A Novel Approach to the 2D Analytic Signal*

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Abstract. The analytic signal of a real signal is a standard concept in 1D signal processing. However, the definition of an analytic signal for real 2D signals is not possible by a straightforward extension of the 1D definition. There rather occur several different approaches in the literature. We review the main approaches and propose a new definition which is based on the recently introduced quaternionic Fourier transform. The approach most closely related to ours is the one by Hahn [8], which defines the analytic signal, which he calls complex signal, to have a single quadrant spectrum. This approach suffers form the fact that the original real signal is not reconstructible from its complex signal. We show that this drawback is cured by replacing the complex frequency domain by the quaternionic frequency domain defined by the quaternionic Fourier transform. It is shown how the new definition comprises all the older ones. Experimental results demonstrate that the new definition of the analytic signal in 2D is superior to the older approaches.

1 Introduction

The notion of the analytic signal of a real one-dimensional signal was introduced in 1946 by Gabor [6]. It can be written as $f_A(x) = f(x) + i f_{\mathcal{H}i}(x)$, where f is the original signal and $f_{\mathcal{H}i}(x)$ is its Hilbert transform. Thus, the analytic signal is the generalization of the complex notation of harmonic signals given by Eulers equation $\exp(i\omega x) = \cos(\omega x) + i\sin(\omega x)$. The construction of the analytic signal can also be understood as suppressing the negative frequency components of f.

The analytic signal plays an important role in one-dimensional signal processing. One of the main reasons for this fact is, that the instantaneous amplitude and the instantaneous phase of a real signal f at a certain position x can be defined as the magnitude and the angular argument of the complex-valued analytic signal f_A at the position x. The analytic signal is a global concept, i.e. the analytic signal at a position x depends on the whole original signal and not only on values at positions near x.

Often local concepts are more reasonable in signal processing: They are of lower computational complexity than global concepts. Furthermore, it is reasonable that the local signal structure, like local phase and local amplitude should

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only depend on local neighborhoods. The "local version" of the analytic signal was also introduced by Gabor in [6]. This is derived from the original signal by applying Gabor filters which are bandpass filters with an approximately one-sided transfer function.

Complex Gabor filters are widely used in 1D signal-processing as well as in image-processing. However, their theoretical basis – the analytic signal – is only uniquely defined in 1D. There have been many attempts to generalize the notion of the analytic signal to higher dimensions. However, there is no unique, straightforward generalization but rather different ones with different advantages and disadvantages. We will propose a new definition of the analytic signal in 2D which is based on the recently introduced quaternionic Fourier transform (QFT) [2,3]. From this definition there follows a new kind of 2D Gabor filters (so-called quaternionic Gabor filters (QGF)) which already have found applications in texture segmentation and disparity estimation [1]. In the present article we restrict ourselves to the motivation, definition and analysis of the analytic signal.

The structure of this article is as follows: In Sect. 2 and Sect. 3 we give a short introduction to the one-dimensional analytic signal and to the main approaches towards a two-dimensional analytic signal, respectively. A short review of the QFT will be given in Sect. 4 followed by the new definition of the analytic signal in Sect. 5. In Sect. 6 we compare the different approaches to a two-dimensional analytic signal and present experimental results. Finally conclusions are drawn.

2 The 1D Analytic Signal

The analytic signal f_A can be derived from a real 1D signal f by taking the Fourier transform F of f, suppressing the negative frequencies and multiplying the positive frequencies by two. Applying this procedure, we do not lose any information about f because of the Hermite symmetry of the spectrum of a real function. The formal definition of the analytic signal is as follows:

Definition 1. *Let f be a real 1D signal. Its analytic signal is then given by*

$$f_A(x) = f(x) + i f_{\mathcal{H}i}(x) = f(x) * \left(\delta(x) + \frac{i}{\pi x} \right) \quad , \tag{1}$$

where the Hilbert transform of f is defined as $f_{\mathcal{H}i} = f * (1/(\pi x))$ and $*$ denotes the convolution operation.

