Algebraically Extended Representations of Multi–Dimensional Signals

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Abstract

In this article we introduce the Clifford Fourier transform (CFT) which is based on the n–dimensional Fourier transform but offers a representation of an n–dimensional signal with values in a 2^n–dimensional Clifford algebra. For $n = 2$ we get the special case of the quaternionic Fourier transform (QFT). The QFT provides information about the symmetry of a real signal in a more explicit way then the Fourier transform does. The QFT is related to the Fourier transform and to the Hartley transform. Based on the QFT we introduce a two–dimensional phase concept. The shift theorem is examined in terms of the QFT and experimental results concerning the estimation of the two–dimensional phase–difference of two images are presented.

1 Introduction

Fourier transforms are among the main tools in both one–dimensional and multi–dimensional signal processing (see e.g. [1]). The reason for this fact is that many frequently occurring tasks are more easily designed and performed in the frequency domain than in the spatial domain. The long history of one–dimensional signal processing has provided us with numerous tools, among which local methods like short–time Fourier transform (e.g. Gabor transform) and the concept of local phase play an important role. All these methods have been successfully adopted to image processing although they are intrinsically one–dimensional. For example the concept of local phase can be applied to lines (two–dimensional extensions of peaks) and edges (two–dimensional extensions of steps), i.e. to two–dimensional signals which are locally constant along one orientation. However, if we apply a two–dimensional Gabor filter to a junction–like signal, the local phase loses its natural interpretation.

The success of intrinsically one–dimensional methods in processing two–dimensional signals is due to the fact that, in natural images, events which are intrinsically two–dimensional occur very rarely [6]. Intrinsic dimensionality is a local signal property and refers to the number of degrees of freedom which are actually used in a signal. Recently Krieger and Zetzsche [6] considered the statistical properties of natural images which respect to intrinsic dimensionality. However, if we turn our attention to the few points in an image which are intrinsically two–dimensional, we should try to find true two–dimensional generalization of the methods mentioned above.

We hope to make a step towards this aim by introducing a transform for n–dimensional signals which is based on the n–dimensional Fourier transform, but offers a representation with values in a Clifford algebra instead of a complex–valued signal representation. The idea of performing Fourier transforms in higher dimensional algebras is not completely new: Recently Chernov [3] considered among other algebras the use of quaternions and Cayley numbers in Fourier transforms, but his aim was different from ours. While we are interested in a richer signal representation Chernov was looking for fast algorithms of discrete Fourier transforms.

The definition of the Clifford Fourier transform (CFT) is the subject of the following section. We will restrict ourselves to the case $n = 2$ where we find that the corresponding Clifford algebra is isomorphic to the algebra of quaternions. Therefore we call the two–dimensional CFT quaternionic Fourier transform (QFT). We show that the QFT splits a real two–dimensional signal into four components of different symmetry analogously to the one–dimensional Fourier transform, where a real even signal is transformed into a real representation in the frequency domain and a real odd signal into an imaginary representation.

In section five we analyze the relation of the QFT to both the Hartley and the Fourier transform and show that the QFT seems to be a very natural extension of the one–dimensional Fourier transform to two dimensions. We will suggest a generalization of the phase–concept for two–dimensional signals. Section six and seven are devoted to the well known shift theorem, which will be analyzed in the case of the QFT. Some experimental results related to the shift theorem will be presented.

2 Definition of the CFT

The basis functions of the n–dimensional Fourier transform are products of basis functions of the one–dimensional Fourier transform. The latter are complex–valued and so are products of them. Regarding the well known advantages of one–dimensional Fourier transform we hold on to its basis functions but, multiplying them, we consider different samples of complex numbers as their ranges.

