An operator for the analysis of superimposed intrinsically two dimensional patterns

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Abstract: In this paper, we present an operator for the analysis of superimposed intrinsically two dimensional (i2D) patterns. Coupling the ideas of the monogenic signal, the tensor algebra and the quadrature filter, a rotationally invariant operator is derived. This operator is able to capture important features (e.g. amplitude, energy, orientation and phase) of the intrinsically 2D structure. Compared with other approaches, like the structure tensor and the boundary tensor, the main contribution of the proposed operator is the possibility to evaluate the phase information. The energy output is considered as a junction strength to detect the points of interest and the estimated orientation represents the main orientation of the local i2D structure. Experimental results illustrate the features extracted from the proposed operator. As an application, some test results of junction detection are also presented.

Keywords: intrinsically two-dimensional structure; phase evaluation; algebraic embedding

Reference

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1 INTRODUCTION

The intrinsic dimensionality is a local property of a multidimensional signal, which expresses the number of degrees of freedom necessary to describe local structure. For 2D images, there exist three types of structures. These are the intrinsically zero dimensional (i0D) signals which are constant signals, intrinsically one dimensional (i1D) signals representing lines and edges and intrinsically two dimensional (i2D) signals which do not belong to the above two cases. The i2D structures consist of curved edges and lines, junctions, corners and line ends, etc. It is well known that the i2D structures capture important information of the image. Therefore, with the correct characterization of the i2D structures, computer vision applications like stereo-vision, motion estimation, object recognition, etc. can be well implemented.

Many approaches have been proposed to characterize the i2D structures. The structure tensor [1] and the boundary tensor [2] estimate the main orientation and the energy of the i2D signal, however, no phase information is contained. In [3], a nonlinear image operator for the evaluation of local intrinsic dimensionality was proposed. Although some i2D image features can be detected, this operator captures no information about the phase. The partial Hilbert transform and the total Hilbert transform [4] provide representations of the phase in 2D. Unfortunately, they are not rotation invariant and are not adequate for detecting i2D features. The monogenic signal [5] enables the simultaneous and rotationally invariant estimation of the amplitude, orientation and phase of the i1D signal, but it delivers no information of the i2D part. Bülow and Sommer [6] proposed the quaternionic analytical signal to evaluate the phase information, however, it is not rotation invariant. A phase model is proposed in [7], where the i2D signal is split into two i1D signals and the corresponding two phases are evaluated. Unfortunately, steering is needed and only i2D patterns with 90^0 opening angle can be correctly handled.

In this paper, an operator, which is derived on the basis of an algebraic embedding, is proposed to characterize the i2D structures. Hence, we refer to it as i2D operator. It is a cascaded operator which consists of two parts. The first one is a tensor pair, and the second one is a determinant operator. The proposed i2D operator has the property of rotation invariance and it enables the simultaneous estimation of amplitude, main orientation and the phase of the i2D structure. Due to the diversity of the i2D structure, our matter of concern is only the double-oriented superimposed pattern with variable opening angle.

2 MATHEMATICAL PRELIMINARIES

The rotation invariant monogenic signal [5] is a quite well model for the i1D signal, and it is based on the algebraic embedding of the 2D signal into a 3D space. The main idea of our approach is to couple the ideas of the monogenic signal, the tensor algebra and the quadrature filter to derive an i2D operator which can describe the features of the i2D structure. Compared with the classical framework of vector algebra, the geometric algebra makes a tremendous extension of modeling capabilities available. By embedding our problem into a certain geometric algebra, more degrees of freedom can be obtained, which makes it possible to extract multiple features of the i2D structure.

2.1 Geometric algebra of 3D Euclidean space $\mathbb{R}^3$

As for the problem we are concerned, the 2D signal is algebraically embedded into the Euclidean 3D space. In this section, we give a brief introduction to the geometric algebra of 3D Euclidean space. For the detail information, please refer to [8,9]. The Euclidean space $\mathbb{R}^3$ is spanned by the orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The geometric algebra of the 3D Euclidean space ($\mathcal{G}_3$) consists of $2^3 = 8$ elements,

$$\mathcal{G}_3 = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{123} = I_3\} \quad (1)$$

