A Lie group approach to steerable filters

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Abstract: Recently Freeman and Adelson (1991) published an approach to steer filters in their orientation by Fourier decompositions with respect to the angular coordinate of a polar representation. Simoncelli et al. (1992) generalized this method to steer other parameters than the orientation. In this paper we formulate the problem of steerability using the Lie group that performs the deformation of the filters. Within the presented theoretical framework we especially discuss the following points: (1) The possible scope and (2) the optimality of steerability by Fourier decompositions, (3) approximate steerability using a limited number of basis functions, (4) the nature of the singularity that occurs when steering the scale.

Contents

1 Introduction

The analysis of local image structure in many early vision tasks can be improved by using the responses of the analysing filters in a continuum of orientations, scales and other parameters as was shown by several authors (Freeman and Adelson (1991), Perona (1991), Simoncelli et al. (1992), Michaelis and Sommer (1994)). To calculate the responses of filters in a continuum of orientations steerability was introduced by Freeman and Adelson (1991) recently. Even though the principles of steerability have been used by others before it was only Freeman and Adelson who addressed this problem explicitly and brought it to the attention of the computer vision community. Nevertheless, steerability is still far from being a standard tool in early vision.

Now we introduce formally what is understood as steerability. In this report the term 'steering' is applied to all deformations and not just to rotations. Let $F_\alpha(\vec{x})$ denote a filter with $\vec{x} \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ a parameter that deforms (translates, rotates, dilates) the filter. Steering means continuously varying the parameter α by an interpolation formula:

$$F_\alpha(\vec{x}) \;=\; \sum_{k=1}^{M} b_k(\alpha)\, A_k(\vec{x}) \tag{1}$$

The whole infinite range of functions for varying α is represented by linear superpositions of a finite number M of some **basis functions** A_k that are independent of α. The parameter α is contained only in the weights b_k that we call **interpolation functions**. Note that if α is viewed as a variable too, steerability means an $\alpha, \vec{x}$-separable decomposition of $F(\alpha, \vec{x})$.

This allows us to calculate the response of an image to F_α in a (quasi) continuum of the parameter α with the computational burden of only M projections $\langle A_k | I\rangle, k = 1 \ldots M$. With I we denote the image and $\langle\cdot|\cdot\rangle$ is the usual scalar product.

In general the basis functions do **not** have to be rotated, scaled, or otherwise deformed copies of the original filter for some fixed values of the parameter, i.e. $A_k \neq F_{\alpha_k}$. If there is no exact decomposition of F_α to a finite number M of basis functions or if M is too large we are interested in the minimal number of basis functions to obtain an acceptable approximation to F_α for all α simultaneously.

Freeman and Adelson (1991) steer orientation by rotated copies of F as basis functions. The interpolation functions are derived by a Fourier decomposition of F with respect to the angular coordinate of a polar representation. Simoncelli et al. (1992) generalized this approach to steerability by Fourier decompositions to other deformations, especially dilating and contracting the filter but there remain some open questions: (1) The Fourier basis is chosen arbitrarily. (2) How do we get approximate steerability with a limited number of basis functions? (3) How is the singularity adequately treated that occurs in the case of scale? (4) What about other basis functions than deformed copies of the original filter? The latter question concerns also two points that are stressed by Simoncelli at al., the self-invertibility and the interpolation by varying only one parameter while leaving others constant in the case of simultaneously steering several deformations. For these requirements to be meaningful the basis functions have to be deformed copies of the filter. From our point of view both things are not necessary for signal analysis purposes. In general the basis functions are not deformed copies of the filter and they have no meaning by themselves but only their superpositions.

Perona (1991) found an optimal solution to approximately steer arbitrary (continuous) deformations for arbitrary filters. He achieves this by a singular value decomposition (SVD) of the linear operator L that has images I as its input and the deformation parameter dependent response function $R(\alpha) = L(I) := \langle F_\alpha | I\rangle$ as its output. The basis functions are the right singular vectors, whereas the interpolation functions are the left singular vectors. A drawback of this approach is that in general (e.g. for dilations/contractions) every filter has different basis functions which have to be calculated numerically.

Recently Beil (1994) published another approach to steerability by using the invariance theory of tensor calculus. Steerable filters are constructed by basic invariant elements which are given by the theory of invariance. In fact, from its practical implications this is close to the Fourier decomposition method.

In this paper we use the Lie group that corresponds to the deformation of the filter as the basis for a deeper understanding of the Fourier decomposition approach of Freeman and Adelson (1991) and Simoncelli et al. (1992). The eigenfunctions of the generating operator of the Lie group are the basis functions and the eigenvalues are the interpolation functions. These functions depend only on the type of the deformation but not on the filter. The basis functions are orthogonal and no deformed copies of the filter. However, it is easy to calculate the interpolation functions for the latter. We also achieve an approximate solution to steerability if the number of basis functions is limited and we show how to treat the singularity that occurs when steering the scale.

2 Steering translations

To give a motivation for the following abstract formalism we start by steering translations in 1D. This is even more helpful as it will turn out that all other continuous one-parameter deformations are isomorphic to translations by appropriate transformations of the coordinates and the deformation parameter. For more details about Lie group theory we refer to the textbook of Hall (1967).

