Quaternionic Gabor Filters for Local Structure Classification*

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Abstract

We introduce quaternionic Gabor filters for the classification of local image structure. These filters are constructed as windowed basis functions of the quaternionic Fourier transform. We show that – in contrast to the 2D complex Gabor filters – the quaternionic Gabor filters are intrinsically 2D filters. A generalized phase concept is introduced and compared to the classical one. It is shown how local image structure can be classified by the value of the local quaternionic phase.

1. Introduction

Gabor filters (GFs) were first introduced in the field of 1D signal processing by D. Gabor [6] for a joint time-frequency analysis. They have the advantage of being optimally localized in the time and in the frequency domain, simultaneously. In recent years GFs have found applications in several different PR and CV tasks such as local phase and frequency estimation for texture segmentation [2] and stereo disparity estimation [7], to name only two possible applications.

There is a close relationship between the local structure of a signal and its local phase. The latter can be estimated as the angular argument of the GF response to the signal [13]. *Here the term local structure mainly refers to local even/odd-symmetries.* For another approach see e.g. [1]. Complex GFs are usually adapted from the analysis of 1D signals to image analysis in an intrinsically 1D way. (For details on intrinsic dimensionality see [11].) Consequently, they are only suited to the detection and classification of 1D features – i.e. lines and edges – in an image across a predefined orientation. In this article we introduce quaternionic GFs which turn out to be intrinsically 2D filters. Quaternionic GFs are defined as Gaussian windowed kernel of the recently introduced quaternionic Fourier transform (QFT) [3, 4, 5].

The structure of this article is as follows: In the following section we give a brief review of the QFT and the quaternion algebra. A novel phase concept for 2D signals which is based on the QFT is introduced in section 3. In section 4 the relation between local structure and local phase is explained and quaternionic GFs are defined. Before the article is closed by a conclusion, results are shown which demonstrate the local structure selectivity of quaternionic GFs.

2. The Quaternionic Fourier Transform

The QFT has recently been introduced in [5] and in [3, 4], independently. Like the Fourier transform (FT) and the Hartley transform the QFT is a linear, invertible harmonic transform. Although the QFT is restricted to 2D signals, its concept can be extended to arbitrary dimensions as Clifford algebra FT with the special cases of the complex FT in 1D and the QFT in 2D [3]. The QFT of a 2D function $f(\mathbf{x})$ (real or quaternion valued) is given by

$$F^q(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi u x} f(\mathbf{x}) e^{-j2\pi v y}\, d^2\mathbf{x}, \tag{1}$$

where $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$. The difference between the FT and the QFT is that the imaginary unit i in the second exponential in (1) is replaced by j. The units i and j are elements of the algebra of quaternions and obey the relations $i^2 = j^2 = -1$ and $ij = -ji = k$. *Note that the quaternionic multiplication is not commutative.*

The quaternion algebra can be seen as a 4D real linear space together with the multiplication defined by the relations given above. The magnitude of a quaternion $q = a + bi + cj + dk$ is defined as $|q| = \sqrt{qq^*}$ where $q^* = a - bi - cj - dk$ is called the conjugate of q. In the definition of the generalized phase concept we will make use of the fact, that unit quaternions represent rotations in $\mathbb{R}^3$: Let $x = ix_1 + jx_2 + kx_3$ represent the vector $\mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, then the rotation through the angle ϕ about the axis given by the unit vector $\mathbf{n} \in \mathbb{R}^3$ can be performed as $x' = qxq^{-1}$ with $q = \exp(n\phi/2)$. For more details an quaternions see [9].

Using the QFT it is possible to generalize the concept of the analytic signal – which stems from 1D signal processing

*This work was supported by the Studienstiftung des deutschen Volkes (Th.B.) and by the DFG (So-320-2-1) (G.S.).

– to 2D in a well defined way which is impossible using the 2D Fourier transform [4]. In [12] the QFT has been proposed for the processing of color images.