In the frequency domain this definition reads:

$$\begin{aligned} F_A(u) &= F(u)(1 + \mathrm{sign}(u)) \quad \text{with} \quad \mathrm{sign}(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1 & \text{if } u < 0 \end{cases} \\ &= F(u) + i F_{\mathcal{H}i}(u) \quad . \end{aligned} \tag{2}$$

As an example we give the analytic signal of $f(x) = \cos(\omega x)$ which is $\cos_A(x) = \cos(x) + i \sin(x) = \exp(i\omega x)$. Thus, *cos* and *sin* constitute a *Hilbert pair*, i.e. one is the Hilbert transform of the other. The effect of the Hilbert transform is, that it shifts each frequency component of frequency $u = 1/\lambda$ by $\lambda/4$ to the right. We state three properties of the 1D analytic signal.

1. The spectrum of an analytic signal is causal ($F_A(u) = 0$ for $u < 0$).
2. The original signal is reconstructible from its analytic signal, particularly, the real part of the analytic signal is equal to the original signal.
3. The envelope of a real signal is given by the magnitude of its analytic signal which is called the *instantaneous amplitude* of f.

While the first property is a construction rule which has not necessarily to be extended to 2D, we expect an extension of the analytic signal to fulfill the last two properties: The second one guarantees that two different signals can never have the same analytic signal, while the third one is the property of the analytic signal which is mainly used in applications.

3 Approaches to an Analytic Signal in 2D

In this section we will mention some of the extensions of the analytic signal to two-dimensional signals which have occurred in the literature . All of these have a straightforward extension to n-dimensional signals. We will use the notation $\boldsymbol{x} = (x, y)$ and $\boldsymbol{u} = (u, v)$.

The first definition is based on the 2D Hilbert transform [9] which is given by

$$f_{\mathcal{H}i}(\boldsymbol{x}) = f(\boldsymbol{x}) ** \left(\frac{1}{\pi^2 xy}\right) \quad , \tag{3}$$

where $**$ denotes the 2D convolution. In analogy to 1D an extension of the analytic signal can be defined as follows:

Definition 2. *The analytic signal of a real 2D signal f is defined as*

$$f_A(\boldsymbol{x}) = f(\boldsymbol{x}) + i f_{\mathcal{H}i}(\boldsymbol{x}) \quad ,$$

where $f_{\mathcal{H}i}$ is given by (3).

In the frequency domain this definition reads

$$F_A(\boldsymbol{u}) = F(\boldsymbol{u})(1 - i\,\mathrm{sign}(u)\mathrm{sign}(v)) \quad .$$

The spectrum of f_A according to Def. 2 is shown in Fig. 1. It does not vanish anywhere such that property 1 from Sect. 2 is not satisfied by this definition. A common approach to overcome this fact can be found e.g. in [7]. This definition starts with the construction in the frequency domain. While in 1D the analytic signal is achieved by suppressing the negative frequencies, in 2D one half-plane of the frequency domain must be set to zero. It is not immediately clear how negative frequencies can be defined in 2D. However, it is possible to introduce a direction of reference defined by the unit vector $\hat{\boldsymbol{e}} = (\cos(\theta), \sin(\theta))$. A frequency $\boldsymbol{u}$ with $\hat{\boldsymbol{e}} \cdot \boldsymbol{u} > 0$ is called positive while a frequency with $\hat{\boldsymbol{e}} \cdot \boldsymbol{u} < 0$ is called negative. The 2D analytic signal can then be defined in the frequency domain.

Definition 3. *Let f be a real 2D signal and F its Fourier transform. The Fourier transform of the analytic signal is defined by:*

$$F_A(\boldsymbol{u}) = \left\{ \begin{array}{cc} 2F(\boldsymbol{u}) & \textit{if}\ \boldsymbol{u} \cdot \hat{e} > 0 \\ F(\boldsymbol{u}) & \textit{if}\ \boldsymbol{u} \cdot \hat{e} = 0 \\ 0 & \textit{if}\ \boldsymbol{u} \cdot \hat{e} < 0 \end{array} \right\} = F(\boldsymbol{u})(1 + \mathit{sign}(\boldsymbol{u} \cdot \hat{e})) \quad . \tag{4}$$

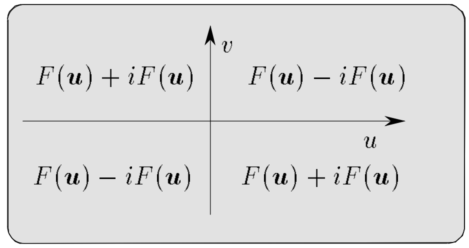

Fig. 1. The spectrum of the analytic signal according to Def. 2.