2.1 The translational Lie group

By $\{\mathcal{L}_\alpha | \alpha \in \mathbb{R}\}$ we denote the Lie group that performs translations in x-direction. The original filter is $F(x) \equiv F_0(x)$ and the deformation is defined by:

$$\mathcal{L}_\alpha F(x) \;=\; F_\alpha(x) \;:=\; F(x+\alpha) \tag{2}$$

The identity operator is $\mathcal{L}_0 = \mathbb{1}$. The group multiplication table is given by the following function f:

$$\begin{aligned} \mathcal{L}_\beta \mathcal{L}_\alpha F(x) \;&=:\; \mathcal{L}_{f(\beta,\alpha)} \;, \qquad \text{with} \\ f(\beta,\alpha) \;&=\; \beta + \alpha \end{aligned} \tag{3}$$

For very small deformations $\varepsilon \to 0$ the following expansion is possible that defines the generating operator $\hat{\mathcal{L}}$ of the Lie group:

$$\mathcal{L}_\varepsilon \approx \mathbb{1} + \varepsilon \hat{\mathcal{L}} \tag{4}$$

We calculate the generating operator for translations by a first order Taylor expansion of the translated function:

$$F(x+\varepsilon) \;=\; F(x) + \varepsilon \partial_x F(x) \;\Longrightarrow\; \hat{\mathcal{L}} \;=\; \partial_x \tag{5}$$

The finite operator $\mathcal{L}_\alpha$ can be derived from $\hat{\mathcal{L}}$ by repeating many small deformations by $\varepsilon = \alpha/n$:

$$\mathcal{L}_\alpha \;=\; \lim_{n\to\infty}(1+\alpha\hat{\mathcal{L}}/n)^n \;=\; e^{\alpha\hat{\mathcal{L}}} \tag{6}$$

The eigenfunctions of $\hat{\mathcal{L}}$ are exponentials e^{zx} with $z \in \mathbb{C}$. We can restrict to the functions e^{jkx} with $k \in \mathbb{R}$ because they are a complete basis for all square integrable functions. The completeness is easily obtained by noting that $j\partial_x$ is hermitean. Hermitean operators have complete sets of eigenfunctions with **real** eigenvalues. An eigenfunction of the generating operator $\hat{\mathcal{L}}$ is also an eigenfunction of $\mathcal{L}_\alpha$ but for another eigenvalue. From (6) we see that:

$$\hat{\mathcal{L}}e^{jkx} = jk\,e^{jkx} \quad\Longrightarrow\quad \mathcal{L}_\alpha e^{jkx} = e^{jk\alpha}e^{jkx} \tag{7}$$

2.2 Steering by Fourier decomposition

We already mentioned that steerability means an α, x-separable decomposition of the filter $F_\alpha(x)$. This is achieved by a decomposition of F into the eigenfunctions (7), i.e. a Fourier decomposition.

$$\begin{aligned} F(x) \;&=\; \sum_{k=-\infty}^{\infty} c_k e^{jkx} \;, \quad \text{with } c_k \;=\; \tfrac{1}{2\pi}\int F(x)e^{-jkx}dx \\ F_\alpha(x) \;&=\; \sum_{k=-\infty}^{\infty} c_k e^{jk\alpha}e^{jkx} \;=:\; \sum_{k=-\infty}^{\infty} b_k(\alpha)A_k(x) \end{aligned} \tag{8}$$

If (8) is compared with the definition of steerability in (1) we see that the basis functions are $A_k(x) = e^{jkx}$, whereas the interpolation functions are $b_k(\alpha) = e^{jk\alpha}$. The weight c_k can be absorbed in either of both functions. In (8) we wrote a sum instead of an integral. That hints to the fact that in practice we have to restrict the functions to a bounded interval (respectively make them periodic). Then α becomes a periodic parameter, the Lie group becomes compact and the eigenfunctions are square integrable. More precisely the basis functions are $A_k(x) = e^{j2\pi kx/L}$, with L the length of the interval. But without loss of generality we assume $L = 2\pi$ in the following.

This application of the Fourier basis is well known and used e.g. in the steerability approach of Freeman and Adelson (1991). With the theoretical framework that is given in this paper we provide the basis for better understanding the scope of this ansatz.

2.3 Optimality of the basis functions

Compared to other sets of basis functions there are two properties of the Fourier base that make it exceptionally suited for steering translations. First, we do not have to calculate the α-dependence of the interpolation functions because it is given by the eigenvalues. Second, the Fourier base is optimal in the sense that it allows the best L^2 approximations with the fewest basis functions, as we will show now.