Let $\mathbf{u}$ and $\mathbf{x}$ denote the vectors $(u_1, u_2, \ldots, u_n)^T$ and $(x_1, x_2, \ldots, x_n)^T$, respectively. We start with the basis–functions of the Fourier transform $e^{-i2\pi\mathbf{u}\cdot\mathbf{x}}$ and replace them with the product of n exponentials where we use n different samples of the complex numbers $\mathbb{R}\{1, i_1\}$ to $\mathbb{R}\{1, i_n\}$. In order to multiply $e^{-i_l 2\pi u_l x_l}$ and $e^{-i_m 2\pi u_m x_m}$ with $l, m \in \{1, \ldots, n\}$ we must look for an algebra that contains the elements 1 and $i_1, \ldots, i_n$ which obey the multiplication rules $i_m^2 = -1$ for $m \in \{1, \ldots, n\}$.

Here we will use a Clifford algebra (for a review on Clifford Algebras see e.g. [7]) which is generated by $\{i_1, \ldots, i_n\}$ according to the multiplication rules $i_l^2 = -1$ for every $l \in \{1, \ldots, n\}$ and $i_l i_m = -i_m i_l$ for $l \neq m$. The elements $i_l i_m$ are called elements of grade two and denoted by $i_l i_m = i_{lm}$. There are $\binom{n}{2}$ basis elements of grade two. Applying the same procedure to the elements of grade two we get elements of grade three: $i_k(i_{lm}) = (i_{kl})i_m = i_{klm}$ for pair–wise different k, l and m. We continue like this and get as element with the highest grade one basis element with grade n. Altogether we get $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ basis elements, which means that the range of the n–dimensional transform is a 2^n–dimensional Clifford algebra. Therefore we call this transform an n–dimensional Clifford Fourier transform (CFT).

The basis functions of the n–dimensional CFT are given by

$$Q_{\mathbf{u}}(\mathbf{x}) = \prod_{k=1}^{n} e^{-i_k 2\pi u_k x_k} \tag{1}$$

and we denote the transformed signal as

$$F^c(\mathbf{u}) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(\mathbf{x}) Q_{\mathbf{u}}(\mathbf{x}) d^n\mathbf{x} \quad .$$

The basis functions are orthogonal and the CFT is a linear and invertible transform. The inverse is given by

$$\mathcal{F}_c^{-1}\{F^c\}(\mathbf{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F^c(\mathbf{u}) \left(Q_{\mathbf{u}}(\mathbf{x})\right)^* d^n\mathbf{u} \quad , \tag{2}$$

where $Q^*_{\mathbf{u}}(\mathbf{x})$ is defined as

$$Q^*_{\mathbf{u}}(\mathbf{x}) = e^{i_n 2\pi u_n x_n} e^{i_{n-1} 2\pi u_{n-1} x_{n-1}} \ldots e^{i_1 2\pi u_1 x_1} \quad .$$

3 The two–dimensional case

We now want to examine the CFT for $n = 2$. In this case the basis functions are given by

$$Q_{\mathbf{u}}(\mathbf{x}) = e^{-i_1 2\pi u x} e^{-i_2 2\pi v y} \quad . \tag{3}$$

$\mathbb{R}\{1, i_1\}$ and $\mathbb{R}\{1, i_2\}$ are two samples of the complex numbers, so we have $i_1^2 = i_2^2 = -1$. According to section two the corresponding Clifford algebra is 2^2–dimensional with the basis elements $\{1, i_1, i_2, i_{12}\}$. This four–dimensional algebra is isomorphic to the algebra of quaternions which was introduced by Hamilton in 1844 and is denoted by $\mathbb{H}$ (see e.g. [5]). In the following we will therefore use the notation $i_1 = i$, $i_2 = j$ and $i_{12} = k$. We call the two–dimensional CFT quaternionic Fourier transform (QFT) and denote it as $\mathcal{F}_q\{f\} = F^q$ with

$$F^q(\mathbf{u}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{x}) Q_{\mathbf{u}}(\mathbf{x}) d^2\mathbf{x} \quad , \tag{4}$$

with $\mathbf{u} = (u, v)^T$ and $\mathbf{x} = (x, y)^T$.

