Multi–Dimensional Signal Processing Using an Algebraically Extended Signal Representation*

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Abstract. Many concepts that are used in multi–dimensional signal processing are derived from one–dimensional signal processing. As a consequence, they are only suited to multi–dimensional signals which are intrinsically one–dimensional. We claim that this restriction is due to the restricted algebraic frame used in signal processing, especially to the use of the complex numbers in the frequency domain. We propose a generalization of the two–dimensional Fourier transform which yields a quaternionic signal representation. We call this transform quaternionic Fourier transform (QFT). Based on the QFT, we generalize the conceptions of the analytic signal, Gabor filters, instantaneous and local phase to two dimensions in a novel way which is intrinsically two–dimensional. Experimental results are presented.

1 Introduction

Realizations of autonomous technical systems which are designed on principles of the perception–action–cycle (PAC) are supposed to act in the real world which can be described as taking place in a four–dimensional Euclidean space–time. Therefore, a PAC–system has to be able to percept events and to organize processes in such a world.

Focusing on the perceptional part of a PAC system we face some serious shortcomings in low–level processing of multi–dimensional signals. These shortcomings have been recognized for a long time but by now are not solved satisfactorily. In the authors opinion the root of the problems seems to lie in the restricted algebraical embedding of multi–dimensional signal–processing which has not yet been recognized. The algebraical embedding of signal processing is meant to be the choice of algebra in which a signal is represented in the frequency domain. Usually this role is played by the algebra of complex numbers but we will show that it is useful to apply algebras of higher dimensions here. Most of the valuable tools that have been developed in one–dimensional signal processing are nowadays used in multi–dimensional signal processing but in a way which leaves them intrinsically one–dimensional.

* This work was supported by the Studienstiftung des deutschen Volkes (Th.B.) and by the DFG (So-320-2-1) (G.S.).

One example for this is the concept of local phase. The local phase of a one–dimensional signal can be estimated by applying a quadrature filter — e.g. a complex Gabor filter — and evaluating the argument of the resulting complex filter response. This concept is usually generalized to two dimensions by defining the two–dimensional Gabor filters as the Gaussian windowed basis function of the Fourier transform for some frequency $\boldsymbol{u}$. Again, we get as the local phase the argument of the complex filter response and we get different values for different orientations of the Gabor filter. Granlund [7] defines an $(n+1)$–dimensional phase vector for n–dimensional signals, consisting of the real phase and the directional vector of the chosen orientation.

In one dimension the local phase yields information about the local structure of the signal. In two dimensions the variety of possible local structures is much higher than in one dimension and so we cannot hope to characterize the local image structure using only one real number. Looking for a concept which yields a higher–dimensional value for the local phase we find that the main restriction of the phase dimension lies in the fact that the responses of the Gabor filters are complex–valued. Thus, we will study filters with responses which are elements of a higher–dimensional algebra than the complex numbers. We will show that a generalization to quaternion–valued filters is possible in two dimensions. A short review on quaternions will be given in the following section.

One–dimensional Gabor filters are based on the Fourier transform. Therefore we will extend the two–dimensional Fourier transform in such a way that it yields a quaternion–valued representation in the frequency domain. We call this transform quaternionic Fourier transform (QFT). We will demonstrate the shift theorem in the case of the QFT, analyze the symmetry properties of the QFT and show its relation to the Fourier transform and to the Hartley transform.

In order to define the instantaneous phase of a two–dimensional signal we will introduce the quaternionic analytic signal of a two–dimensional signal via the QFT. Finally we introduce the quaternionic Gabor filters based on the QFT and the two–dimensional local phase and demonstrate some experimental results.

2 Quaternions

As motivated in the introduction we need as the range of a generalized two–dimensional Fourier transform an algebra whose dimension is higher than the dimension of the algebra of complex numbers. In the following we will use the four–dimensional $\mathbb{R}$–algebra

$$\mathbb{H} = \{q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\} \quad , \tag{1}$$

where i, j and k obey the following multiplication rules:

$$i^2 = j^2 = -1, \quad k = ij = -ji \quad \Longrightarrow \quad k^2 = -1 \quad . \tag{2}$$

The algebra $\mathbb{H}$ was invented in 1843 by Hamilton[2] [8] who called it the *algebra of quaternions*. There is a whole lot of literature on quaternions (see e.g. [9, 10]). For the sake of brevity we will only introduce the properties which will be needed in the course of this article.