A proof of the optimality was given by Perona (1991) by investigating the operator $L := \langle F_\alpha|\cdot\rangle$. We want to give here another proof that might be more familiar in the signal processing community and that in fact is a finite dimensional representation of Perona's proof. We investigate the set of all deformed filters $\{F_\alpha(x)|\alpha\}$ where α is running through all possible values. All these functions can be reconstructed by superpositions from a set of functions that has the same linear span as $\{F_\alpha(x)|\alpha\}$. To find such a set we sample α and x and obtain the matrix C with the following elements:

$$C_{kl} \;=\; F_{\alpha_k}(x_l) \tag{9}$$

We assume this matrix to be square and of size $N \times N$. By definition of F_α as a translation of F_0 this matrix is **circulant** and hence, it can be diagonalized by a Fourier transform (for a proof see Hall (1979, Appendix B), Λ is diagonal).

$$C = W \Lambda W^{-1} \quad , \quad \text{with} \quad W_{kl} = N^{-1} e^{j2\pi kl/N} \tag{10}$$

For a diagonalizable matrix diagonalization is equivalent to an SVD. From the properties of the SVD we know that there is no sum of n dyadic products of two vectors (which is an α, x-separable decomposition) that approximates the original matrix better in the L^2 sense than the first n components of the SVD of the matrix. Approximating the matrix means approximating all steered filters simultaneously. By making the size of the matrix N very large we can approximate the continuous case to an arbitrary precision. The distance measure for the approximation is $d(F,G) = ||F - G||_{x,\alpha}$, where the L^2 norm is with respect to x **and** α (see Perona (1991)).

2.4 Approximate steerability

To steer a filter F approximately with a limited number M of basis functions we take only the first M eigenvectors of the matrix C as basis functions. The vectors are orthogonal and hence, their L^2 norms, that are given by the elements of Λ (10), sum up to the norm of F_α. From (10) the norms can be calculated. We have $\Lambda = W^{-1} C W$ and hence (we switch now back to the continuous case):

$$\begin{aligned} \Lambda_{k'k} &= \tfrac{1}{2\pi} \int e^{-jk'\alpha} \int F_\alpha(x) e^{jkx} dx d\alpha \overset{(t = x + \alpha)}{=} \\ = \tfrac{1}{2\pi} \int e^{j(k-k')\alpha} d\alpha & \int F(t) e^{jkt} dt = 2\pi \delta(k - k') \;\; c_k \end{aligned} \tag{11}$$

where c_k is the k'th Fourier coefficient from equation (8). We want to give also another derivation of the relative importance of the basis functions that yields an interesting result. The basis functions A_k from (8) can be obtained by $A_k(x) = \frac{1}{2\pi} \int F(x+\alpha) e^{-jk\alpha} d\alpha = c_k e^{jkx}$. The weight c_k is now absorbed in the basis function. The L^2 norm of A_k renders its relative importance to approximate F_α. Clearly the norm is given by c_k but we can also write it in the following way :

$$\begin{aligned} ||A_k||^2 &= \tfrac{1}{4\pi^2} \int \int F(x+\alpha) e^{-jk\alpha} d\alpha \int F^*(x+\alpha') e^{jk\alpha'} d\alpha' \; dx \overset{(a)}{=} \\ = \tfrac{1}{4\pi^2} \int d\alpha & \int \int F(z) F^*(z+\beta) dz \;\; e^{jk\beta} d\beta = \tfrac{1}{2\pi} \int \langle F_\beta | F_0 \rangle e^{jk\beta} d\beta \end{aligned} \tag{12}$$

In (a) we changed the order of integration and we applied the substitutions $z = x + \alpha$ and $\beta = \alpha' - \alpha$. The integrand does only depend on $\alpha' - \alpha$ and hence, we can factor out the α integration that merely gives a factor of 2π. Again, at this point we have to assume the Lie group to be compact.

As the result, the L^2 norm of A_k is given by the k'th Fourier coefficient of the function:

$$h(\alpha) := \langle F_\alpha | F_0 \rangle = \int F^*(x+\alpha) F(x) dx \tag{13}$$

$h(\alpha)$ is the autocorrelation function of F. In wavelet theory it is called the reproducing kernel which governs the sampling scheme for complete wavelet bases (Antoine et al. (1993)).

3 Steering other parameters

The steering scheme for translations becomes powerful by the fact that all one-parameter Lie groups are isomorphic to the translation group by appropriate transformations of the coordinates and the parameter (Hall (1967)). A general one-parameter Lie group is given by its elements $\mathcal{L}_\alpha$ and the group multiplication table $f(\beta,\alpha)$ with $\mathcal{L}_\beta \mathcal{L}_\alpha = \mathcal{L}_{f(\beta,\alpha)}$. There exists a special value α_0 of the parameter α with $\mathcal{L}_{\alpha_0} = \mathbb{1}$. α_0 is the neutral element with $f(\alpha_0,\alpha) = f(\alpha,\alpha_0) = \alpha$. The generating operator $\hat{\mathcal{L}}$ is defined by

$$\hat{\mathcal{L}} \;:=\; \frac{d\mathcal{L}_\alpha}{d\alpha}\Big|_{\alpha_0} \tag{14}$$

where $|_{\alpha_0}$ means that the expression to the left is evaluated at α_0. For the infinitesimal transformation at other values of the parameter we have

$$\frac{d\mathcal{L}_\alpha}{d\alpha}\Big|_\alpha \;=\; \hat{\mathcal{L}}\mathcal{L}_\alpha \left(\frac{\partial f(\beta,\alpha)}{\partial \beta}\Big|_{\beta_0}\right)^{-1} \tag{15}$$

