Riesz Transforms for the Isotropic Estimation of the Local Phase of Moiré Interferograms

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Abstract. The estimation of the local phase and local amplitude of 1-D signals can be realized by the construction of the analytic signal. This includes the evaluation of the signal's Hilbert transform, which performs a phase shift. In the past, different definitions of the analytic signal of multidimensional signals have been proposed, all of which are based on different combinations of partial and total Hilbert transforms. None of these approaches is isotropic. We propose the use of Riesz transforms which are known to mathematicians as appropriate generalizations of the Hilbert transform to n-D. This approach allows the isotropic estimation of the intrinsically 1-D local image phase. Applications to Moiré interferograms are shown.

1 Introduction

The estimation of the local phase and the local amplitude is an important step in many signal and image processing tasks. A second crucial task in image processing is the estimation of the local orientation. Usually these two tasks are treated separately. The methods used for the phase and amplitude estimation are based on the evaluation of the analytic signal of the input signal[1] which involves the calculation of the signal's Hilbert transform. In practical problems either the analytic signal itself is evaluated, or the analytic signal of a band-pass filtered version of the input signal is considered. The latter can be constructed by the application of quadrature filters or, approximately, by using Gabor filters.

2-D Gabor filters are now widely used in image processing. These filters are orientation selective and allow the estimation of the local phase provided that the local orientation is known or has been estimated in a previous processing step. Actually, 2-D Gabor filters, like their 1-D correspondents, rely on an analytic signal, the *partial analytic signal*, which is defined as the line-wise evaluation of the 1-D analytic signal wrt. a predefined orientation. This presents, besides others, one possible extension of the analytic signal to 2-D. Unfortunately, none of these extensions allows the estimation of a smooth phase map of images containing arbitrary orientations.

[1] We assume all signals to be real-valued.

In this article we propose to replace the Hilbert transform by the Riesz transform in 2-D (and generally in n-D). We will show how the analytic signal constructed using the Riesz transform allows the estimation of the local orientation, local phase and local amplitude at the same time. Evaluation of the local phase from the partial analytic signal leads to an undesirable effect. At the positions where the local orientation flips from $-\pi/2$ to $\pi/2$ the phase is inverted: $\phi(\boldsymbol{x}) \rightarrow -\phi(\boldsymbol{x})$. We call this effect *sudden phase inversion*. This is not to be confused with the 2π wrap arounds, which are typical for phase images. We will apply the Riesz transform to the phase estimation of Moiré interferograms and provide a way to obtain smooth phase maps without sudden phase inversions. This is possible if we use the additional orientation information given by the Riesz transform.

In Sect. 2 we first shortly recap the notions of the Hilbert transform and the analytic signal. Afterwards the Riesz transform is presented along with a way of combining it with the original signal to a kind of analytic signal. In Sect. 3 we deal with two technical aspects concerning the orientation map as provided by the Riesz transform. These aspects are crucial for the estimation of really smooth phase images without sudden inversions. Experimental results on Moiré interferograms are demonstrated in Sect. 4.

2 Isotropic Phase Estimation in 2-D

2.1 The Hilbert Transform and the Analytic Signal

The 1-D analytic signal is derived from the input signal by suppressing its negative frequency components, while multiplying the positive ones by two [5]. This transforms a real-valued signal f into a complex-valued signal f_A. The real part of f_A is identical to the input signal, while the imaginary part is a $(-\pi/2)$-phase-shifted version (or the *Hilbert transform* f_{Hi}) of f. In the frequency domain the Hilbert transform is defined by $F_{Hi}(u) = -iu/|u|F(u)$, where F and F_{Hi} are the Fourier transforms of f and f_{Hi}, respectively. The analytic signal can be written as $f_A(x) = |f_A(x)|\exp(i\phi(x))$. Here $|f_A(x)|$ is called the local amplitude and $\phi(x)$ the local phase of f. E.g. $f(x) = \cos(\omega x), \omega > 0$ yields $f_A(x) = \exp(i\omega x)$ and thus the local amplitude of f is $|f_A(x)| \equiv 1$ and the local phase is $\phi(x) = \omega x$.

