = \widehat{f}(u\boldsymbol{n}_\theta) \ . \tag{20}$$

where $\widehat{F}(u,\theta)$ is the 1D Fourier transform of $F(t,\theta)$ and $\widehat{f}(\boldsymbol{u})$ is the 2D Fourier transform of $f(\boldsymbol{x})$.

[5] Actually, also the 1D phase includes geometric information: you cannot distinguish if the direction of the 1D signal has changed or if the phase itself is getting negative. This problem is solved for frequency modulation by adding a pilot tone.

[6] Note the difference between *direction* and *orientation* in this context: a direction corresponds to a vector, an orientation to a 1D subspace.

Proof:
The Radon transform is a convolution of the signal with a line orthogonal to $\boldsymbol{n}_\theta$. Applying the convolution theorem of the 2D Fourier transform, we get the pointwise product of $\widehat{f}$ with the line oriented at $\boldsymbol{n}_\theta$ through the origin. ■

Having the Radon transform as a relation between 1D and 2D signals, one can pose the following interesting question: what is the 1D correspondence of the monogenic signal? In order to answer this question, we reformulate it as: what is the Radon transform of the Riesz transform?

As it turns out, the Radon transform relates the Riesz transform to the Hilbert transform. Accordingly, we get a direct interpretation of intrinsically 1D signals in 2D space. But also for intrinsically 2D signals we obtain an interpretation by decomposing the signal into intrinsically 1D parts. Indeed, the Radon transform is the connecting link between the 1D and 2D approaches. We formulate this amazing fact in the subsequent theorem:

Theorem 3 (correspondence of Hilbert and Riesz transform)
The Radon transform of the Riesz transform of a 2D signal $f(\boldsymbol{x})$ is identical to $(i,j)\boldsymbol{n}_\theta$ times the Hilbert transform of the Radon transform of $-if(\boldsymbol{x})$:

$$F_R(t,\theta) = (1,k)\boldsymbol{n}_\theta h(t) * F(t,\theta) \ , \tag{21}$$

where $F_R(t,\theta)$ is the Radon transform of $f_R(\boldsymbol{x})$ and $h(t)$ is the kernel of the Hilbert transform.

Proof:
In the Fourier domain, due to the linearity of the Radon, Riesz, Hilbert, and Fourier transforms we have

$$\begin{aligned} \widehat{F}_R(u,\theta) &= (i,j)\boldsymbol{n}_\theta \operatorname{sign}(u)\widehat{F}(u,\theta) \\ &= -(i,j)\boldsymbol{n}_\theta i\widehat{h}(u)\widehat{F}(u,\theta) \ . \end{aligned}$$

Note that $\operatorname{sign}(u)$ must be introduced since $\theta \in [0,\pi)$ is an orientation (see also footnote 6). The frequencies where u is negative correspond to the angles in $[\pi, 2\pi)$ which inverts the vector $\boldsymbol{n}_\theta$. ■

As we just showed, the Radon transform allows us to calculate the Riesz transform (and therefore also the monogenic signal) using the Hilbert transform (see also fig. 7). This amazing fact can be used to circumvent the application of the Riesz transform in the Fourier domain (actually, the application in the spatial domain is not very sensible due to the infinite extend of the impulse response, see (13)). Especially in applications where the data is given in the Radon domain (e.g. X-ray), it is advantageous to have this theorem. By the following algorithm, we get directly the monogenic signal from data given in the Radon domain (denoted by F):

1. calculate the Hilbert transform F_H,
2. multiply F_H by $\cos\theta$ and $\sin\theta$,
3. calculate the inverse Radon transform of F, $\cos\theta\, F_H$ and $\sin\theta\, F_H$.

Having the monogenic signal, we can apply further algorithms for estimation of local properties, feature detection etc. (see [8]).