The coordinates (x,y) (e.g. in 2D) are transformed under the group to $x' = x'(x,y,\alpha) = \mathcal{L}_\alpha x$ and $y' = y'(x,y,\alpha) = \mathcal{L}_\alpha y$. The infinitesimal transformations are described by the following derivatives:

$$\begin{aligned} & dx = \xi d\alpha \qquad dy = \eta d\alpha \;, \quad \text{with} \\ \xi(x,y) \;&:=\; \tfrac{\partial x'}{\partial \alpha}|_{\alpha_0} \qquad \eta(x,y) \;:=\; \tfrac{\partial y'}{\partial \alpha}|_{\alpha_0} \\ \tfrac{\partial x'}{\partial \alpha}|_\alpha \;&=\; \xi(x',y')\left(\tfrac{\partial f(\beta,\alpha)}{\partial \beta}\right)|_{\beta_0} \qquad \tfrac{\partial y'}{\partial \alpha}|_\alpha \;=\; \eta(x',y')\left(\tfrac{\partial f(\beta,\alpha)}{\partial \beta}\right)|_{\beta_0} \end{aligned} \tag{16}$$

Equations (15) and (16) are simplified if we change to the **canonical parameter** τ. The group table for the τ parametrization $f(\tau_a,\tau_b)$ must have the property

$$\frac{\partial f(\tau_a,\tau_b)}{\partial \tau_a}\Big|_{\tau_0} \;=\; 1 \tag{17}$$

This is especially true for the canonical choice

$$f(\tau_a,\tau_b) \;=\; \tau_a + \tau_b \;, \qquad \tau_0 = 1 \tag{18}$$

For the canonical parametrization we have simple representations of the generating operator $\hat{\mathcal{L}}$ and the group elements $\mathcal{L}_\tau$:

$$\hat{\mathcal{L}} \;=\; \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y \tag{19}$$

$$\mathcal{L}_\tau \;=\; e^{\tau\hat{\mathcal{L}}} \tag{20}$$

In addition we can always find curvilinear **canonical coordinates** u, v that make the representation of the group especially simple and equivalent to a translation. For the canonical coordinates we have:

$$\begin{aligned} u' = \mathcal{L}_\tau u = u + \tau \qquad & v' = \mathcal{L}_\tau v = v \\ \hat{\mathcal{L}} \;=\; & \partial_u \end{aligned} \tag{21}$$

Hence, if we have any continuous one-parameter transformation we can achieve steerability by transforming the problem to canonical parametrization and canonical coordinates and apply the same formalism as in section 2.

3.1 Deformations in two dimensions

Hoffman (1966) points out six generators of a basic 2D Lie algebra of visual perception. The six transformations together with their canonical parametrizations are:

$$\begin{array}{rclll}
\hat{\mathcal{L}}^x &=& \partial_x & \left\{\begin{array}{l} x' = x + \tau \\ y' = y \end{array}\right. & \text{Translation} \\
\hat{\mathcal{L}}^y &=& \partial_y & \left\{\begin{array}{l} x' = x \\ y' = y + \tau \end{array}\right. & \text{Translation} \\
\hat{\mathcal{L}}^r &=& -y\partial_x + x\partial_y & \left\{\begin{array}{l} x' = x\cos\tau - y\sin\tau \\ y' = x\sin\tau + y\cos\tau \end{array}\right. & \text{Rotation} \\
\hat{\mathcal{L}}^s &=& x\partial_x + y\partial_y & \left\{\begin{array}{l} x' = xe^{\tau} \\ y' = ye^{\tau} \end{array}\right. & \text{Dilation/Contraction} \\
\hat{\mathcal{L}}^b &=& x\partial_x - y\partial_y & \left\{\begin{array}{l} x' = xe^{\tau} \\ y' = ye^{-\tau} \end{array}\right. & \text{Hyperbolic Rotation} \\
\hat{\mathcal{L}}^B &=& y\partial_x + x\partial_y & \left\{\begin{array}{l} x' = x\cosh\tau + y\sinh\tau \\ y' = x\sinh\tau + y\cosh\tau \end{array}\right. & \text{Hyperbolic Rotation}
\end{array} \tag{22}$$

$\mathcal{L}^x$ and $\mathcal{L}^y$ are by definition in canonical form and x, y are the canonical variables. For $\mathcal{L}^r$ and $\mathcal{L}^s$ the canonical coordinates are log-polar $(\varphi = \arctan(y/x), t = \ln\sqrt{x^2+y^2})$. This is known to be a conformal mapping. Steering rotations follows straight forward the standard scheme of section 2 if it is expressed in canonical coordinates. In this case the group is already compact (periodic) and no arbitrary restriction to finite intervals for the coordinate and the parameter is necessary. In so far it is the simplest case of all. The case of steering dilations and contractions is treated in some detail in section 4. Finally, for the hyperbolic deformations usually the transformed coordinates $u = \sqrt{x^2 - y^2}, v = \sqrt{2xy}$ are used. Note, that these are not the canonical coordinates because the generating operators do not have the canonical form of (21).

$$\hat{\mathcal{L}}^b = \frac{r^2}{u}\partial_u \neq \partial_u \qquad\qquad \hat{\mathcal{L}}^B = \frac{r^2}{v}\partial_v \neq \partial_v \tag{23}$$

However, the deformations are along the lines $v = \text{const}, u = \text{const}$ but with deformation 'speedterms' of r^2/u and r^2/v.