3. A Novel Phase Concept

In polar representation a signal's FT can be decomposed in an amplitude and a phase component. In 2D the phase component carries the main part of image information [8]. For the QFT the polar representation has to be extended. Since the quaternions constitute a 4D algebra we can represent $F^q(\mathbf{u})$ in a polar representation of the form $(|F^q(\mathbf{u})|, \phi(\mathbf{u}), \theta(\mathbf{u}), \psi(\mathbf{u}))$, where $|F^q(\mathbf{u})|$ is the magnitude and the angles $(\phi(\mathbf{u}), \theta(\mathbf{u}), \psi(\mathbf{u}))$ represent a novel kind of phase vector. Among several possibilities we choose the following definition:

$$F^q(\mathbf{u}) = |F^q(\mathbf{u})| e^{i\phi(\mathbf{u})} e^{k\psi(\mathbf{u})} e^{j\theta(\mathbf{u})}. \qquad (2)$$

The meaning of ϕ, θ and ψ has to be clarified below. The angles $2\phi(\mathbf{u}), 2\theta(\mathbf{u})$ and $2\psi(\mathbf{u})$ are one of several possible definitions of the Euler angles [10] when $F^q/|F^q|$ is interpreted as a quaternion, representing a rotation in 3D as mentioned in section 2. Using this relation we can recover the angles (ϕ, θ, ψ) which yield the representation $q = |q| e^{i\phi} e^{k\psi} e^{j\theta}$ of a quaternion given as $q = a + ib + jc + kd$. Unlike the complex phase which is unique in the interval $[-\pi, \pi[$ the quaternionic phase (ϕ, θ, ψ) can only be evaluated within the interval $[-\pi, \pi[\times [-\pi/2, \pi/2[\times [-\pi/4, \pi/4]$. To clarify the meaning of the quaternionic phase we present the QFT version of the shift theorem:

Theorem (shift theorem): Let $F^q(\mathbf{u})$ and $F^q_T(\mathbf{u})$ be the QFTs of the 2D signals f and f_T, respectively. Here f_T is a shifted version of f, namely $f_T(\mathbf{x}) = f(\mathbf{x} - \mathbf{d})$ and $\mathbf{d} = (d_1, d_2)^\top$. Then, $F^q(\mathbf{u})$ and $F^q_T(\mathbf{u})$ are related by

$$F^q_T(\mathbf{u}) = e^{-i2\pi u d_1} F^q(\mathbf{u}) e^{-j2\pi v d_2}. \qquad (3)$$

If we denote the phase of $F^q(\mathbf{u})$ by $(\phi(\mathbf{u}), \theta(\mathbf{u}), \psi(\mathbf{u}))$ then, as a result of the shift, the first and the second component of the phase undergo a phase-shift and the phase of $F^q_T(\mathbf{u})$ is given by $(\phi(\mathbf{u}) - 2\pi u d_1, \theta(\mathbf{u}) - 2\pi v d_2, \psi(\mathbf{u}))$. This result shows that the ϕ-component (θ-component) of the phase corresponds to the horizontal (vertical) Fourier phase while the ψ-component represents a new entity, which is not effected by a mere shift of the image. The meaning of ψ will be demonstrated in section 5.

4. Local Structure from Local Phase

The *local phase* of a signal can be computed from the filter response of a complex GF. The impulse response of a 1D GF as shown in figure 1 is given by

$$G(x) = e^{-x^2/2\sigma^2} e^{icx/\sigma}, \quad c = \sigma\omega, \qquad (4)$$

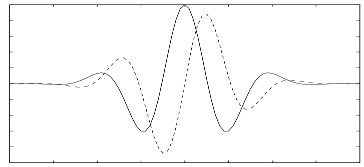


Figure 1. A complex GF with $c = 3$ (real part solid, imaginary part dashed).

i.e. it is a windowed basis function of the FT. To different local structures there correspond different values of the local phase as shown in table 1. E.g. for *local phase* $= 0$ the local structure is a positive peak, while for *local phase* $= \pi/2$ the local structure is a falling step edge and so on.

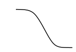
local structure				
local phase	$-\pi$	$-\pi/2$	0	$\pi/2$

Table 1. Local structure and local phase in 1D.

For tasks of image analysis, the 1D GF (4) usually is extended to 2D in the following way:

$$G(\mathbf{x}) = e^{(-x^2/2\sigma_1^2 - y^2/2\sigma_2^2)} e^{ixc/\sigma_1}, \quad c = \sigma_1\omega. \qquad (5)$$