Generalizations of the analytic signal to higher dimensions are based on the same construction principle: Instead of one half-axis in the frequency domain, one half-space which is chosen wrt. a reference orientation of the n-D frequency domain can be suppressed. The resulting complex-valued signal is the sum of the original signal and its *partial Hilbert transform* [6]. Evaluating the local phase from the partial Hilbert transforms directly leads to undesirable sudden phase inversions as shown in Fig. 5 (b). More recently, it has been suggested to construct an n-D extension of the analytic signal by keeping merely one orthant (in 2-D = quadrant) of the frequency domain and suppressing the rest. Depending on the type of Fourier transform used this leads either to a complex-valued signal [6] or a Clifford-algebra-valued signal [1, 2]. These approaches show advantages in the analysis of intrinsically multidimensional signal. However, they suffer from the fact that they do not yield isotropic results.

2.2 The Riesz Transform

As shown above, the 1-D Hilbert transform has the transfer function[2] $H(u) = -iu/|u|$. A surprisingly straightforward extension yields the transfer function $\boldsymbol{R}(\boldsymbol{u})$ of the n-D Riesz transform [10]: $\boldsymbol{R}(\boldsymbol{u}) = -i\boldsymbol{u}/|\boldsymbol{u}|$, with $\boldsymbol{u} = (u_1, \dots u_n)^T$. In this compact notation we combined the transfer functions of the n Riesz transforms $R_k(\boldsymbol{u}) = -iu_k/|\boldsymbol{u}|$ into the vector $\boldsymbol{R} = (R_1, \dots, R_n)^T$. In Riesz transform of a 2-D signal f is given by

$$\begin{pmatrix} f_{r1}(\boldsymbol{x}) \\ f_{r2}(\boldsymbol{x}) \end{pmatrix} =: \boldsymbol{f}_r(\boldsymbol{x}) \circ\!\!-\!\!\bullet\ \boldsymbol{R}(\boldsymbol{u})F(\boldsymbol{u}) = -i \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} F(\boldsymbol{u}), \tag{1}$$

where α is the angle between the $\boldsymbol{u}$ and the x-axis and F is the Fourier transform of f. A simple example reveals the key properties of the 2-D Riesz transform. Consider an arbitrarily oriented straight cosine-grating: $f(\boldsymbol{x}) = \cos(|\boldsymbol{u}_0|(x_1 \cos\beta + x_2 \sin\beta))$. The Riesz transform of f is then given by

$$\boldsymbol{f}_r(\boldsymbol{x}) = \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix} \sin(|\boldsymbol{u}_0|(x_1 \cos\beta + x_2 \sin\beta)). \tag{2}$$

Thus, the Riesz transform of f can be expressed as a unit vector multiplied by the intrinsically 1-D Hilbert transform of f. Furthermore, the unit vector points into the normal direction of the oriented structure!

We obtain a $\pi/2$-phase shifted version of the signal and the orientation information at the same time. Because of the linearity of the transform this applies as well if the image contains more than one frequency component. As long as the image is locally straight and coherent, we will find a $\pi/2$-phase shift normal to the local orientation.

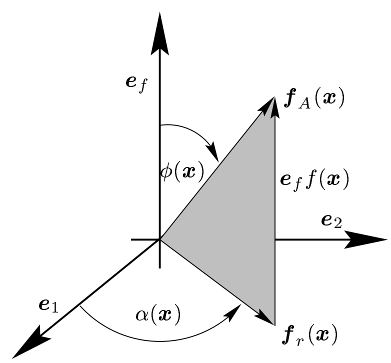


Fig. 1. The combination of the input signal f and its Riesz transform $\boldsymbol{f}_r$ into the vector-valued function $\boldsymbol{f}_A$.