3.2 Steering multiple parameters

If several deformations have to be steered simultaneously the generating operators in general do not commute and the canonical coordinate axes of the involved one-parameter groups will not be mutually locally orthogonal. This is for example the case for the 3D rotation group $SO(3)$, where we can not follow the full scheme of section 2. However, it is possible to use the canonical basis functions of the irreducible representations of the group to derive a steering equation. In case of $SO(3)$ these are the spherical harmonics Y_l^m, which are used by Freeman and Adelson (1991) to steer 3D functions. The spherical harmonics and the canonical basis functions in general (for unitary representations) are orthogonal and hence it is easy to calculate the interpolation functions. If A_k are the orthogonal basis functions the interpolation functions b_k are given by

$$b_k(\alpha) = \langle F_\alpha | A_k \rangle \tag{24}$$

We will not go further into the details of the general case in this paper. However, in the special case of steering two parameters where the associated canonical coordinates u, v are locally orthogonal, the generating operators ∂_u and ∂_v commute and we can steer both parameters simultaneously according to the scheme given in section 2. This is the case for the three pairs $(\hat{\mathcal{L}}^x, \hat{\mathcal{L}}^y), (\hat{\mathcal{L}}^s, \hat{\mathcal{L}}^r)$, and $(\hat{\mathcal{L}}^b, \hat{\mathcal{L}}^B)$.

In general a pair of locally orthogonal coordinates is given by a solution of the Cauchy-Riemann equations

$$\partial_x u = \partial_y v \qquad \partial_y u = -\partial_x v \tag{25}$$

If u, v are differentiable these equations are equivalent to the complex differentiability of $f := u + jv$, i.e. every complex differentiable function defines a pair of orthogonal coordinates.

4 Steering scale

As an example of steering filters according to the method of section 3 we demonstrate how to steer the scale. Especially we show how to treat the singularity that occurs in this case.

Let $F(r, \vec{\varphi})$ be an N-dimensional (ND) function in polar representation. The variable $\vec{\varphi}$ denotes the angular components that are omitted if no use is made of them. We define the scaling operator in ND to be

$$\mathcal{L}_\alpha(F(r)) := e^{-\frac{\alpha N}{2}} F(e^{-\alpha} r) \tag{26}$$

This definition differs from the deformation that is generated by $\hat{\mathcal{L}}^s$ from (22) by the normalization factor $e^{-\alpha N/2}$. The parameter α is the canonical scaling parameter: $\mathcal{L}_\alpha \mathcal{L}_\beta = \mathcal{L}_{\alpha+\beta}, \mathcal{L}_0 = \mathbb{1}$. We obtain the generating operator $\hat{\mathcal{L}}$ by a first order Taylor approximation of $\mathcal{L}_\alpha F$:

$$e^{-\frac{\alpha N}{2}} F(e^{-\alpha} r) = F(r) - \alpha(\tfrac{N}{2} + r\partial_r)F(r) + o(\alpha^2) \tag{27}$$

Hence, the generating operator for scaling (including the normalization $e^{-\frac{\alpha N}{2}}$) in polar coordinates is:

$$-\hat{\mathcal{L}} = \tfrac{N}{2} + r\partial_r \tag{28}$$

The eigenvalue equation $(\frac{N}{2} + r\partial_r)E_z(r) = zE_z(r)$ with the complex eigenvalue $z = a + jk$ is a simple differential equation with separated variables and the solution $E_z(r) = Cr^{z-\frac{N}{2}} = Cr^{a-(N/2)}e^{jk\ln r}$ with C as an integration constant.

4.1 Construction of a complete and orthogonal eigenbasis

In this section we are concerned with the question how to choose the parameters a, k so that the set of eigenfunctions is complete and orthogonal. We can use the property of hermitean operators to have complete and orthogonal eigenbases with **real** eigenvalues to construct such a basis. The generating operator $\hat{\mathcal{L}}$ is not hermitean but we can construct the following symmetrized hermitean operator $\hat{\mathcal{L}}^S$ (^{H} denotes hermitean conjugation).

$$\hat{\mathcal{L}}^S := \frac{1}{2}(j\hat{\mathcal{L}} + (j\hat{\mathcal{L}})^H) = \frac{j}{2}(r\partial_r + r^{-N+1}\partial_r r^N) \tag{29}$$

where the hermitean conjugate operator $(j\hat{\mathcal{L}})^H) = j(-\frac{N}{2} + r^{-N+1}\partial_r r^N)$ is calculated by

$$
\begin{aligned}
\langle F | j\hat{\mathcal{L}} G\rangle &= \int_0^\infty F^* \,(j(\tfrac{N}{2} + r\partial_r)G) r^{N-1} dr = \\
&= \int_0^\infty (j(-\tfrac{N}{2} + r^{-N+1}\partial_r r^N)F)^* \, G \, r^{N-1} dr = \langle (j\hat{\mathcal{L}})^H F | G\rangle
\end{aligned}
\tag{30}
$$