A second consequence of this theorem is that it enables us to *identify the monogenic phase with the phase of the analytic signal*. Obviously, we get from theorem 3 the identity

$$\begin{aligned} F_M(t,\theta) &= F(t,\theta) - F_R(t,\theta)i \\ &= F(t,\theta) - (i,j)\boldsymbol{n}_\theta h(t) * F(t,\theta) \\ &= (i,j,1)R_3(\theta)(-h(t) * F(t,\theta), 0, F(t,\theta))^\top \ , \end{aligned}$$

where $R_3(\theta)$ is again the rotation about the real axis by θ. Having a closer look at $(i,j,1)(-h(t) * F(t,\theta), 0, F(t,\theta))^\top$, one observes that this is just the analytic signal of the Radon transform (for every θ). Hence, the Radon transform of the monogenic signal is just the analytic signal of the Radon transform but with the imaginary unit i rotated by θ (the orientation orthogonal to the projection direction).

What does this mean for the interpretation of the monogenic phase? For linear structures with a large support (lines, edges in images), the Radon transform is dominated by this structure (see fig. 7). Hence, the monogenic phase is mainly given by the 1D phase in orthogonal projection to the structure and by the orientation of the structure.

Therefore, we get the following interpretation of the phase vector (16): the orientation of $\boldsymbol{\varphi}^\perp(\boldsymbol{x})$ represents the *local orientation* of the 2D signal and $\mathrm{sign}(\langle\boldsymbol{\varphi}(\boldsymbol{x}), (0,1)^\top\rangle)\|\boldsymbol{\varphi}(\boldsymbol{x})\|$ represents the *local 1D phase* of the 2D signal (see also fig. 5). These interpretations are consistent with the former definition of local phase and local orientation in [10, 9].

Note that these definitions do not yield a unique phase representation, since a rotation of the signal by π yields the same orientation and a negated phase. This ambiguity can be visualized by two different decompositions of a rotation with its rotation axis in a fixed plane (see fig. 6). The same problem also occurs in the context of oriented quadrature filters (see [11]), where Granlund and Knutsson claim that there is no local way to get the direction from the orientation. Bülow [5] applies a global algorithm that removes all orientation-jumps which are greater than $\pi/2$ (modulo 2π). He argues that this approach yields a consistent phase representation and that it is unique up to negation of the whole phase.

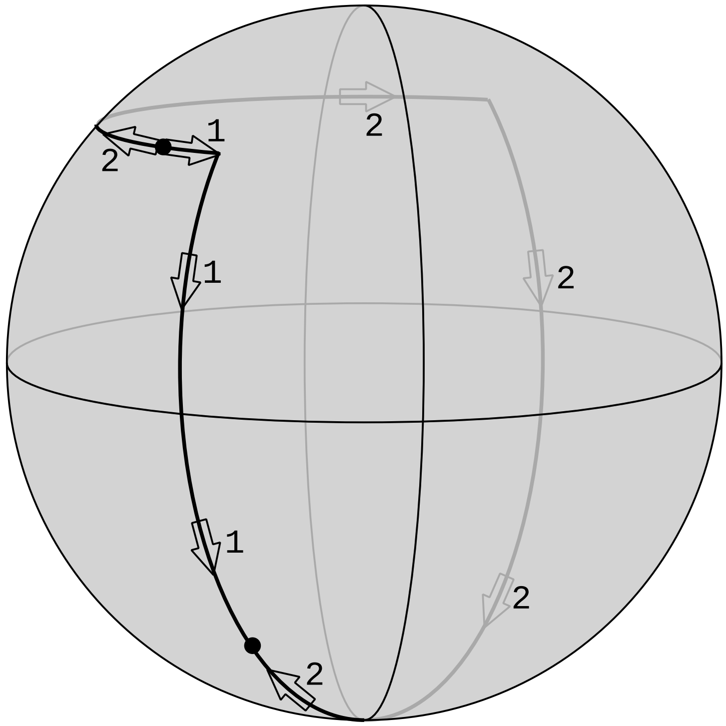

Figure 6: Two ways to decompose a 3D rotation by rotations about two fixed axes. First way: from the top dot to the right and 'directly' down to the bottom dot. Second way: to the left (to the backside), down, to the frontside again and to the bottom dot.