Here $\mathbf{e}_{23}$, $\mathbf{e}_{31}$ and $\mathbf{e}_{12}$ are the unit bivectors and the element $\mathbf{e}_{123}$ is called a trivector. A general combination of these elements is called a multivector, e.g. $M = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3 + e\mathbf{e}_{23} + f\mathbf{e}_{31} + g\mathbf{e}_{12} + hI_3$. The k-grade part of a multivector is obtained from the grade operator $\langle M \rangle_k$. Hence, $\langle M \rangle_0$ is the scalar part of M, $\langle M \rangle_1$ represents the vector part, $\langle M \rangle_2$ indicates the bivector part and $\langle M \rangle_3$ is the trivector part. If only the scalar and the bivectors are involved, the combined result is called a spinor, i.e. $S = a + e\mathbf{e}_{23} + f\mathbf{e}_{31} + g\mathbf{e}_{12}$. The geometric product of two multivectors M_1 and M_2 is indicated by juxtaposition of M_1 and M_2, i.e. $M_1 M_2$. The multiplication results of the basis elements are shown in table 1. The geometric product of two vectors $\mathbf{x}$ and $\mathbf{u}$ can be decomposed into their inner product ($\cdot$) and outer product ($\wedge$), i.e. $\mathbf{xu} = \mathbf{x} \cdot \mathbf{u} + \mathbf{x} \wedge \mathbf{u}$. It is shown in table 1 that the square of the bivector equals -1, therefore, the imaginary unit i of the complex numbers can be substituted by a bivector, yielding an algebra isomorphism. Accordingly, a rotation by angle θ can be represented by $\cos\theta + \sin\theta \mathbf{e}_{ij} = \exp(\theta \mathbf{e}_{ij})$, where i is not equal to j. The modulus of a multivector is obtained by $|M| = \sqrt{\langle M\widetilde{M} \rangle_0}$, where $\widetilde{M}$ represents the reverse of the multivector, it is defined as $\widetilde{M} = \langle M \rangle_0 + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3$.

Table 1 *The geometric product of basis elements.*

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	I_3
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	I_3
$\mathbf{e}_1$	$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	I_3	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_{23}$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$\mathbf{e}_{23}$	$\mathbf{e}_3$	I_3	$-\mathbf{e}_1$	$\mathbf{e}_{31}$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	1	$-\mathbf{e}_2$	$\mathbf{e}_1$	I_3	$\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	I_3	$-\mathbf{e}_3$	$\mathbf{e}_2$	-1	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$-\mathbf{e}_1$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_3$	I_3	$-\mathbf{e}_1$	$\mathbf{e}_{12}$	-1	$-\mathbf{e}_{23}$	$-\mathbf{e}_2$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	I_3	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	-1	$-\mathbf{e}_3$
I_3	I_3	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	-1

2.2 Basis functions

In order to analyze superimposed i2D patterns, we choose the 2D spherical harmonics as basis functions according to the proposal in [7]. Since the angular behavior of a signal can be regarded as band limited, only spherical harmonics of order zero to three are applied, otherwise, aliasing would occur. In the frequency domain, they take the following form:

$$H_n = \cos(n\alpha) + \sin(n\alpha)\mathbf{e}_{12} \quad (2)$$

where n indicates the order of the spherical harmonic, and α represents the angle in polar coordinates. Every spherical harmonic consists of two orthogonal components. The first order spherical harmonic is basically identical to the Riesz transform [7]. In practice, spherical harmonics are combined with radial bandpass filters such that polar separable filters are used. In this paper, the difference of Possion (DOP) [7] is employed as the bandpass filter. Polar separable filters are separable both in the spatial and spectral domains. Let $K(\rho, \alpha)$ represent the polar separable filter in the spectral domain, it can be written as

$$K(\rho, \alpha) = K(\rho)K(\alpha) = K(\rho)H_n \quad (3)$$

where ρ and α denote polar coordinates in the Fourier domain, $K(\alpha)$ is the angular part and $K(\rho)$ indicates the radial function. Using the Hankel transform, the spatial domain representation of it can be derived as

$$\begin{aligned} \mathcal{F}^{-1}\{K(\rho, \alpha)\} &= ck(r)(\cos(n\beta) + \sin(n\beta)\mathbf{e}_{12}) \\ &= ck(r)k(\beta) \end{aligned} \quad (4)$$