For a quaternion $q = a + bi + cj + dk$ the component a is called the *scalar part* of q, whereas $bi + cj + dk$ is called the *vector part* of q. A quaternion consisting only of a vector part is called a *pure quaternion*. Like in the algebra of complex numbers we can define the operation of conjugation for quaternions. The conjugate of a quaternion $q = a + bi + cj + dk$, denoted by q^*, is defined by changing the sign of the vector part of q:

$$q^* = a - bi - cj - dk \quad . \tag{3}$$

The operation of conjugation is a vector space involution. The *magnitude* of q is defined as

$$|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2} \quad . \tag{4}$$

By ϵ, α, β and γ we denote the four algebra involutions of $\mathbb{H}$. They are given by

$$\begin{array}{ll} \epsilon : q \mapsto q, & \epsilon(q) = a + bi + cj + dk, \\ \alpha : q \mapsto -iqi, & \alpha(q) = a + bi - cj - dk, \\ \beta : q \mapsto -jqj, & \beta(q) = a - bi + cj - dk, \\ \gamma : q \mapsto -kqk, & \gamma(q) = a - bi - cj + dk. \end{array}$$

In analogy to a Hermitian function $f : \mathbb{R} \to \mathbb{C}$ with $f(x) = f^*(-x)$ for every $x \in \mathbb{R}$ we introduce the notion of a *quaternionic Hermitian function* for a function $f : \mathbb{R}^2 \to \mathbb{H}$ which obeys the rules

$$f(-x, y) = \beta(f(x, y)) \quad \text{and} \quad f(x, -y) = \alpha(f(x, y)) \quad , \tag{5}$$

for each $(x, y) \in \mathbb{R}^2$. For a quaternionic Hermitian function also the relation

$$f(-x, -y) = \gamma(f(x, y)) \tag{6}$$

holds true.

We will need the exponential function $exp : \mathbb{H} \to \mathbb{H}$ of a quaternion q which is defined via the series

$$\exp(q) = \sum_{k=0}^{\infty} \frac{q^k}{k!}, \quad q \in \mathbb{H} \quad . \tag{7}$$

It can be shown that this sum converges for every quaternion q. Let us write the quaternion q in the form $q = s + \boldsymbol{v}$, where s and $\boldsymbol{v}$ denote the scalar part and

[2] Blaschke [3] states that they were already known to Euler — who used them to describe rotations in $\mathbb{R}^3$ — in 1748.

the vector part of q, respectively. We can then evaluate $exp(q)$ in the following way:

$$\exp(q) = \exp(s + \boldsymbol{v}) = \exp(s)\left(\cos(|\boldsymbol{v}|) + \frac{\boldsymbol{v}}{|\boldsymbol{v}|}\sin(|\boldsymbol{v}|)\right) \quad . \tag{8}$$

In the last step we used the fact that the Euler formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \tag{9}$$

is not only valid if i is the imaginary unit of the complex numbers but also in the form

$$e^{r\psi} = \cos(\psi) + r\sin(\psi) \quad ,$$

where r is an arbitrary pure unit quaternion[3].

3 Quaternionic Fourier Transform

Here we want to give a review of the recently introduced quaternionic Fourier transform (QFT) [5]. The QFT is a transform for two–dimensional signals which on the first glance seems to be only a slight modification of the well–known two–dimensional Fourier transform. For detailed information on Fourier transform see e.g. [4]. The two–dimensional Fourier transform is given by

$$F(\boldsymbol{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi ux} f(\boldsymbol{x}) e^{-i2\pi vy} d^2\boldsymbol{x} \quad , \tag{10}$$

whereas the quaternionic Fourier transform is defined as

$$F^q(\boldsymbol{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi ux} f(\boldsymbol{x}) e^{-j2\pi vy} d^2\boldsymbol{x} \quad , \tag{11}$$