As shown in the above example, $\boldsymbol{f}_r(\boldsymbol{x})$ contains the local orientation information and the 1-D Hilbert transform of the intrinsically 1-D structure normal to its orientation at the same time. However, the splitting of $\boldsymbol{f}_r(\boldsymbol{x})$ into the orientation component and the Hilbert transform component is not as straightforward as it may seem at the first sight. If we factorize the $\boldsymbol{f}_r$ as

$$\boldsymbol{f}_r(\boldsymbol{x}) = \hat{\boldsymbol{f}}_r(\boldsymbol{x})|\boldsymbol{f}_r(\boldsymbol{x})|, \tag{3}$$

where $\hat{\boldsymbol{f}}_r(\boldsymbol{x})$ is a unit vector representing the orientation and $|\boldsymbol{f}_r(\boldsymbol{x})|$ is supposed to be the Hilbert transform component, we do not get the desired result: In the above example the factorization (3) yields $|\sin(\dots)|$ as the Hilbert transform component of the cosine-function and not the sine-function as we should expect.

[2] We can say *transfer function of a transform* since the Hilbert and the Riesz transform are linear and shift-invariant and thus can be considered as LSI-filters.

Furthermore, according to (3) $\hat{\boldsymbol{f}}_r(\boldsymbol{x})$ is uniquely defined on the unit circle and thus represents orientations in the range $[-\pi, \pi[$ although the orientation of a straight structure is merely defined in the interval $[-\pi/2, \pi/2[$. Defining the local phase on this stage leads to

$$\tilde{\phi}(\boldsymbol{x}) = \text{atan2}(|\boldsymbol{f}_r(\boldsymbol{x})|, f(\boldsymbol{x})), \tag{4}$$

where atan2 is the sign dependent arc-tangent function with range $[-\pi, \pi[$. Figure 2 illustrates the results of (3) and (4). Obviously, Fig. 2 (c) does not represent the local

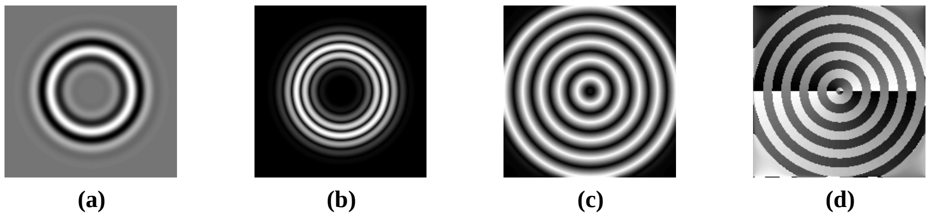


Fig. 2. Effects of the Riesz transform factorization according to Eqs. (3). **(a)** A synthetic test image. **(b)** The Hilbert transform component $\hat{\boldsymbol{f}}_r(\boldsymbol{x})$ according to (3). **(c)** The local phase according to (4). **(d)** The orientation vector $|\boldsymbol{f}_r(\boldsymbol{x})|$. The angular component of the vector (i.e. the orientation) is coded by gray values representing orientations between $-\pi$ (black) and π (white).

phase of Fig. 2 (a). Before we deal with this problem in the following section, we show how the Riesz transform and the original signal can be combined into a kind of multidimensional analytic signal.

The 1-D analytic signal is constructed as a combination of the input signal f and its Hilbert transform f_{Hi}. This combination is realized either as a vector-valued function $f\boldsymbol{e}_f + f_{Hi}\boldsymbol{e}_H$, where $\{\boldsymbol{e}_f, \boldsymbol{e}_H\}$ is an orthonormal basis of $\mathbb{R}^2$, or, more often, as a complex-valued function $f_A = f + i f_{Hi}$. Analogously, the Riesz transform can be combined with the original signal into a vector $\boldsymbol{f}_A = f\boldsymbol{e}_f + f_{r1}\boldsymbol{e}_1 + f_{r2}\boldsymbol{e}_2$, as visualized in Fig. 1, where $\{\boldsymbol{e}_f, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an orthonormal basis of $\mathbb{R}^3$. A combination into a quaternion-valued function is possible as well. This was first proposed by Nabighian [7] and more recently by Felsberg [4].