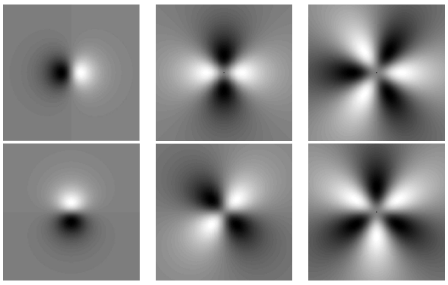

Figure 1 *From left to right are the spherical harmonics of order 1 to 3 in the spatial domain, every spherical harmonic consists of two orthogonal components. The white color indicates a value of positive one and the black color represents negative one.*

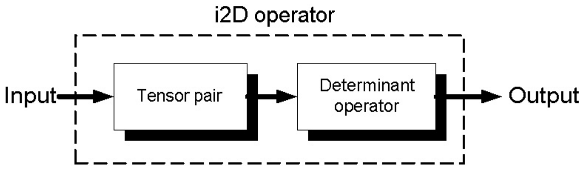

Figure 2 *The system structure diagram.*

where r, β are polar coordinates in the spatial domain, c is a constant. Note that the radial part $k(r)$ is derived from $K(\rho)$ and $K(\alpha)$. Combined with the bandpass filters, spherical harmonics of order 1 to 3 in the spatial domain are illustrated in Figure 1.

3 THE INTRINSICALLY 2D OPERATOR

The proposed i2D operator is a cascaded operator, i.e. $O_{i2D}\{s\} = O_2\{O_1\{s\}\}$, where s represents the original signal. As shown in Figure 2, the cascaded operator consists of two parts, i.e. a tensor pair (O_1) and a determinant operator (O_2).

The first part is a tensor valued filter which is composed of the even and the odd parts. In the frequency domain, the filter can be represented as $H = H_e + H_o$. In order to evaluate the phase information of the i2D structure, even and odd filters are designed to capture the corresponding even and odd information. The frequency domain representation of the even part takes the following form:

$$\begin{aligned} H_e &= \begin{bmatrix} H_0 + \mathbf{e}_1 H_2 \cdot \mathbf{e}_1 & \mathbf{e}_1 H_2 \cdot \mathbf{e}_2 \\ \mathbf{e}_1 H_2 \cdot \mathbf{e}_2 & H_0 - \mathbf{e}_1 H_2 \cdot \mathbf{e}_1 \end{bmatrix} \\ &= \begin{bmatrix} 2\cos^2(\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & 2\sin^2(\alpha) \end{bmatrix} \end{aligned} \tag{5}$$

For the convenience of analysis, the radial functions are ignored. In this tensor, two angular windowing functions, which can yield two perpendicular i1D components of the 2D image along the $\mathbf{e}_1$ and $\mathbf{e}_2$ coordinates, are obtained from $\cos^2(\alpha)$ and $\sin^2(\alpha)$, respectively. The i1D signals along the two diagonals of the 2D image are captured by the sine component of the second order spherical harmonic, i.e. $\sin(2\alpha)$. Since $\sin(2\alpha) = \cos^2(\alpha - \frac{\pi}{4}) - \sin^2(\alpha - \frac{\pi}{4})$, the two angular windowing functions yield again two i1D components. Therefore, i1D signals at different orientations in the 2D image can be extracted by applying the even filter. With the proper choice of the radial function, this tensor is related to the Hessian matrix, which is sensitive to the points of high Gaussian curvature. For example, according to the derivative theorem of Fourier theory, the horizontal second derivative of a rotationally symmetric spatial filter w is

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2} w\right\} = -\cos^2(\alpha)\rho^2 W = -\frac{1+\cos(2\alpha)}{2}\rho^2 W \tag{6}$$

where $\mathcal{F}$ indicates the Fourier transform, W is the Fourier transform of w, ρ and α are the polar coordinates. The analogous arguments apply to the other components of H_e.

As the even filter is able to yield differently oriented i1D signals, the Riesz transform is employed to evaluate the corresponding odd parts as it does for the monogenic signal. In the frequency domain, the odd part of the tensor valued filter reads

$$H_o = \begin{bmatrix} H_1(1 + \mathbf{e}_1 H_2 \cdot \mathbf{e}_1) & H_1(\mathbf{e}_1 H_2 \cdot \mathbf{e}_2) \\ H_1(\mathbf{e}_1 H_2 \cdot \mathbf{e}_2) & H_1(1 - \mathbf{e}_1 H_2 \cdot \mathbf{e}_1) \end{bmatrix} \tag{7}$$

For the computation of the tensor valued filter response, only three real valued and three complex valued convolutions are involved.