with the only difference of using two different imaginary units in the exponential functions. Here $\boldsymbol{x}$ denotes the vector (x, y) in the image plane and $\boldsymbol{u}$ denotes the two–dimensional frequency vector (u, v). The units i and j are supposed to be two of the imaginary units of the quaternion algebra defined above. This leads to a significant difference between the two–dimensional Fourier transform and the QFT. Let $\mathcal{F}$ and $\mathcal{F}_q$ denote the operators of the Fourier transform and the QFT, respectively. For a two–dimensional signal $f : \mathbb{R}^2 \to \mathbb{R}$ the Fourier transform $F = \mathcal{F}\{f\}$ maps the image signal to a complex–valued representation whereas the QFT $F^q = \mathcal{F}_q\{f\}$ maps the image signal onto a quaternion–valued spectral representation.

[3] For the proof of the Euler formula we only need the definition of the exponential function, the cosine and the sine function as series and the algebraic properties of i, i.e. $i^2 = -1$. Hence, to proof the Euler formula for quaternions we must only show that $r^2 = -1$ which is straightforward.

Also Chernov [6] used quaternions in the Fourier transform. However, his aim was to find fast algorithms for the evaluation of the two–dimensional discrete Fourier transform, while we are interested in constructing a more complex phase representation.

The QFT of a real signal is a quaternionic Hermitian function as defined in the previous section. The inverse of the QFT is given by

$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi ux} F^q(\boldsymbol{u}) e^{j2\pi vy} d^2\boldsymbol{u} \quad . \tag{12}$$

In order to translate some of the properties of the Fourier transform to the QFT, we will consider the shift–theorem here. Let $F^q(\boldsymbol{u})$ and $F^q_T(\boldsymbol{u})$ be the QFT's of a real signal $f(\boldsymbol{x})$ and the translated signal $f_T(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{d})$. It follows easily that

$$F^q_T(\boldsymbol{u}) = e^{-i2\pi u d_1} F^q(\boldsymbol{u}) e^{-j2\pi v d_2} \quad . \tag{13}$$

In the following we will write this down in matrix–notation representing the quaternion $F^q(\boldsymbol{u}) = F^q_0(\boldsymbol{u}) + iF^q_1(\boldsymbol{u}) + jF^q_2(\boldsymbol{u}) + kF^q_3(\boldsymbol{u})$ as the vector $\mathbf{F}^q(\boldsymbol{u}) = (F^q_0(\boldsymbol{u}), F^q_1(\boldsymbol{u}), F^q_2(\boldsymbol{u}), F^q_3(\boldsymbol{u}))^T$. Eq. (13) then reads $\mathbf{F}^q_T(\boldsymbol{u}) = T(\phi,\theta)\mathbf{F}^q(\boldsymbol{u})$, with

$$T(\phi,\theta) = \begin{pmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) & -\cos(\phi)\sin(\theta) & -\sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & -\cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) \end{pmatrix} . \tag{14}$$

Here ϕ and θ denote $-2\pi d_1 u$ and $-2\pi d_2 v$, respectively. Now we can show explicitly how the translation vector $\boldsymbol{d}$ can be recovered from a pair $\mathbf{F}^q(\boldsymbol{u})$ and $\mathbf{F}^q_T(\boldsymbol{u})$ for a single value of $\boldsymbol{u}$. It is straightforward to show that

$$\mathbf{F}^q_T(\boldsymbol{u}) = \begin{pmatrix} F^q_0(\boldsymbol{u}) & -F^q_1(\boldsymbol{u}) & -F^q_2(\boldsymbol{u}) & F^q_3(\boldsymbol{u}) \\ F^q_1(\boldsymbol{u}) & F^q_0(\boldsymbol{u}) & -F^q_3(\boldsymbol{u}) & -F^q_2(\boldsymbol{u}) \\ F^q_2(\boldsymbol{u}) & -F^q_3(\boldsymbol{u}) & F^q_0(\boldsymbol{u}) & -F^q_1(\boldsymbol{u}) \\ F^q_3(\boldsymbol{u}) & F^q_2(\boldsymbol{u}) & F^q_1(\boldsymbol{u}) & F^q_0(\boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\sin\theta \end{pmatrix} =: \boldsymbol{F}(\boldsymbol{u}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