Please note the similarity of this definition with (2). In the spatial domain (4) reads

$$f_A(\boldsymbol{x}) = f(\boldsymbol{x}) ** \left(\delta(\boldsymbol{x} \cdot \hat{\boldsymbol{e}}) + \frac{i}{\pi \boldsymbol{x} \cdot \hat{\boldsymbol{e}}}\right) \delta(\boldsymbol{x} \cdot \hat{\boldsymbol{e}}_\perp) \ . \tag{5}$$

The vector $\hat{\boldsymbol{e}}_\perp$ is a unit vector which is orthogonal to $\hat{\boldsymbol{e}}$: $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}}_\perp = 0$.

According to this definition the analytic signal is calculated line-wise along the direction of reference. The lines are processed independently. Hence, Def. 3 is intrinsically 1D, such that it is no satisfactory extension of the analytic signal to 2D. Another definition of the 2D analytic signal was introduced by Hahn [8][1].

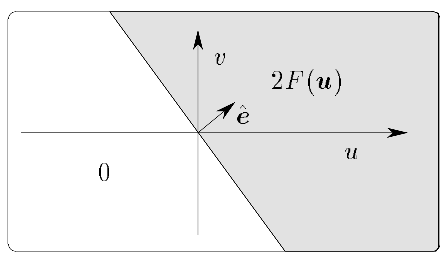

Fig. 2. The spectrum of the analytic signal according to Def. 3.

Definition 4. *The 2D analytic signal is defined by*

$$f_A(\boldsymbol{x}) = f(\boldsymbol{x}) ** \left(\delta(x) + \frac{i}{\pi x}\right)\left(\delta(y) + \frac{i}{\pi y}\right) \tag{6}$$

$$= f(\boldsymbol{x}) - f_{\mathcal{H}i}(\boldsymbol{x}) + i(f_{\mathcal{H}i_1}(\boldsymbol{x}) + f_{\mathcal{H}i_2}(\boldsymbol{x})) \ , \tag{7}$$

where $f_{\mathcal{H}i}$ is the Hilbert transform according to Def. 2 and $f_{\mathcal{H}i_1}$ and $f_{\mathcal{H}i_2}$ are the so called partial Hilbert transforms, which are the Hilbert transforms according to Def. 3 with $\hat{\mathbf{e}}^\top = (1,0)$ and $\hat{\mathbf{e}}^\top = (0,1)$, respectively:

$$f_{\mathcal{H}i} = f ** \frac{1}{\pi^2 xy}, \quad f_{\mathcal{H}i_1} = f ** \frac{\delta(y)}{\pi x} \quad \text{and} \quad f_{\mathcal{H}i_2} = f ** \frac{\delta(x)}{\pi y} \ . \tag{8}$$

[1] Hahn avoids the term "analytic signal" and uses "complex signal" instead.

The meaning of Def. 4 becomes clearer in the frequency domain: Only the frequency components with $u > 0$ and $v > 0$ are kept, while the components in the three other quadrants are suppressed (see Fig. 3):

$$F_A(\boldsymbol{u}) = F(\boldsymbol{u})(1 + \mathrm{sign}(u))(1 + \mathrm{sign}(v)) \ .$$

A main problem of Def. 4 is the fact that the original signal is not reconstructible

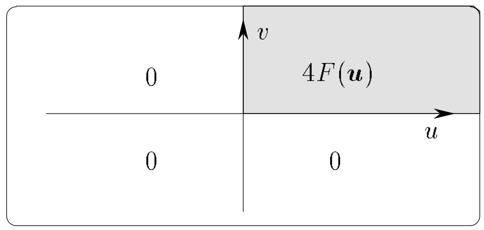

Fig. 3. The spectrum of the analytic signal according to Hahn [8] (Definition 4).

from the analytic signal, since due to the Hermite symmetry only one half-plane of the frequency representation of a real signal is redundant. For this reason Hahn proposes to calculate not only the analytic signal with the spectrum in the upper right quadrant but also another analytic signal with its spectrum in the upper left quadrant. It can be shown that these two analytic signals together contain all the information of the original signal. Thus, the complete analytic signal consists of two real parts and two imaginary parts or, in polar representation, of two amplitude- and two phase-components which makes the interpretation, especially of the amplitude, difficult.