Let r_e represent the even part of the tensor valued filter response in the spatial domain, we have the following form:

$$r_e = \mathcal{F}^{-1}\{H_e S\} = \begin{bmatrix} r_{e11} & r_{e12} \\ r_{e21} & r_{e22} \end{bmatrix} \tag{8}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and S indicates the Fourier transform of the original signal. Analogously, the odd part of the tensor valued filter response r_o can be computed.

The second operation of the i2D operator is implemented by the determinant. Similar to the idea of [3], the second operator makes sure that the output is selective to the i2D structures. Therefore, the combination of the two i1D signals along the $\mathbf{e}_1$ and $\mathbf{e}_2$ coordinates is needed. However, these two i1D signals may not be independent, which becomes apparent if the operation is not done in the principle axes. In such case, the false contributions from i1D signals need to be cancelled out and this results in a rotation invariant operation. Hence, the even information of the i2D structure, which is related with the determinant of the Hessian, can be written as

$$O_e = r_{e11} r_{e22} - r_{e12} r_{e21} = A\mathbf{e}_3 \tag{9}$$

where O_e denotes the even information of the i2D structure. The computation result A is scalar valued. As in the case of the monogenic signal, the even information is embedded as the $\mathbf{e}_3$ component in the 3D Euclidean space. Analogously, the odd information can be obtained with the following form:

$$\begin{aligned} O_o &= r_{o11} r_{o22} - r_{o12} r_{o21} \\ &= \mathbf{e}_1(B + C\mathbf{e}_{12}) = B\mathbf{e}_1 + C\mathbf{e}_2 \end{aligned} \tag{10}$$

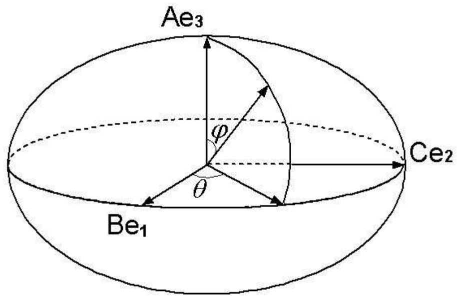

Figure 3 *The geometric model for the i2D operator, φ represents the phase and θ indicates twice of the main orientation.*

where O_o represents the odd information. The result $B + C\mathbf{e}_{12}$ is spinor valued, and the values of B and C are proportional to $\cos(2\beta)$ and $\sin(2\beta)$. Therefore, the main orientation is obtained as half of the angle (θ) which is computed from the spinor. By multiplying the $\mathbf{e}_1$ basis, O_o is converted to a vector valued representation. Consequently, the i2D operator can be derived as

$$O_{i2D} = O_e + O_o = A\mathbf{e}_3 + B\mathbf{e}_1 + C\mathbf{e}_2 \tag{11}$$

With the algebraic embedding, a geometric model for the i2D operator can be visualized as Figure 3. The geometric model for the i2D operator is an ellipsoid, which looks very similar to that of the monogenic signal. However, each axis encodes totally different meaning. The even information of the i2D structure is encoded within the $\mathbf{e}_3$ axis, and the odd information is encoded within the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes. The amplitude of the i2D operator response is then written as

$$|O_{i2D}| = \sqrt{O_{i2D}\widetilde{O_{i2D}}} = \sqrt{A^2 + B^2 + C^2} \tag{12}$$

According to the algebra isomorphism mentioned before, the phase information is evaluated by the following form:

$$\varphi = arg(A + sign(B)\sqrt{B^2 + C^2}\mathbf{e}_{12}) \qquad \varphi \in (-\pi, \pi] \tag{13}$$

where arg means the argument of the expression and $sign(B)$ represents the sign of B. The main orientation ψ is accordingly estimated as

$$\psi = \frac{\theta}{2} = \frac{arg(B + C\mathbf{e}_{12})}{2} \qquad \psi \in (-\frac{\pi}{2}, \frac{\pi}{2}] \tag{14}$$

4 EXPERIMENTAL RESULTS

In this section, we show some experimental results by applying the proposed i2D operator. The energy output (the square of the amplitude) and the estimated main orientation are illustrated in Figure 4, where the test image consists of a superposition of an angular and a radial modulation.