We want to mention a few facts about quaternions and their exponential function. The exponential function for quaternions, $\exp : \mathbb{H} \to \mathbb{H}$, is defined in the same way as the exponential function for the real and the complex numbers via the infinite sum

$$\exp(q) = \sum_{k=0}^{\infty} \frac{q^k}{k!}, \quad q \in \mathbb{H} \quad . \tag{5}$$

It can be shown that this sum converges for every quaternion q. For each quaternion $q = a + bi + cj + dk$ we define the conjugate $q^* = a - bi - cj - dk$. The magnitude of a quaternion q is defined by $|q| = \sqrt{qq^*}$. If we consider quaternions of the form $p = bi + cj + dk$ (so called *pure quaternions*) with magnitude $|p| = 1$, the set $\mathbb{R}\{1, p\}$ is an algebra which is isomorphic to the algebra of the complex numbers. Therefore the Euler formula

$$e^{px} = \cos(x) + p\sin(x) \quad , \quad x \in \mathbb{R} \tag{6}$$

holds for each pure unit quaternion p. Also the relation

$$e^{px} e^{py} = e^{p(x+y)} \tag{7}$$

remains valid for such p. If we multiply exponentials containing different pure quaternions p and p' the above rules are no longer true, that means we have

$$e^{px} e^{p'y} \neq e^{px + p'y}, \quad x, y \in \mathbb{R}, \quad p \neq \pm p' \tag{8}$$

especially

$$e^{ix} e^{jy} \neq e^{ix+jy} \quad , \tag{9}$$

and as another consequence of the non–commutativity of quaternions the exponentials of quaternions also do not commute in general

$$e^{ix} e^{jy} \neq e^{jy} e^{ix} \quad . \tag{10}$$

For this reason it is necessary to fix the order of the factors in the QFT. One important feature of the Fourier transform is the orthogonality of the basis functions $e^{-i2\pi \mathbf{x}\cdot\mathbf{u}}$. For each two quaternions q_1 and q_2 the equation $q_1^* q_2^* = (q_1 q_2)^*$ holds. Using this relation we find

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_{\mathbf{u}}(\mathbf{x}) Q_{\mathbf{u}'}^*(\mathbf{x}) = \delta^2(\mathbf{u} - \mathbf{u}') \quad , \tag{11}$$

so the orthogonality–relation holds for the basis functions of the QFT. Another important feature of the Fourier transform is the existence of the inverse transform $\mathcal{F}^{-1}$ with $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$, which is also provided by the QFT:

$$\mathcal{F}^{-1}\{F^q\}(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^q(\mathbf{u}) Q_{\mathbf{u}}^*(\mathbf{x}) d^2\mathbf{u} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^q(\mathbf{u}) e^{j2\pi vy} e^{i2\pi ux} d^2\mathbf{u} \tag{12}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{x}') \delta^2(\mathbf{x} - \mathbf{x}') d^2\mathbf{x}' = f(\mathbf{x}) \quad . \tag{13}$$

Of course, like the CFT in general and the Fourier transform, the QFT is a linear transform.

4 Symmetry selectivity of the QFT

In this section we will show how the QFT deals with signals of different combinations of even and odd symmetries. It is a well known fact that the Fourier transform of the even part of a real one–dimensional signal is real and even. The Fourier transform of the odd part of the same signal is imaginary and odd.

A two–dimensional signal can be split into even and odd parts along the x–axis and along the y–axis as well. So every real two–dimensional signal can be written in the form $f = f_{ee} + f_{oe} + f_{eo} + f_{oo}$, where f_{ee} denotes the part of

	$f=f_{ee}$				$f=f_{oe}$				$f=f_{eo}$				$f=f_{oo}$			
f	r	i	j	k	r	i	j	k	r	i	j	k	r	i	j	k
$\tilde{f}$	r	i	j	k	i	r	k	j	r	i	j	k	i	r	k	j
F	r	i	j	k	i	r	k	j	j	k	r	i	k	j	i	r

Table 1: Symmetry properties of the QFT. r stands for the real part, i for the i–imaginary and so on.

f which is even with respect to x and y, f_{oe} denotes the part which is odd with respect to x and even with respect to y and so on. Because the two–dimensional Fourier transform has only two components — one real component and one imaginary component — it is not able to separate these four components of different symmetry.