The boundary term of the partial integration vanishes if F, G are not singular at the origin. $\hat{\mathcal{L}}^S$ has the same eigenfunctions as $\hat{\mathcal{L}}$ but with the eigenvalues $-k + ja$:

$$
\hat{\mathcal{L}}^S r^{a-\frac{N}{2}+jk} = (-k + ja)\, r^{a-\frac{N}{2}+jk} \tag{31}
$$

For $\hat{\mathcal{L}}^S$ to have real eigenvalues we must choose $a = 0$ what results in the following complete and orthogonal set of eigenfunctions:

$$
E_k(r) = C r^{-\frac{N}{2}} e^{jk\ln r} \tag{32}
$$

C can be chosen to normalize the eigenfunctions. The eigenfunction for $N = 1$ and $k = 5$ is depicted in fig.1.

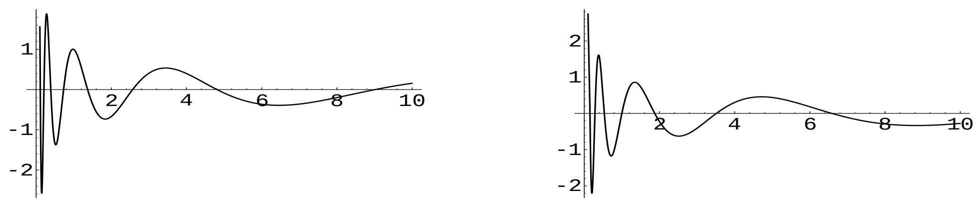

Figure 1: Example of an eigenfunction of the scaling operator: $\Re = r^{-\frac{1}{2}} \cos(5 \ln r)$ (left) and $\Im = r^{-\frac{1}{2}} \sin(5 \ln r)$ (right).

The orthogonality (33) and completeness (34) of the eigenfunctions is stated by the following formulas (the angular integration merely gives a constant factor that is omitted):

$$
\tfrac{1}{2\pi} \int_0^\infty r^{-N} e^{-jk\ln r} e^{+jk'\ln r} r^{N-1} dr = \delta(k - k') \tag{33}
$$

$$
\tfrac{1}{2\pi} \int_{-\infty}^\infty (rr')^{-\frac{N}{2}} e^{-jk\ln r'} e^{+jk\ln r} dk = r^{-N+1}\delta(r - r') = e^{-Nt}\delta(t - t') \;, \quad r, r' > 0 \tag{34}
$$

$t := \ln r$ is the canonical scaling variable according to (21). In t-space the dilation/contraction becomes a simple shift (beside the normalization factor $e^{-\alpha N/2}$): $\mathcal{L}_\alpha F(r) = e^{-\alpha N/2} F(e^{-\alpha} r) = e^{-\alpha N/2} F(e^{t-\alpha})$. With the substitution $t = \ln r$ (33) and (34) are the ordinary orthogonality and completeness relations for complex exponentials. In (34) we applied the formula $\delta(\ln r - \ln r') = r'\delta(r - r')$. This formula is a special case of the general formula $\delta(f(x)) = \sum_n \delta(x - x_n)/|\partial_x f(x_n)|$, where f has only first order zeros x_n: $f(x_n) = 0, f'(x_n) \neq 0$. The factor r^{-N+1} at the r.h.s. of (34) compensates the factor r^{N-1} of the ND integration measure. Concerning the optimality of the eigenfunctions (32) for steering the scale, remember that according to section 2.3 they are optimal only in t-space where scaling is a translation but not in the original r-space.

From the eigenfunctions (32) the basis functions A_k for steering the scale of a filter F according to (1) are most easily obtained by interpreting (1) as an $\alpha, \vec{x}$-separable decomposition of $F(\alpha, \vec{x})$ that is orthogonal in α and in $\vec{x}$. Therefore, the basis functions $A_k(\vec{x})$ can be obtained as the coefficients when $F(\alpha, \vec{x})$ is projected to the interpolation functions $b_k(\alpha) = e^{-jk\alpha}$:

$$
\begin{aligned}
2\pi A_k(r,\vec{\varphi}) &= \int e^{-\frac{\alpha N}{2}} F(e^{-\alpha}r,\vec{\varphi}) e^{jk\alpha} d\alpha \quad \overset{(t=\ln r,\ z=t-\alpha)}{=} \\
&= e^{-\frac{tN}{2}} e^{jkt} \int e^{\frac{zN}{2}} F(e^{z},\vec{\varphi}) e^{-jkz} dz \ =: \ r^{-\frac{N}{2}} e^{jk\ln r}\, C_k(\vec{\varphi})
\end{aligned} \tag{35}
$$

This shows that the basis functions are $r,\vec{\varphi}$-separable. This is a general result that is valid for other deformations too because it simply uses the fact that the deformation is a translation in the canonical coordinate and that the interpolation function is an exponential. The translation is transfered to the interpolation function by the substitution $z = t - \alpha$ where the shift is separable: $e^{jk(t-z)} = e^{jkt}e^{-jkz}$.