If the monogenic phase is decomposed into local orientation and local phase, the split of identity (the third property of the analytic signal) is also preserved wrt. geometric and structural information. Though the local phase and the local orientation are only 'nearly' orthogonal (see ambiguity above), the local direction *is* orthogonal to the phase. If we can recover the 'correct' local direction, we have really a split of identity wrt. energetic, geometric and structural information. The problem with the 'correct' local direction is that there is no absolute solution, only a relative one. This relative solution can be obtained by constraints on the smoothness of the phase and orientation.

A further consequence of theorem 3 is that 2D signals which are intrinsically 1D signals, i.e. they are constant in one direction [18], have a prefect match between the monogenic phase and the 1D phase of the underlying 1D signal: the frequency domain representation of an intrinsically 1D signals is a line. Therefore, the Radon transform is zero everywhere except for the orientation in which the signal changes (orthogonal to the constant lines in the spatial domain). Consequently, the factor $(1,k)\boldsymbol{n}_\theta$ in (21) can be replaced by a constant and the inverse Radon transform of (21) yields $f_R(\boldsymbol{x}) = (1,k)\boldsymbol{n}_\theta(\delta_0(\langle\boldsymbol{x},\boldsymbol{n}_\theta^\perp\rangle)h(\langle\boldsymbol{x},\boldsymbol{n}_\theta\rangle)) * f(\boldsymbol{x})$. Therefore, the monogenic phase reads $\boldsymbol{\varphi}(\boldsymbol{x}) = \varphi(\langle\boldsymbol{x},\boldsymbol{n}_\theta\rangle)\boldsymbol{n}_\theta^\perp$, where φ indicates the complex local phase (see also [10]). Hence, theorem 3 is the generalization of theorem 1 in [10] which is only related to intrinsically 1D signals. Consequently, we have shown that the monogenic signal is not only well suited for intrinsically 1D signal but also gives sensible (and interpretable) results for intrinsically 2D signals.

7 Conclusions

In the present paper, we have analytically derived the monogenic signal, a capable approach for the analytic signal in two dimensions. This new 2D analytic signal is based on the Riesz transform and preserves the properties of the 1D analytic signal. In contrast to the known approaches it is isotropic and therefore performs a split of identity. The information included in the signal is orthogonally decomposed into energetic, structural, and geometric information by means of local amplitude, local phase, and local orientation. We have established a theorem which directly relates the 1D analytic signal and the 2D monogenic signal. The Radon transform emerged to be the appropriate tool for shifting the 1D Hilbert transform to 2D.

A wide field of possible applications of the monogenic signal is imaginable. Up to now, only few applications have been realized. For example: estimation of the local orientation, contrast independent edge detection (see both in [7]), Moire interferograms [5], texture analysis [10]. Currently, we are working on curvature estimation, corner detection, 3D correspondence, and noise suppression. Further applications will follow.

Both, the monogenic signal and its applications are easier to formulate in geometric algebra. It is even possible to generalize the approach to arbitrary dimensions [7]. Nevertheless, we chose once more the vector notation which is easier to understand for most of the readers. Only some details had to be formulated using quaternions which can be identified within geometric algebra. Future publications will be completely formulated in geometric algebra.

Acknowledgments

This work has been supported by German National Merit Foundation and by DFG Graduiertenkolleg No. 357 (M. Felsberg) and by DFG Grant So-320-2-2 (G. Sommer). The authors would like to thank N. Krüger for many fruitful discussions.