However, the QFT has symmetry–splitting properties that are analogous to the properties of the one–dimensional Fourier transform: The transform of the f_{ee}–part of a real two–dimensional signal is real, the f_{oe}–part is transformed into an i–imaginary part, f_{eo} into the j–imaginary and f_{oo} into the k–imaginary part. The symmetry of the signal is preserved by the QFT.

We can easily see this by looking at the QFT as two sequentially performed one–dimensional Fourier transforms: First we perform a one–dimensional Fourier transform on $f(\mathbf{x})$ with respect to x keeping y fixed and call the result $\tilde{f}$:

$$\tilde{f}(u,y) = \int_{-\infty}^{\infty} f(x,y) e^{-i2\pi ux} dx \quad . \tag{14}$$

In a second step we perform a Fourier transform on $\tilde{f}$ with respect to y keeping u fixed

$$F^q(\mathbf{u}) = \int_{-\infty}^{\infty} \tilde{f}(u,y) e^{-j2\pi vy} dy \quad . \tag{15}$$

Notice that the second Fourier transform is realized using the imaginary unit j instead of i. (That leads to the fact that the order of the factors under the integral can not be chosen arbitrarily: $\int \tilde{f} e^{-j2\pi vy} dy \neq \int e^{-j2\pi vy} \tilde{f} dy$). An overview of the symmetry properties of the QFT is given in table 1. That shows for example that the QFT of an i–imaginary signal which is even with respect to x and odd with respect to y is k–imaginary.

5 Relation to the Hartley transform

The Hartley transform of a one–dimensional signal f is defined by

$$H(u) = \int_{-\infty}^{\infty} f(x)\{\cos(2\pi ux) + \sin(2\pi ux)\} dx \quad . \tag{16}$$

It is related to the Fourier transform of f by

$$F(u) = H_e(u) - iH_o(u) \quad , \tag{17}$$

where H_e and H_o denote the even and odd part of H, respectively. Thus the Fourier transform separates the parts of different symmetry — which are mixed in the Hartley transform — by putting them into different components.

The two–dimensional Hartley transform is given by

$$H(\mathbf{u}) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\{\cos(2\pi\mathbf{u}\cdot\mathbf{x}) + \sin(2\pi\mathbf{u}\cdot\mathbf{x})\} d^2\mathbf{x}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\cos(2\pi ux)\cos(2\pi vy)d^2\mathbf{x} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\sin(2\pi ux)\sin(2\pi vy)d^2\mathbf{x} \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\cos(2\pi ux)\sin(2\pi vy)d^2\mathbf{x} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\sin(2\pi ux)\cos(2\pi vy)d^2\mathbf{x} \\
&= H_{ee}(\mathbf{u}) + H_{oo}(\mathbf{u}) + H_{eo}(\mathbf{u}) + H_{oe}(\mathbf{u}) \quad .
\end{aligned}
$$

Again it is possible to get the Fourier transform of f from the Hartley transform:

$$F(\mathbf{u}) = \big(H_{ee}(\mathbf{u}) + H_{oo}(\mathbf{u})\big) - i\big(H_{eo}(\mathbf{u}) + H_{oe}(\mathbf{u})\big) \quad . \tag{18}$$

Hence, in this case, too, the Fourier transform separates parts of different symmetry which are mixed in the Harley transform, but this separation is only halfway. The complete separation is only provided by the QFT:

$$F^q(\mathbf{u}) = H_{ee}(\mathbf{u}) - iH_{oe}(\mathbf{u}) - jH_{eo}(\mathbf{u}) - kH_{oo}(\mathbf{u}) \quad . \tag{19}$$

From this point of view the construction of the QFT looks so natural that one might ask why Fourier himself did not invent his transform in this way and the only answer we can give is that the quaternions were invented by Hamilton in 1844 whereas Fourier had already died in 1830.