3 Improving the Orientation Map

There are two further effects that we should pay attention to. **(I)** As mentioned in Sect. 2 direct application of (3) does lead to the absolute value of the Hilbert transform component rather than to the Hilbert transform component itself. As a consequence we cannot use this definition for the extraction of the local phase. **(II)** The orientation vector is not well-defined if the magnitude of the Riesz transform is zero, i.e. $\boldsymbol{f}_A(\boldsymbol{x}) \parallel \boldsymbol{e}_f$ (see Fig. 1).

In order to avoid **(I)** we apply an unwrapping to the orientation image. The situation can be clarified as follows. The Riesz transform $\boldsymbol{f}_r(\boldsymbol{x})$ of f is uniquely defined. However, the factorization of $\boldsymbol{f}_r(\boldsymbol{x})$ into a scalar part and a vector of unit length is defined only up to a sign. Thus, at each position of the image, there are two possible definitions of the orientation vector and the scalar part. This corresponds to the fact, that the local orientation can only be known in an interval of length π. Problems occur at those positions where the orientation jumps from α to $\alpha \pm \pi$. At those positions the sign of the scalar component has to change in order to represent the same value $\boldsymbol{f}_r(\boldsymbol{x})$. Thus, the solution to our problem is to use the freedom in the factorization of $\boldsymbol{f}_r(\boldsymbol{x})$ in order to smooth out the π-jumps of the orientation. We call this *orientation unwrapping*. We use a simple procedure, processing the orientation-image line-wise. Each pixel is compared to its already processed predecessor. The decision on whether to flip the pixel by π or not is made such that the distance of the two pixels on the unit circle is minimized. If the orientation image is noisy it is more stable to compare each pixel to a whole neighborhood which votes whether to flip the pixel or not. This method has been used in the experiments in Sect. 4. The new orientation vector is denoted by $\boldsymbol{n}$.

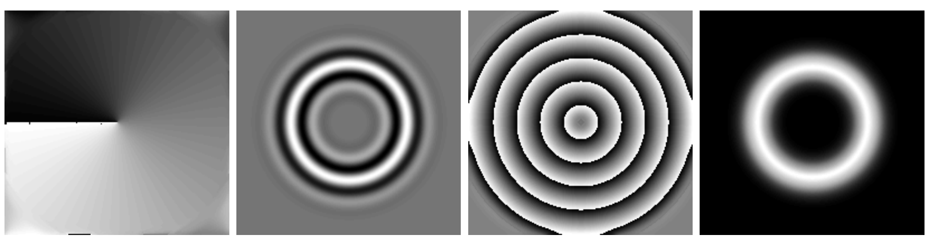
Each time an orientation values, during orientation unwrapping, is flipped by π, we have to replace $|\boldsymbol{f}_r(\boldsymbol{x})|$ by $-|\boldsymbol{f}_r(\boldsymbol{x})|$. The so modified scalar component of $\boldsymbol{f}_r$ will be called the Hilbert transform component of $\boldsymbol{f}_r$ and is denoted by f_H such that

$$\boldsymbol{f}_r(\boldsymbol{x}) = f_H(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}). \tag{5}$$

The local phase is now defined by

$$\phi(\boldsymbol{x}) := \mathrm{atan2}(f_H(\boldsymbol{x}), f(\boldsymbol{x})). \tag{6}$$

The results on our test image are shown in Fig. 3. In Fig. 3 (b) the Hilbert transform



(a) **(b)** **(c)** **(d)**

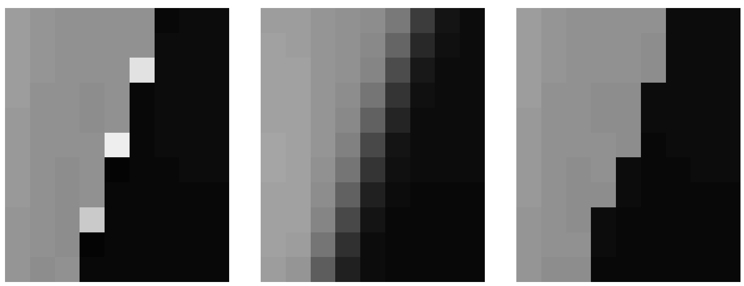
Fig. 3. Results according to the test image shown in Fig. 2 (a) **(a)** The unwrapped orientation image (black $= -\pi$, white $= \pi$). **(b)** The Hilbert transform component f_H of $\boldsymbol{f}_r$. **(c)** The phase image. **(d)** The local amplitude of f.