A rotated version of (5) is given by $G_\alpha(x, y) := G(x', y')$ with $x' = x\cos(\alpha) + y\sin(\alpha)$ and $y' = -x\sin(\alpha) + y\cos(\alpha)$. Analogous to the 2D complex GF we define the quaternionic GF as the windowed kernel of the QFT:

$$G_q(\mathbf{x}) = e^{(-x^2/2\sigma_1^2 - y^2/2\sigma_2^2)} e^{i(c_1 x/\sigma_1)} e^{j(c_2 y/\sigma_2)}, \qquad (6)$$

with $c_i = \sigma_i \omega_i$. Again, we can construct rotated versions of the quaternionic GFs as $G_{q,\alpha}(x, y) := G_q(x', y')$ with (x', y') like given above. Since G^q contains two orthogonal frequencies it depends on one more parameter than G. In figure 2 a complex and a quaternionic GF are shown with $\sigma_1 = \sigma_2$, $c = c_1 = c_2 = 2$ and $\alpha = 0$. The 2D complex GFs suffer from the disadvantage that they can only distinguish elongated versions of the structures shown in table 1, i.e. straight lines and edges. In the following section it will be shown how this restriction can be overcome using quaternionic GFs.

5. Results

Table 2 shows, as an 2D-analogue to table 1, the relation between the local 2D structure and the local quaternionic phase. The local quaternionic phase of the shown structures is computed from the response of a quaternionic GF. Table 2 shows four of sixteen primitive structures that can be discriminated from their local phase. The value $\psi = \pm\pi/4$ indicates that the patterns respond optimally to a linear combination of two components of the quaternionic GF. These

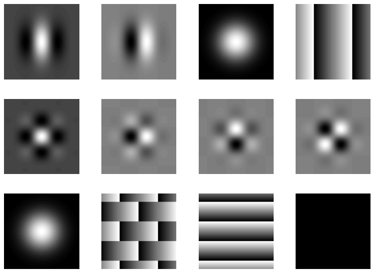


Figure 2. A complex and a quaternionic GF. The rows show from left to right: Real and imaginary part, magnitude and phase of the complex GF (top), real, i–, j– and k–imaginary part of the quaternionic GF (middle) and magnitude, $\phi-$, $\theta-$ and $\psi-$phase of the quaternionic GF (bottom).

are either the i- and the j-imaginary part (1^{st} pattern) or the real and the k-imaginary part (2^{nd} pattern). Lines and step edges can be discriminated in analogy to the 1D case where $\phi + \theta$ corresponds to the 1D phase (compare to table 1.). The value $\psi = 0$ in the other cases indicates that the patterns respond only to one component of the quaternionic GF (e.g. k-imaginary part for the chess-board structure). Thus, $\psi = 0$ corresponds to intrinsically 2D structures. Inverting a pattern always results in a shift of the ϕ-component by $\pm\pi$.

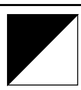
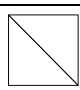
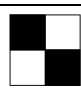
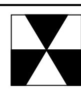
			
$\begin{bmatrix} 0 \\ \pi/2 \\ \pi/4 \end{bmatrix}$	$\begin{bmatrix} -\pi \\ 0 \\ -\pi/4 \end{bmatrix}$	$\begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \pi/2 \\ 0 \end{bmatrix}$

Table 2. Local structure and quaternionic phase in 2D.

The ψ-component of the quaternionic phase provides new information which is not given by the phase when estimated with two complex GFs. This is clarified in figure 3. We regard a class of structures which have all the same horizontal and vertical local phase ($\phi = \theta = 0$), independently of the way it is computed: with two complex GF or with one quaternionic GF. This class is parameterized by λ. The additional phase value ψ resulting from the quaternionic filtering allows the distinction of the patterns.

6. Summary and Conclusions

In this article we introduced quaternionic GFs. It could be shown, how a local phase concept can be defined based on these filters. By evaluating the quaternion phase of the filter response intrinsically 1D and 2D structures can be distinguished applying only one quaternionic GF. Whereas in this article we presented only the basic ideas we see many

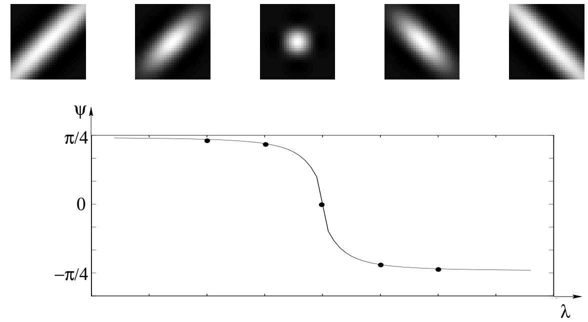


Figure 3. Five structures and the ψ-component of their local quaternionic phase. The dots indicate the positions of the structures shown above.

potential applications such as texture segmentation or image matching.

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