If the Hartley transform of a real two–dimensional image signal is known, it is possible to calculate the QFT of the signal directly from its Hartley transform using (19) with

$$
\begin{aligned}
H_{ee}(\mathbf{u}) &= \frac{1}{4}\big(H(u,v) + H(-u,-v) + H(-u,v) + H(u,-v)\big) && (20)\\
H_{oe}(\mathbf{u}) &= \frac{1}{4}\big(H(u,v) - H(-u,-v) - H(-u,v) + H(u,-v)\big) && (21)\\
H_{eo}(\mathbf{u}) &= \frac{1}{4}\big(H(u,v) - H(-u,-v) + H(-u,v) - H(u,-v)\big) && (22)\\
H_{oo}(\mathbf{u}) &= \frac{1}{4}\big(H(u,v) + H(-u,-v) - H(-u,v) - H(u,-v)\big) \quad . && (23)
\end{aligned}
$$

Obviously the QFT contains no information about the image, which is missing in the Hartley transform. So why do we need the QFT?

We will try to explain our point of view about this question by first looking at the one–dimensional case: Also in one dimension we could ask, why we should use the Fourier transform with its complex values instead of the one–dimensional Hartley transform which provides the same information content in a more compact representation. Here we want to give only one argument which is related to the properties of the short time Fourier transform in its special form of a Gabor transform.

The complex response of a Gabor filter can be represented as the magnitude of the response multiplied by a phase factor. The value of the magnitude shows whether there is an event at the position of the signal under consideration, independent of the symmetry of the event, whereas the phase determines the symmetry of the event, i.e. whether there is a peak or a step in the signal.

In two–dimensional signals besides one–dimensional structures like lines and edges we meet more complicated symmetries. In [2] we show the usefulness of the (short–time) QFT in distinguishing between different types of symmetry of two–dimensional signals. We are mainly interested in local neighborhoods which are intrinsically two–dimensional.

Thus we see that we must look for a polar representation of F^q in order to generalize the concept of phase to the QFT. The question how the concept of phase can be extended to higher dimensions is not trivial (see for example [4]). Although there is still some work to be done we want to suggest the following representation here: Every quaternion q can be written in the form

$$q = re^{s\psi} \tag{24}$$

where r and ψ are real numbers and s is the pure unit quaternion

$$s = i\cos\phi\sin\theta + j\sin\phi\sin\theta + k\cos\theta \quad . \tag{25}$$

For the magnitude of $e^{s\psi}$ is 1, we have $|q| = r$. So we can represent the QFT of an image by the magnitude $|F^q|(\mathbf{u})$ and the three angles $\phi(\mathbf{u})$, $\psi(\mathbf{u})$ and $\theta(\mathbf{u})$ which could be regarded as a generalization of the one–dimensional phase. In the following section we will introduce a more natural concept of two–dimensional phase.

6 The QFT shift theorem

The well known shift theorem of the two–dimensional Fourier transform reads

$$F_T(\mathbf{u}) = e^{-i2\pi\mathbf{u}\cdot\mathbf{d}} F(\mathbf{u}) \quad , \tag{26}$$

where F and F_T are given by

$$F(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi\mathbf{u}\cdot\mathbf{x}} d^2\mathbf{x} \tag{27}$$

and

$$F_T(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_T(\mathbf{x}) e^{-i2\pi\mathbf{u}\cdot\mathbf{x}} d^2\mathbf{x}, \quad \text{with} \quad f_T(\mathbf{x}) = f(\mathbf{x}-\mathbf{d}) \tag{28}$$

Hence, the magnitude of the Fourier transform remains invariant under the translation of the signal. In one dimension it is possible to recover the amount of translation from the phase difference, which is impossible in two dimensions if we regard (26) only for one value of $\mathbf{u}$.