If we would steer also the orientation of the filter, only the functions $C_k(\vec{\varphi})$ are affected. We could also start by steering the orientation to obtain again $r,\vec{\varphi}$-separable basis functions with r dependent coefficients $C_k(r)$ that can be steered in scale by the above procedure. Hence, scale and orientation are treated in the same manner. In the steering scheme of Perona (1992) the orientation is steered first and it is not straight forward to start by steering the scale. This is because he uses optimal basis functions from an SVD that are of different (and unknown) analytical form for every filter and every Fourier component $C_k(r)$.

Finally we apply the general result of section 2.4 to find the most important basis functions (in the L^2 sense) to steer the scale approximately using only a limited number of basis functions. With $A_k(r,\vec{\varphi}) = \frac{1}{2\pi}\int e^{-\frac{\alpha N}{2}} F(e^{-\alpha}r,\vec{\varphi})e^{jk\alpha}d\alpha$ we can apply exactly the same calculation as in (12) if we first substitute the canonical variable $t = \ln r$. It is again the autocorrelation function $\langle F_\alpha | F_0\rangle$ that governs the importance of the basis functions:

$$
\begin{aligned}
4\pi^2 ||A_k||^2 &= \int\int e^{-\frac{\alpha N}{2}} F(e^{-\alpha}r,\vec{\varphi}) e^{jk\alpha} d\alpha \int e^{-\frac{\alpha' N}{2}} F^*(e^{-\alpha'}r,\vec{\varphi}) e^{-jk\alpha'} d\alpha' r^{N-1} dr d\vec{\varphi} \\
&= \int\int\int F(e^{-\alpha}r,\vec{\varphi}) F^*(e^{-\alpha'}r,\vec{\varphi}) r^{N-1} dr d\vec{\varphi} e^{jk(\alpha-\alpha')} e^{-\frac{(\alpha+\alpha')N}{2}} d\alpha d\alpha' \quad \overset{(t=\ln r)}{=} \\
&= \int\int\int F(e^{t-\alpha},\vec{\varphi}) F^*(e^{t-\alpha'},\vec{\varphi}) e^{Nt} dt d\vec{\varphi} e^{jk(\alpha-\alpha')} e^{-\frac{(\alpha+\alpha')N}{2}} d\alpha d\alpha' \quad \overset{(z=t-\alpha)}{=} \\
&= \int\int\int F(e^{z},\vec{\varphi}) F^*(e^{z+\alpha-\alpha'},\vec{\varphi}) e^{N(z+\alpha)} dz d\vec{\varphi} e^{jk(\alpha-\alpha')} e^{-\frac{(\alpha+\alpha')N}{2}} d\alpha d\alpha' \quad \overset{(\beta=\alpha'-\alpha)}{=} \\
&= \int d\alpha \int\int F(e^{z},\vec{\varphi}) F^*(e^{z-\beta},\vec{\varphi}) e^{Nz} dz d\vec{\varphi} e^{-jk\beta} e^{-\frac{\beta N}{2}} d\beta \quad \overset{(r=e^{z})}{=} \\
&= 2\pi \int\int F(r,\vec{\varphi}) e^{-\frac{\beta N}{2}} F^*(e^{-\beta}r,\vec{\varphi}) r^{N-1} dz d\vec{\varphi} e^{-jk\beta} d\beta \ = \\
&= 2\pi \int \langle F_\beta | F_0 \rangle e^{-jk\beta} d\beta
\end{aligned} \tag{36}
$$

4.2 Treating the singularity

The eigenfunctions E_k from equation (32) are singular at the origin (fig.1). First, they are infinite as $r^{-N/2}$ and second, they oscillate infinitely fast at the origin. In this section we analyse this singularity and we show how to deal with it.

In fact the eigenfunctions E_k (or the basis functions A_k (35) respectively) themselves are not of interest but only their projections to the filters $\langle F_\alpha | E_k\rangle$. This regularizes the singularity what is evident when the projection is transformed to the canonical coordinate $t = \ln r$ (the $\vec{\varphi}$ integration is omitted).

$$\begin{aligned} \langle F_\alpha | E_k \rangle &= \int_0^\infty e^{-\frac{\alpha N}{2}} F(e^{-\alpha} r)\, r^{-\frac{N}{2}} e^{jk \ln r}\, r^{N-1} dr = \\ &= \int_{-\infty}^{\infty} F(e^{t-\alpha})\, e^{\frac{N(t-\alpha)}{2}}\, e^{jkt} dt = \int_{-\infty}^{\infty} \tilde{F}(t-\alpha)\, e^{jkt} dt \\ &\text{with} \quad \tilde{F}(t) := e^{\frac{Nt}{2}} F(e^t) \end{aligned} \tag{37}$$

The function $\tilde{F}$ contains the transformed filter as well as the powers of r from the eigenfunction and the integration measure. We call $\tilde{F}$ for short the **warped filter**. The warped eigenfunctions become the Fourier base, i.e. if $F = E_k$ then $\tilde{F}(t) = e^{jkt}$. For general F the projection $\langle F_\alpha | E_k \rangle$ becomes an ordinary Fourier transform of the warped filter. This means that **the singularity of the eigenfunctions is not worse than the singularity of ordinary complex exponentials**. The unbounded support and the infinite number of oscillations of the latter are logarithmically compressed. Scaling the original function means translating the warped function: to the right for larger functions, to the left for smaller functions. The singularity is treated as usually by restricting to bounded support functions and finite shifts (scalings) in t-space. The warped filter for a 2D isotropic Gaussian is depicted in figure 2.