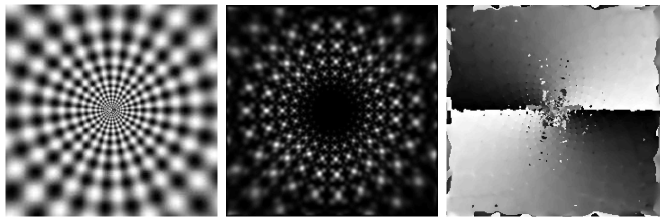

Figure 4 *From left to right: the original image, the energy output from the operator and the estimated main orientation.*

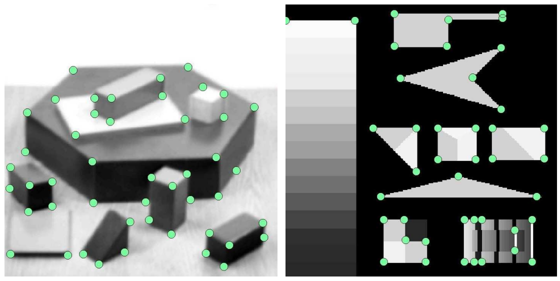

Figure 5 *The points of interest detection with our approach.*

It can be shown that our i2D operator gives a rotation invariant energy output, and the estimated orientation indicates the main orientation of the pattern. The energy output can be regarded as the junction strength to detect the points of interest and the detection results for two test images are demonstrated in Figure 5. The experimental results show that our approach can produce a satisfied detection. However, at the junctions where the intensity contrasts are low, nothing can be detected. Some points with high curvature are also not checked, this is caused by the tradeoff between the chosen scale parameter and the structure information. The detection may be improved by employing a coarse to fine tracking skill in the scale space.

The evaluated phase information is shown in Figure 6, where we use a synthetic image which is basically the superposition of two cosine signals with different frequencies, different amplitudes and orientations. A detail information of the evaluated phase is also given. The results show that at the centers of the blobs, non separated i2D phases are estimated. As the i2D structure is superimposed by two i1D patterns, the evaluated result illustrates clearly that the non separated phase contains simultaneously both of the two i1D phase information. The experimental results indicate that our i2D operator is able to evaluate the phase information of a superimposed pattern which has flexible opening angle. This is a remarkable advantage when compared with the classical structure tensor which contains no phase information, or with the structure multivector [7] which is restricted to the perpendicular superposition.

5 CONCLUSIONS

In this paper, we propose a rotationally invariant i2D op-

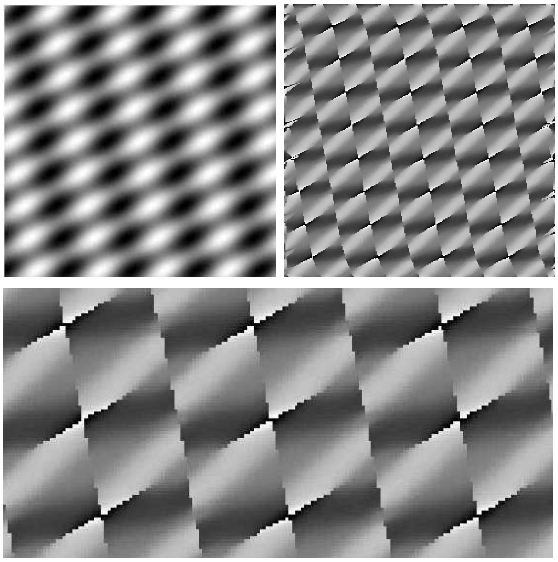

Figure 6 *Top row: from left to right are the original image and the evaluated phase information. Bottom row: detail information of the phase.*

erator for the analysis of the superimposed patterns with flexible opening angle. Based on an algebraically extended signal representation, the i2D operator is obtained and no steering is needed. It is a cascaded operator consisting of a tensor pair and a determinant operator. The presented i2D operator enables the simultaneous evaluation of the amplitude, main orientation and phase information of the i2D structure and it fulfills the split of identity. Compared with other approaches, the best advantage of it lies in the phase evaluation, which has never been well developed in the past. Although the local phase representation has been figured out, further investigation is needed in order to apply it in the real application and the symmetry concept in the current case ought to be identified in detail.

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