Using the QFT, the situation is the following. Let

$$F^q(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi ux} e^{-j2\pi vy} d^2\mathbf{x}$$

and

$$F^q_T(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_T(\mathbf{x}) e^{-i2\pi ux} e^{-j2\pi vy} d^2\mathbf{x} \quad .$$

It follows easily that

$$F^q_T(\mathbf{u}) = e^{-i2\pi ud_1} F^q(\mathbf{u}) e^{-j2\pi vd_2} \quad . \tag{29}$$

But what does this mean in geometrical terms? In one–dimensional Fourier transform a translation of the signal by d causes a rotation of the complex number $F(u)$ by the angle $-2\pi du$ in the $(1,i)$–plane for any value of u. If we consider the QFT we find out, that a translation of the signal by d_1 in x–direction causes a rotation of the quaternion $F^q(\mathbf{u})$ by the angle $-2\pi d_1 u$ in the $(1,i)$–plane and in the (j,k)–plane as well. A translation by d_2 in y–direction effects a rotation of $F^q(\mathbf{u})$ by $-2\pi d_2 v$ in the $(1,j)$–plane and in the (i,k)–plane, instead. In the following we will write this down in matrix–notation representing the quaternion $F^q(\mathbf{u}) = F^q_0(\mathbf{u}) + iF^q_1(\mathbf{u}) + jF^q_2(\mathbf{u}) + kF^q_3(\mathbf{u})$ as the vector $\vec{F}^q(\mathbf{u}) = (F^q_0(\mathbf{u}), F^q_1(\mathbf{u}), F^q_2(\mathbf{u}), F^q_3(\mathbf{u}))^T$. Let f_{T_1} be defined by $f_{T_1}(x,y) = f(x-d_1, y)$. We then obtain

$$\vec{F}^q_{T_1}(\mathbf{u}) = T_1(-2\pi d_1 u)\vec{F}^q(\mathbf{u}) \tag{30}$$

with

$$T_1(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix} \quad . \tag{31}$$

Analogously defining f_{T_2} by $f_{T_2}(x,y) = f(x, y-d_2)$ we get

$$\vec{F}^q_{T_2}(\mathbf{u}) = T_2(-2\pi d_2 v)\vec{F}^q(\mathbf{u}) \tag{32}$$

with

$$T_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \quad . \tag{33}$$

As we should expect the matrices T_1 and T_2 commute, i.e.

$$T_1(\phi)T_2(\theta) = T_2(\theta)T_1(\phi) \quad . \tag{34}$$

Now we can show explicitly how the translation vector $\mathbf{d}$ can be recovered from a pair $\vec{F}^q(\mathbf{u})$ and $\vec{F}^q_T(\mathbf{u})$. Defining $T(\phi,\theta) = T_1(\phi)T_2(\theta)$ we have

$$\vec{F}^q_T(\mathbf{u}) = T(\phi,\theta)\vec{F}^q(\mathbf{u}) \quad .$$

Here ϕ and θ denote $-2\pi d_1 u$ and $-2\pi d_2 v$, respectively. It is straightforward to show that

$$T(\phi,\theta)\vec{F}^q(\mathbf{u}) = \mathbf{F}(\mathbf{u})(\alpha,\beta,\gamma,\delta)^T$$

where $\mathbf{F}(\mathbf{u})$ denotes the matrix

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} F^q_0(\mathbf{u}) & -F^q_1(\mathbf{u}) & -F^q_2(\mathbf{u}) & F^q_3(\mathbf{u}) \\ F^q_1(\mathbf{u}) & F^q_0(\mathbf{u}) & -F^q_3(\mathbf{u}) & -F^q_2(\mathbf{u}) \\ F^q_2(\mathbf{u}) & -F^q_3(\mathbf{u}) & F^q_0(\mathbf{u}) & -F^q_1(\mathbf{u}) \\ F^q_3(\mathbf{u}) & F^q_2(\mathbf{u}) & F^q_1(\mathbf{u}) & F^q_0(\mathbf{u}) \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \end{pmatrix} \quad . \tag{36}$$

$\mathbf{F}(\mathbf{u})$ is invertible if and only if $\det(\mathbf{F}(\mathbf{u})) \neq 0$, i.e. $F^q(\mathbf{u})(F^q)^*(\mathbf{u}) \neq \sqrt{2}|F^q_3(\mathbf{u})F^q_0(\mathbf{u}) - F^q_1(\mathbf{u})F^q_2(\mathbf{u})|$. If $\mathbf{F}(\mathbf{u})$ is invertible we get α,β,γ and δ directly from

$$\mathbf{F}^{-1}(\mathbf{u})\vec{F}^q_T(\mathbf{u}) = (\alpha,\beta,\gamma,\delta)^T \quad . \tag{37}$$