component f_H is shown. On the first glance this looks very similar to the original signal f. However, careful consideration shows that f_H is phase shifted by $\pi/2$, from the center of the circle outwards, against f. Fig. 3 (d) shows the root of the squared sum of f and f_H which is the local amplitude of f. This result also visualizes the isotropy of the Riesz transform.

Due to effect **(II)** there occur instabilities in the orientation image, that have to be smoothed out before orientation unwrapping is feasible. Felsberg [3] proposes to smooth the orientation image via a weighted averaging:

$$\alpha_{mn}^{s} = \frac{\sum\limits_{(i,j)\in\mathcal{N}(m,n)} \sin^2(\phi_{ij})\alpha_{ij}}{\sum\limits_{(i,j)\in\mathcal{N}(m,n)} \sin^2(\phi_{ij})}, \tag{7}$$

i.e. the more reliable an orientation value is, the stronger is its weight in the averaging. We modify this method in order to cure one problem: In (7) there occurs smoothing across π-jumps which leads to spurious orientation and furthermore makes orientation unwrapping infeasible. Thus, we propose *controlled smoothing*: Before averaging in a neighborhood $\mathcal{N}(m,n)$, we apply orientation unwrapping in $\mathcal{N}(m,n)$ with respect to the pixel $(i,j) \in \mathcal{N}(m,n)$ that maximizes $\sin^2(\phi_{ij})$. Furthermore, we avoid smoothing of reliable orientation values. The controlled averaging is only applied in those neighborhoods $\mathcal{N}(m,n)$ with $\sin^2(\phi_{mn}) < \tau$ for a certain threshold $\tau \in [0,1]$. See Fig. 4 for results.



(a) **(b)** **(c)**

Fig. 4. **(a)** Detail from the very left part of Fig. 2 (d). Inaccurate orientation values near the π-jump are visible. **(b)** The same detail of the smoothed orientation image (according to (7), neighborhood size was 5×5.) The unstable orientation estimates are smoothed out. However, the π-jump has been smoothed as well. **(c)** The same subregion, after controlled smoothing ($\tau = 0.04$, neighborhood size 5×5). The errors are smoothed out, while the π-jump is preserved.

4 Experiments

Moiré interferometry uses the interference of periodic patterns to measure the topology of a given surface. The principle of a projection Moiré interferometer is described in detail in [8]. A ruling is projected onto the surface under investigation. The image of the ruling is observed from a different direction and focused onto a second reference ruling, superimposing to a Moiré pattern. As an example, Fig. 5 shows the interferogram of a tilt plane plate and the results of our method. Lines of equal intensity represent lines of equal elevation. By determining the phase function of the pattern the surface of the object can be evaluated. So this technique can also be used to measure translations and deformations of the surface. We apply the methods developed in Sects. 2 and 3 in order to estimate the phase image ϕ. Fig. 6 shows an application of the Moiré interferometry:

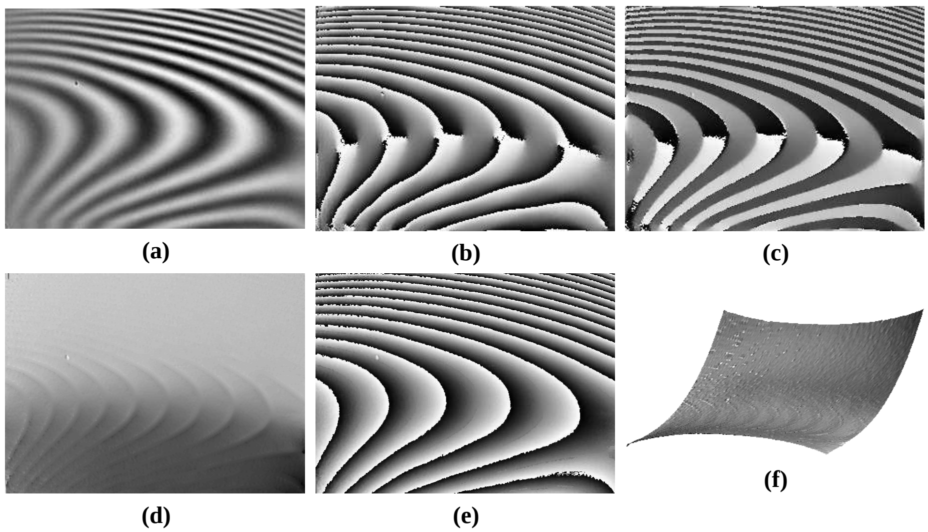


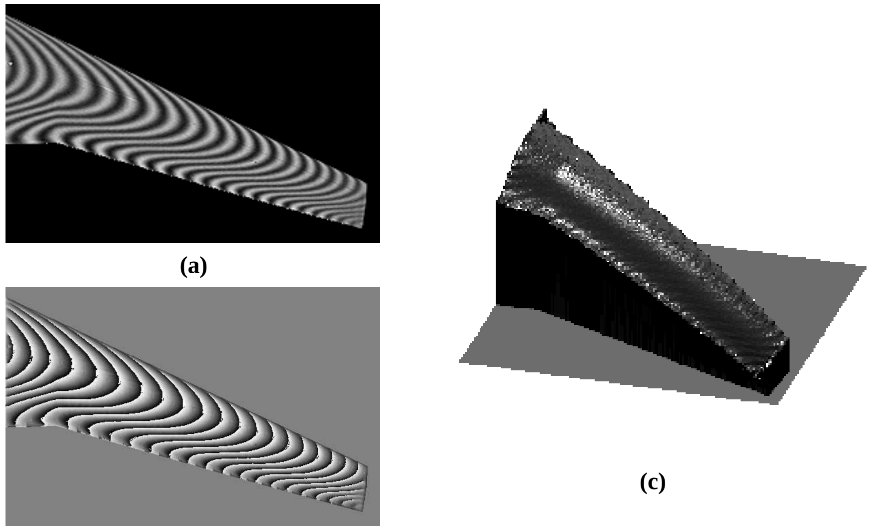
Fig. 5. Estimation of the local phase of a Moiré interferogram. **(a)** The interferogram. **(b)** The local phase wrt. the horizontal partial Hilbert transform. The typical phase inversions are visible. **(c)** The raw orientation image. **(d)** The smoothed and unwrapped orientation image. **(e)** The local phase image ϕ. **(f)** The unwrapped phase image as surface plot.

the instantaneous deformations and movements due to aerodynamic load of a model wing of a transport aircraft were measured in the cryogenic European Transonic Wind Tunnel (ETW) in Köln.

5 Conclusion

We proposed to replace the Hilbert transform by the Riesz transform in order to construct a multidimensional analytic signal for the use with intrinsically 1-D signal. This differs from the recently introduced Clifford-valued signal [2] which is constructed in order to divulge properties of intrinsically multidimensional signals. The Riesz transform is isotropic and yields orientation, phase and amplitude information at the same time. Its isotropy is based on the fundamental fact that the n Riesz transforms provide the basis functions of a steerable filter[3]. We presented experimental results in phase estimation from Moiré interferometry. However, the use of Riesz transforms is certainly not limited to this application and to 2-D signals. First applications in another area of image processing using local filters based on the Riesz transform can be found in [3].

[3] The details are outside the scope of this article. Compare e.g. [9].



(a) **(b)** **(c)**

Fig. 6. (a) The Moiré interferogram of the model wing. **(b)** The phase image according to (6). **(c)** The unwrapped phase image as surface plot. This corresponds to the profile of the model wing.

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