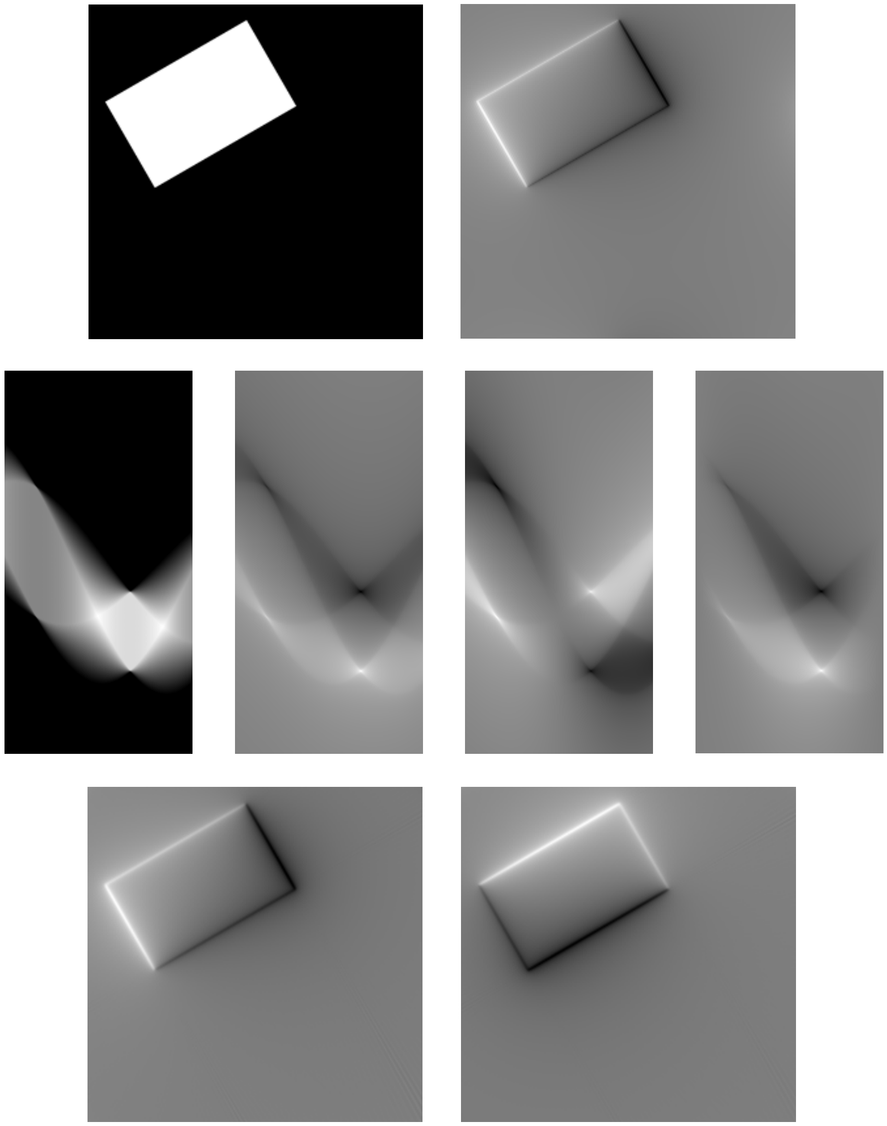

Figure 7: The top left image shows an object with dominant oriented structures. The top right image shows one component of the corresponding Riesz transform. The second row shows the following signals in the Radon domain (from left to right): original signal (note the sharp edges at those angles which correspond to the orientations of the edges of the rectangle), Hilbert transform, cosine- and sine-weighted Hilbert transform. The abscissa indicates the orientation angle and the ordinate indicates the 1D coordinates. The images in the bottom show the Riesz transform obtained from the weighted Hilbert transform (compare the left image to that one in the upper right).

References

[1] BRACEWELL, R. N. *The Fourier transform and its applications*. McGraw Hill, 1986.

[2] BRACEWELL, R. N. *Two-Dimensional Imaging*. Prentice Hall Signal Processing Series. Prentice Hall, Englewood Cliffs, 1995.

[3] BRACKX, F., DELANGHE, R., AND SOMMEN, F. *Clifford Analysis*. Pitman, Boston, 1982.

[4] BÜLOW, T. *Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images*. PhD thesis, Christian-Albrechts-University of Kiel, 1999. http://www.ks.informatik.uni-kiel.de/~vision/doc/TechnicalReports/1999_tr03.ps.gz.

[5] BÜLOW, T., PALLEK, D., AND SOMMER, G. Riesz transforms for the isotropic estimation of the local phase of moire interferograms. In *22. DAGM Symposium Mustererkennung, Kiel* (2000), G. Sommer, Ed., Springer-Verlag, Heidelberg. accepted.