If $\boldsymbol{F}(\boldsymbol{u})$ is invertible we get α, β, γ and δ directly from

$$\boldsymbol{F}^{-1}(\boldsymbol{u})\mathbf{F}^q_T(\boldsymbol{u}) = (\alpha,\beta,\gamma,\delta)^T \quad . \tag{15}$$

It is possible to recover (ϕ,θ) within the interval $[0, 2\pi[\times [0, \pi[$ from $(\alpha,\beta,\gamma,\delta)$ by a function $arg : \mathbb{H}\backslash\{0\} \mapsto \mathbb{R}^2$, $arg(\alpha,\beta,\gamma,\delta) = (\phi,\theta)$. Because of the bulkiness of the definition of arg we will give it in the appendix A. From ϕ and θ we get d_1 and d_2 by

$$d_1 = \frac{\phi}{2\pi u}, \quad d_2 = \frac{\theta}{2\pi v} \quad . \tag{16}$$

Thus, we recovered the translation vector $\boldsymbol{d}$ from the QFT's of f and f_T for one specific value of $\boldsymbol{u}$.

4 Symmetries of the QFT

The concept of symmetry of a signal is well known in one dimension. Globally as well as locally signals can be split into an even and an odd component. While the global symmetry is not an inherent signal property but depends on the choice of the origin, the local symmetry describes the local structure of the signal; it is even for a peak–like structure and odd for step–like structure in the signal.

In this section we will show how the QFT deals with signals of different combinations of even and odd symmetries. It is a well known fact that the Fourier transform of the even part of a real one–dimensional signal is real and even. The Fourier transform of the odd part of the same signal is imaginary and odd. As said above this splitting depends on the choice of the origin. It is independent of scaling, though.

A two–dimensional signal can be split into even and odd parts along the x–axis and along the y–axis as well. So, every real two–dimensional signal can be written in the form $f = f_{ee} + f_{oe} + f_{eo} + f_{oo}$, where f_{ee} denotes the part of f which is even with respect to x and y, f_{oe} denotes the part which is odd with respect to x and even with respect to y and so on. In this case the splitting is not only dependent of the choice of the origin but also of the orientation of the image. Because the two–dimensional Fourier transform has only two components — one real and one imaginary component — we are not able to immediately recognize the four components of different symmetry.

However, the quaternionic Fourier transform has symmetry–splitting properties that are analogous to the properties of the one–dimensional Fourier transform: The transform of the f_{ee}–part of a real two–dimensional signal is real, the f_{oe}–part is transformed into a i–imaginary part, f_{eo} into the j–imaginary and f_{oo} into the k–imaginary part. The symmetry of the signal is preserved by the quaternionic Fourier transform. We can see this easily by looking at the quaternionic Fourier transform as two sequentially performed one–dimensional Fourier transforms: First we perform a one–dimensional Fourier transform on $f(\boldsymbol{x})$ with respect to x keeping y fixed and call the result $\tilde{f}$:

$$\tilde{f}(u, y) = \int_{-\infty}^{\infty} e^{-i2\pi ux} f(x, y) dx \quad . \tag{17}$$

In a second step we perform a Fourier transform on $\tilde{f}$ with respect to y keeping u fixed:

$$F^q(\boldsymbol{u}) = \int_{-\infty}^{\infty} \tilde{f}(u, y) e^{-j2\pi vy} dy \quad . \tag{18}$$

Actually, this two step procedure is the way we implemented the QFT on the computer. Hence, the implementation is similar to the one of the two–dimensional Fourier transform. The difference is that while calculating the two–dimensional Fourier transform we add up some components which we keep separately when calculating the QFT. An overview over the symmetry properties of the QFT is given in table 1.

	$f = f_{ee}$				$f = f_{oe}$				$f = f_{eo}$				$f = f_{oo}$			
f	r	i	j	k	r	i	j	k	r	i	j	k	r	i	j	k
$\tilde{f}$	r	i	j	k	i	r	k	j	r	i	j	k	i	r	k	j
F^q	r	i	j	k	i	r	k	j	j	k	r	i	k	j	i	r

Table 1. Symmetry properties of the QFT. r stands for the real part, i for the i–imaginary and so on.