4 The Quaternionic Fourier Transform

Since our definition of the analytic signal is based on the quaternionic Fourier transform (QFT), we will briefly review this transform. The QFT was recently introduced in [2, 3, 1] and [5], independently. The QFT of a 2D signal $f(\boldsymbol{x})$ is defined as

$$F^q(\boldsymbol{u}) = \int_{\perp\infty}^{\infty} \int_{\perp\infty}^{\infty} e^{\perp i 2\pi u x} f(\boldsymbol{x}) e^{\perp j 2\pi v y} d^2\boldsymbol{x} \ , \tag{9}$$

where i and j are elements of the algebra of quaternions $\mathbb{H} = \{q = a + bi + cj + dk \,|\, a, b, c, d \in \mathbb{R}, i^2 = j^2 = -1, ij = -ji = k\}$. Note that the quaternionic multiplication is not commutative. The magnitude of a quaternion $q = a + bi + cj + dk$ is defined as $|q| = \sqrt{qq^*}$ where $q^* = a - bi - cj - dk$ is called the conjugate of q.

The 1D Fourier transform separates the symmetric and the antisymmetric part of a real signal by transforming them into a real and an imaginary part, respectively. In real 2D a signal splits into four symmetry parts (symmetric and antisymmetric with respect to each argument). These four symmetry components are decoupled by the QFT and mapped to the four algebraic components of the quaternions [3].

The phase concept can be generalized using the QFT: In 1D the phase of a frequency component is represented by one real number. In the 2D quaternionic frequency domain a triple of real numbers can be defined, which can be regarded as the generalized phase in 2D [4, 1].

The operation of conjugation in $\mathbb{C}$ is a so-called algebra involution, i.e. it fulfills the two following properties: Let $z, w \in \mathbb{C} \Rightarrow (z^*)^* = z$ and $(wz)^* = w^* z^*$. In $\mathbb{H}$ there are three nontrivial algebra involutions:

$$\begin{aligned} \alpha &: q \mapsto -iqi, & \alpha(q) &= a + bi - cj - dk, \\ \beta &: q \mapsto -jqj, & \beta(q) &= a - bi + cj - dk \quad \text{and} \\ \gamma &: q \mapsto -kqk, & \gamma(q) &= a - bi - cj + dk \ . \end{aligned}$$

Using these involutions we can extend the definition of Hermite symmetry: A function $f : \mathbb{R}^2 \to \mathbb{H}$ is called quaternionic Hermitian if:

$$f(-x, y) = \beta(f(x, y)) \quad \text{and} \quad f(x, -y) = \alpha(f(x, y)) \ , \tag{10}$$

for each $(x, y) \in \mathbb{R}^2$. *The QFT of a real 2D signal is quaternionic Hermitian!*

5 The Quaternionic 2D Analytic Signal

Using the QFT we can follow the arguments of Hahn [8] and keep only one of the four quadrants of the frequency domain. Since the QFT of a real signal is quaternionic Hermitian (see Sect. 4) we do not lose any information about the signal in this case (see Fig. 4).

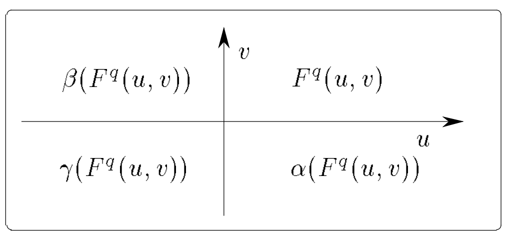

Fig. 4. The quaternionic spectrum of a real signal can be reconstructed from only one quadrant.

Definition 5. *In the frequency domain we define the quaternionic analytic signal of a real signal as*

$$F_A^q(\boldsymbol{u}) = (1 + sign(u))(1 + sign(v)) F^q(\boldsymbol{u}) \ ,$$

where F^q is the QFT of the real two-dimensional signal $f(\boldsymbol{x})$ and F_A^q denotes the QFT of the quaternionic analytic signal of f.