[6] ELL, T. A. *Hypercomplex Spectral Transformations*. PhD thesis, University of Minnesota, 1992.

[7] FELSBERG, M., AND SOMMER, G. The multidimensional isotropic generalization of quadrature filters in geometric algebra. In *Proc. Int. Workshop on Algebraic Frames for the Perception-Action Cycle, Kiel* (2000), G. Sommer and Y. Zeevi, Eds., vol. 1888 of *Lecture Notes in Computer Science*, Springer-Verlag, Heidelberg. accepted.

[8] FELSBERG, M., AND SOMMER, G. A new extension of linear signal processing for estimating local properties and detecting features. In *22. DAGM Symposium Mustererkennung, Kiel* (2000), G. Sommer, Ed., Springer-Verlag, Heidelberg. accepted.

[9] FELSBERG, M., AND SOMMER, G. Structure multivector for local analysis of images. Tech. Rep. 2001, Institute of Computer Science and Applied Mathematics, Christian-Albrechts-University of Kiel, Germany, February 2000. http://www.ks.informatik.uni-kiel.de/~mfe/Techn_Report.ps.gz.

[10] FELSBERG, M., AND SOMMER, G. Structure multivector for local analysis of images. In *Proceedings of Dagstuhl Seminar No. 00111, Multi-Image Search, Filtering, Reasoning and Visualisation* (2000), A. Bruckstein and

R. Klette, Eds., Lecture Notes in Computer Science, Springer-Verlag. to appear.

[11] GRANLUND, G. H., AND KNUTSSON, H. *Signal Processing for Computer Vision*. Kluwer Academic Publishers, Dordrecht, 1995.

[12] HAHN, S. L. *Hilbert Transforms in Signal Processing*. Artech House, Boston, London, 1996.

[13] HESTENES, D., AND SOBCZYK, G. *Clifford algebra to geometric calculus, A Unified Language for Mathematics and Physics*. Reidel, Dordrecht, 1984.

[14] JÄHNE, B. *Digitale Bildverarbeitung*. Springer, Berlin, 1997.

[15] KANTOR, I. L., AND SOLODOVNIKOV, A. S. *Hypercomplex Numbers*. Springer Verlag, New-York, 1989.

[16] KOVESI, P. *Invariant Measures of Image Features from Phase Information*. PhD thesis, University of Western Australia, 1996.

[17] KRANTZ, S. G. *Handbook of Complex Variables*. Birkhäuser, Boston, 1999.

[18] KRIEGER, G., AND ZETZSCHE, C. Nonlinear image operators for the evaluation of local intrinsic dimensionality. *IEEE Transactions on Image Processing 5*, 6 (June 1996), 1026–1041.

[19] LOUNESTO, P. *Clifford Algebras and Spinors*, vol. 239 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1997.

[20] MILKEREIT, B., AND SPENCER, C. Multiattribute processing of seismic data: Application to dip displays. *Canadian Journal of Exploration Geophysics 26*, 1 & 2 (December 1990), 47–53.

[21] NABIGHIAN, M. N. Toward a three-dimensional automatic interpretation of potential field data via generalized Hilbert transforms: Fundamental relations. *Geophysics 49*, 6 (June 1984), 780–786.

[22] OPPENHEIM, A., AND LIM, J. The importance of phase in signals. *Proc. of the IEEE 69*, 5 (May 1981), 529–541.

[23] PORTEOUS, I. R. *Clifford Algebras and the Classical Groups*. Cambridge University Press, 1995.

[24] RADON, J. On the determination of functions from their integral values along certain manifolds. *IEEE Transactions on Medical Imaging 5*, 4 (December 1986), 170–176. Translation of the original German text by P.C.Parks.

[25] STEIN, E., AND WEISS, G. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, New Jersey, 1971.

[26] ZAYED, A. I. *Handbook of Function and Generalized Function Transformations*. Mathematical Science Reference Series. CRC Press, Boca Raton, 1996.