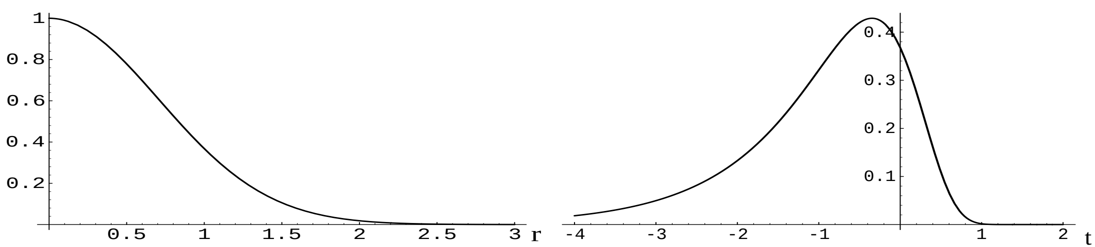

Figure 2: Original filter $F(r) = e^{-r^2}$ (left) and warped filter $\tilde{F}(t) = \exp(-e^{2t} + t)$ (right) for an isotropic Gaussian in 2D.

In t-space the example is easy to understand. The support of $\tilde{F}$ is not bounded at negative t if the original filter F is not zero in a finite neighborhood of the origin. But if F is not infinite at the origin (what is never the case for vision kernels), $\tilde{F}$ has a strong decay towards negative t from the factor $e^{\frac{N(t-s)}{2}}$ in (37). Hence, we can treat $\tilde{F}$ as bounded by allowing a little error. In addition we must restrict to a bounded range of scales that is steered. Then, the t-axis has to be sampled in the interval $[t_{\min}, t_{\max}]$ where $t_{\min}, t_{\max}$ are given by the support of the warped function $\tilde{F}$ plus and minus the translations from scaling the function.

The sampling density Δt in t-space is given by the sampling theorem. Translating this back to r-space the sampling is $r_n := e^{n\Delta t + t_{\min}}$. If a (suboptimal) homogeneous sampling is required in r-space the highest sampling rate of this logarithmic sampling has to be taken. Concerning the oscillatory singularity of the eigenfunctions the interpretation of this sampling is as follows: only frequencies below the Nyquist frequency of the smallest steered filter have to be considered what wipes out the infinitely rapid oscillations at the origin. According to this frequency we have in r-space the first sample some distance away from the origin and we don't have to consider anything closer to the origin than this sample. An exception is the origin itself for which we have trivially $F_s(0) = e^{-Ns/2} F_0(0)$. Hence, for steering scale we don't need arbitrary linear distortions of the logarithm near the origin as in the frequency space approach of Simoncelli et al. (1992).

As in (8) the integral of the steering equation

$$F_\alpha(r, \vec{\varphi}) \;=\; \int_{-\infty}^{\infty} e^{jk\alpha}\, r^{-\frac{\alpha N}{2}} e^{-jk\ln r} C_k(\vec{\varphi}) dk \tag{38}$$

can be substituted by a sum by making the warped filter periodic in t-space. The same discrete frequencies k that are derived there can be applied in r-space.

5 Discussion

We applied the Lie group formalism to the problem of steering filters. This approach provides a theoretical basis and a generalization of the steering schemes of Freeman and Adelson (1991) and Simoncelli et al. (1992). The generalization allows the same formalism to be applied to all one-parameter continuous deformations as well as to two-parameter deformations for which the canonical coordinates are orthogonal.

Our approach does not give the most parsimonious set of basis functions (except for rotations and translations) as the steering scheme of Perona (1991). But our approach has the advantage that the basis functions and the interpolation functions are given analytically and that they are the same for all filters. Only their relative weights depend on the filter. In addition all deformations are treated in the same manner and compared to Perona's method it is straight forward to steer first the scale and then the orientation.

Our approach does not use deformed copies of the steered filter as basis functions. However, those deformed filters can easily be obtained by superpositions of our basis functions. By the same superposition formulas the appropriate interpolation functions are easily calculated. The fact that our basis functions are given analytically and that they are the same for all filters is advantageous compared to Perona's method for calculating these interpolation functions. This is not only of interest for calculating the interpolation functions for deformed copies of the filter but also in general to make existing analysing schemes steerable that use more or less complete sets of filters.

The log-polar canonical coordinates that are used to steer the scale and orientation are well known from the Mellin transform and the conformal mappings that are used in invariance theory. The Lie group formalism has been applied also by Lenz (1990) but he focused on invariant pattern recognition and did not explicitly address the problem of steerability.

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