In order to clarify the position of the QFT among the existing transforms, we relate the QFT to the Fourier transform and to the Hartley transform (see e.g. [4]):

The Hartley transform of a one–dimensional signal f is defined by

$$H(u) = \int_{-\infty}^{\infty} f(x)\{\cos(2\pi ux) + \sin(2\pi ux)\}dx \quad . \tag{19}$$

It is related to the Fourier transform of f by

$$F(u) = H_e(u) - iH_o(u) \quad , \tag{20}$$

where H_e and H_o denote the even and odd part of H respectively. So the Fourier transform separates the parts of different symmetry — which are mixed in the Hartley transform — by putting them into different components.

The two–dimensional Hartley transform is given by

$$\begin{aligned} H(\boldsymbol{u}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x})\{\cos(2\pi\boldsymbol{u}\cdot\boldsymbol{x}) + \sin(2\pi\boldsymbol{u}\cdot\boldsymbol{x})\}d^2\boldsymbol{x} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x})\,\{\cos(2\pi ux)\cos(2\pi vy) - \sin(2\pi ux)\sin(2\pi vy) \\ &\qquad + \cos(2\pi ux)\sin(2\pi vy) + \sin(2\pi ux)\cos(2\pi vy)\}\,d^2\boldsymbol{x} \\ &= H_{ee}(\boldsymbol{u}) + H_{oo}(\boldsymbol{u}) + H_{eo}(\boldsymbol{u}) + H_{oe}(\boldsymbol{u}) \quad . \end{aligned} \tag{21}$$

Again it is possible to get the Fourier transform of f from the Hartley transform:

$$F(\boldsymbol{u}) = \big(H_{ee}(\boldsymbol{u}) + H_{oo}(\boldsymbol{u})\big) - i\big(H_{eo}(\boldsymbol{u}) + H_{oe}(\boldsymbol{u})\big) \quad . \tag{22}$$

Hence, also in this case the Fourier transform separates parts of different symmetry, which are mixed in the Hartley transform, but this separation is only halfway. The complete separation is only given by the quaternionic Fourier transform:

$$F^q(\boldsymbol{u}) = H_{ee}(\boldsymbol{u}) - kH_{oo}(\boldsymbol{u}) - jH_{eo}(\boldsymbol{u}) - iH_{oe}(\boldsymbol{u}) \; . \tag{23}$$

It follows from this that the two–dimensional Fourier transform stands between the QFT and the two–dimensional Hartley transform in the sense that we can derive the QFT from the two–dimensional Fourier transform in a similar way as we derive the Fourier transform from the Hartley transform.

As shown in section 2 we can represent the QFT of a signal also in polar representation. Thus, we can write $F^q(\boldsymbol{u})$ in the form

$$F^q(\boldsymbol{u}) = |F^q(\boldsymbol{u})| \exp(s(\boldsymbol{u})\psi(\boldsymbol{u})) \tag{24}$$

with

$$s(\boldsymbol{u}) = i\cos(\phi(\boldsymbol{u}))\sin(\theta(\boldsymbol{u})) + j\sin(\phi(\boldsymbol{u}))\sin(\theta(\boldsymbol{u})) + k\cos(\theta(\boldsymbol{u})) \tag{25}$$

The three angles ψ, ϕ and θ could be regarded as the phase of a two–dimensional signal. Nevertheless, the phase (ψ, ϕ, θ) for special values of the angles is an element of the three–dimensional hypersphere S^3 which makes an interpretation of the values complicated. Another approach to a multidimensional phase concept will be represented in the next section.

5 The Analytic Signal

The analytic signal plays an important role in one–dimensional signal processing. One of the main reasons for this fact is, that it is possible to read the instantaneous amplitude and the instantaneous phase from a signal f at a certain position x simply by taking the magnitude and the phase of the analytic signal f_A at the position x, where $f_A(x)$ is a complex number. The analytic signal f_A of a real signal f is defined as $f_A = f - i\mathcal{H}\{f\}$ where $\mathcal{H}\{f\}$ is the Hilbert transform of f. It can be derived from f by taking the Fourier transform F of f, suppressing the negative frequencies and multiplying the positive frequencies by two. Applying this procedure, we do not lose any information about f.