If we can assume that we have $\phi,\theta \in]-\pi/2,\pi/2[$, once we know α,β,γ and δ, ϕ and θ can directly be determined:

$$\phi = \arctan\left(\frac{\beta}{\alpha}\right), \qquad \theta = \arctan\left(\frac{\gamma}{\alpha}\right) \quad . \tag{38}$$

Generally it is possible to recover (ϕ,θ) within the interval $[-\pi,\pi[\times[-\pi/2,\pi/2[$ by a function $arg : \mathbb{H}\backslash\{0\} \mapsto \mathbb{R}^2$ which is based on (38). Because of the bulkiness of the definition of arg it will not be given here (for a definition see [2]).

From ϕ and θ we recover d_1 and d_2 by

$$d_1 = \frac{\phi}{2\pi u}, \quad d_2 = \frac{\theta}{2\pi v} \quad . \tag{39}$$

Using the arg–function we can define the two–dimensional phase of the component with frequency $\mathbf{u}$ of an image signal f as $arg(F^q(\mathbf{u}))$.

7 Experimental results

In this section we want to test the algorithm given in the previous section for recovering the translation vector $\mathbf{d}$ of two synthetic images. We use two periodic images, where the second image is a translated copy of the first one (see figure 1).

The periods are 32 pixels in x–direction and 64 pixels in y–direction. The translation vector is $\mathbf{d} = (d_1,d_2)^T$ with $d_1 = 9$, pixels (translation to the right) and $d_2 = 15$ pixels (downwards).

We evaluate their QFT's at the frequencies of the periodic structure, which is $\mathbf{u} = (u,v)^T$ with $u = \frac{1}{32}$ pixels^{-1} and $v = \frac{1}{64}$ pixels^{-1}, and get in vector notation:

$$\vec{F}^q(\mathbf{u}) = (4.253, -6.365, -1.290, 1.931)^T, \qquad \vec{F}^q_T(\mathbf{u}) = (1.442, 0.597, 7.249, 3.003)^T$$

for the first and the second image, respectively. Applying (37), (38) and (39) we get the estimate for the translation vector $\mathbf{d} = (8.999, 15.004)^T$ pixels.

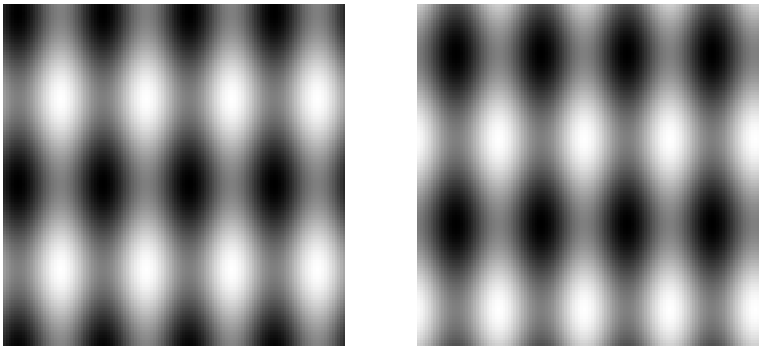

Figure 1: The original and the translated image

8 Conclusion

In this article we introduced the quaternionic Fourier transform (QFT), which is a linear integral transform for two–dimensional signals as the two–dimensional Clifford Fourier transform (CFT). We showed that the four components of the quaternionic representation of an image in the frequency domain are selective to signal components with different symmetry. We suggest a generalized concept of the phase of a two–dimensional signal. We worked out the shift theorem for the QFT and proved that it is possible to recover the translation vector $\mathbf{d}$ that relates two translated copies of an image from the QFT's of the images. We presented an algorithm for the recovering of the translation vector and demonstrated its applicability in an experiment on two translated copies of a periodic image.

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