Definition 5 can be expressed in the spatial domain as follows:

$$f_A^q(\boldsymbol{x}) = f(\boldsymbol{x}) + \boldsymbol{n} \cdot \boldsymbol{f}_{\mathcal{H}i}(\boldsymbol{x}) \ , \tag{11}$$

where $\boldsymbol{n} = (i, j, k)^\top$ and $\boldsymbol{f}_{\mathcal{H}i}$ is a vector which consists of the partial and the total Hilbert transforms of f according to (8):

$$\boldsymbol{f}_{\mathcal{H}i}(\boldsymbol{x}) = \left(f_{\mathcal{H}i_1}, \ f_{\mathcal{H}i_2}, \ f_{\mathcal{H}i}\right)^\top \ . \tag{12}$$

Note that, formally, (11) resembles (1).

6 Comparison of the Different Approaches

The 2D analytic signal according to all approaches is most easily constructed in the frequency domain. While the first approach Def. 2 does not suppress any parts of the frequency domain, all other approaches do have this analogy to the 1D case. According to Def. 3 one half of the frequency domain is suppressed. According to Def. 4 and Def. 5 only one quarter of the frequency domain is kept while the rest is suppressed. The difference between the last two definitions is that Def. 4 uses the complex frequency domain while Def. 5 uses the quaternion-valued frequency domain of the QFT.

The requirement that the original signal be reconstructible from the analytic signal is fulfilled is by all approaches except for Def. 4. The main requirement is that the magnitude of the analytic signal (the instantaneous amplitude of the original signal) should be the envelope of the original oscillating signal. We demonstrate the instantaneous amplitude according to the four definitions of the 2D analytic signal on an example image containing oscillations in all directions of the image plane. In Fig. 5 we show the magnitudes of the different analytic

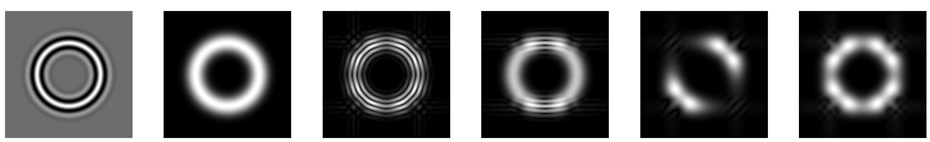

Fig. 5. The instantaneous amplitude according to the different definitions of the analytic signal. From left to right: Original image, real envelope, inst. amplitude according to Def. 2, Def. 3 (oriented along the x-axis), Def. 4, and Def. 5, respectively.

signals of a test-images containing oscillations in all orientations of the image plane. Definition 3 is applied with $\hat{\boldsymbol{e}} = (1, 0)$. Obviously Def. 2 is not successful in providing the envelope of the test-image. The quality of Def. 3 depends strongly on the orientations of the local structure. If it is orthogonal to the chosen $\hat{\boldsymbol{e}}$ the envelope is constructed well, while the Def. 3 fails as soon as the local structure is parallel to $\hat{\boldsymbol{e}}$. Definition 4 looses information about the signal and so yields the envelope only for structures that correspond to frequencies in the upper right quadrant. Also Def. 5 does not yield the perfect envelope. However, compared to the other approaches this one is the most satisfactory.

7 Conclusion

The quaternionic analytic signal combines in itself the earlier approaches in a natural way: The vector $\boldsymbol{f}_{\mathcal{H}i}$ in (12) is not constructed artificially. The quaternionic analytic signal is rather defined in a simple way in the frequency domain and everything falls in place automatically. In our opinion the successful definition of the analytic signal in 2D, which can only be obtained using the QFT, shows also that the QFT is a reasonable definition of a 2D harmonic transform.

As mentioned in the introduction the analytic signal is the theoretical foundation of Gabor filter techniques. These techniques are widely used in image processing. Based on the quaternionic analytic signal introduced here it is possible to define quaternion-valued Gabor filters instead of complex-valued ones. These filters have already been successfully applied in image processing. Based on quaternion-valued image representations the concept of the local phase has been extended and used for local structure analysis as well as for disparity estimation [4, 1].

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