One way to extend the concept of the analytic signal to two dimensions is to split the frequency plane into two half planes with respect to a direction $\boldsymbol{e} = (\cos(\theta), \sin(\theta))$. A frequency $\boldsymbol{u} = (u, v)$ with $\boldsymbol{e} \cdot \boldsymbol{u} > 0$ is called positive while a frequency $\boldsymbol{u} = (u, v)$ with $\boldsymbol{e} \cdot \boldsymbol{u} < 0$ is called negative. With this definition the one–dimensional construction rule for the analytic signal can be applied to two–dimensional signals [7]. Using this construction, the conception of the analytic signal remains a one–dimensional one, though.

We will present here another extension of the analytic signal conception using the QFT: For the one–dimensional analytic signal it is important that the Fourier transform of a real signal is a Hermitian function, i.e. that the equation

$$F(-u) = F^*(u) \tag{26}$$

holds, where F^* is the complex conjugate function of F. Therefore, if we want to examine what the notion of the analytic signal means in the conception of QFT we have to remember the notion of a quaternionic Hermitian function which was introduced in section 2.

In section 4 we found out that the QFT of a real image f obeys some symmetry rules, e.g. that the real part of the QFT is even with respect to both arguments of F^q. We can restate these properties in the form

$$F(-u,v) = \beta(F(\boldsymbol{u})) \tag{27}$$
$$F(u,-v) = \alpha(F(\boldsymbol{u})) \tag{28}$$
$$F(-\boldsymbol{u}) = \gamma(F(\boldsymbol{u})) \quad , \tag{29}$$

where α, β and γ are the nontrivial involutions of $\mathbb{H}$ defined in section 2. Writing $F(\boldsymbol{u}) = F_0(\boldsymbol{u}) + iF_1(\boldsymbol{u}) + jF_2(\boldsymbol{u}) + kF_3(\boldsymbol{u})$ we can restate (27) as

$$\begin{aligned}
& F(-u,v) = \beta(F(\boldsymbol{u})) = -jF(\boldsymbol{u})j \\
\Longrightarrow \quad & F_0(-u,v) = F_0(\boldsymbol{u}), \qquad F_1(-u,v) = -F_1(\boldsymbol{u}) \\
& F_2(-u,v) = F_2(\boldsymbol{u}), \qquad F_3(-u,v) = -F_3(\boldsymbol{u}) \quad ,
\end{aligned}$$

which means that the i–imaginary and the k–imaginary part of F are odd with respect to the first argument whereas the real and the j–imaginary part are even with respect to the first argument. Analogously we can find from (28) that with respect to the second argument the real and the i–imaginary component are even while the j–imaginary and the k–imaginary part are odd. These are the symmetry properties of the QFT we found in the previous section.

Hence, it follows that the QFT of a real signal is a quaternionic Hermitian function. It is easy to see that a quaternionic Hermitian function contains redundant information in three quadrants of its domain and, therefore, can be reconstructed from the values $f(x,y)$ for $x \geq 0$ and $y \geq 0$. In order to reconstruct the function f from these values we need only to apply the equations (27), (28) and (29).

For this reason it seams reasonable to define the quaternionic analytic signal of a real 2–dimensional signal in the following way: We suppress all frequencies $\boldsymbol{u}$ in the quaternionic frequency domain for which either u or v or both of them are negative. The values at the double positive frequencies are multiplied by four. By the inverse QFT we transform the result into the spatial domain again and get the quaternionic analytic signal which, of course, is quaternion valued. The three imaginary components of the quaternionic analytic signal can be seen as the *quaternionic Hilbert transform* of f.

Definition: The quaternionic analytic signal of a two–dimensional signal f is given by

$$f_A^q(x,y) = \mathcal{F}_q^{-1}\{Z^q(u,v)\} \quad , \tag{30}$$

where Z^q is defined as

$$Z^q(u,v) = \begin{cases} 4F^q(u,v) & \text{if } u \geq 0 \text{ and } v \geq 0 \\ 0 & \text{else.} \end{cases} \tag{31}$$

We will prove that the real part of the quaternionic analytic signal of a real signal is equal to the signal itself.

Proof: In the following we will use the fact that for each quaternion q the relations $Re(q) = Re(\alpha(q)) = Re(\beta(q)) = Re(\gamma(q))$ hold. Following our definition the quaternionic analytic signal f_A^q of f is given by

$$f_A^q(x, y) = 4 \int_0^\infty \int_0^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv \quad . \tag{32}$$

Regarding only the real part of f_A^q and omitting the factor four we find

$$\begin{aligned} & Re\left(\int_0^\infty \int_0^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv\right) \\ = \ & Re\left(\int_0^\infty \int_0^\infty \alpha(e^{i2\pi ux} F^q(u, v) e^{j2\pi vy}) du\, dv\right) \\ = \ & Re\left(\int_{-\infty}^0 \int_0^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv\right) . \end{aligned} \tag{33}$$

Analogously using the involutions β and γ instead of α we get

$$\begin{aligned} & Re\left(\int_0^\infty \int_0^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv\right) \\ = \ & Re\left(\int_0^\infty \int_{-\infty}^0 e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv\right) \quad (34) \\ = \ & Re\left(\int_{-\infty}^0 \int_{-\infty}^0 e^{i2\pi ux} F^q(u, v) e^{j2\pi vy} du\, dv\right) \quad , \quad (35) \end{aligned}$$

respectively. Substituting (33) and (35) in (32) completes the proof:

$$\begin{aligned} Re(f_A^q(x, y)) &= 4Re\left(\int_0^\infty \int_0^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy}\right) du\, dv \\ &= Re\left(\int_{-\infty}^\infty \int_{-\infty}^\infty e^{i2\pi ux} F^q(u, v) e^{j2\pi vy}\right) du\, dv = f(x, y) \end{aligned} \tag{36}$$

□

Like in the one–dimensional case also here we can use the (quaternionic) analytic signal to define the instantaneous phase of a signal. We will demonstrate this in the following section.

6 Two–dimensional Phase

In one dimension the analytic signal of the cosine function $f(x) = \cos(x)$ is $f_A(x) = e^{ix}$. Hence, for each $x \in \mathbb{R}$ we can get the instantaneous phase of f by evaluating the argument of f_A at the position x. For the cosine function we simply get $\arg(f_A(x)) = x$. We can generalize this concept (see [7]) to all functions $f : \mathbb{R} \to \mathbb{R}$ for which the analytic signal f_A exists. We call $\arg(f_A(x))$ the *instantaneous phase* of f at x.

We want to generalize this concept to two dimensions. In order to start with the same motivation as in the one–dimensional case we consider the function $f(x, y) = \cos(x)\cos(y)$ first. We will show that the quaternionic analytic signal of f is $f_A(x, y) = e^{ix}e^{jy}$.

Proof: In the one–dimensional case we know that

$$f(x) = \cos(x) \Rightarrow f_A(x) = e^{ix} \quad ,$$

which follows from:

$$e^{ix} = 2\int_0^\infty \int_{-\infty}^\infty \left(e^{i2\pi ux}e^{-i2\pi ux'}\cos(x')\right) dx' du \quad . \tag{37}$$

Therefore, using (37), we obtain

$$\begin{aligned} f_A(x, y) &= 4\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i2\pi ux}e^{-i2\pi ux'}\cos(x')\cos(y')e^{-j2\pi vy'}e^{j2\pi vy} dx'\, dy'\, du\, dv \\ &= 2e^{ix}\int_0^\infty \int_{-\infty}^\infty \cos(y')e^{-j2\pi vy'}e^{j2\pi vy} dy'\, dv = e^{ix}e^{jy} \quad . \end{aligned} \tag{38}$$

□

Since we are looking for a concept of two–dimensional phase it is now straightforward to define the phase of $f(x, y) = \cos(x)\cos(y)$ at position (x, y). In section 3 we already mentioned the *arg*–function that maps the quaternions without zero to $\mathbb{R}^2$ in such a way that $arg(|q|e^{ix}e^{jy}) = (x, y)$ for $(x, y) \in [0, 2\pi[\times [0, \pi[$. The function *arg* is defined in the appendix A.

In one dimension the phase is defined within the interval $[0, 2\pi[$. In order to clarify why the two–dimensional phase is only defined within $[0, 2\pi[\times [0, \pi[$, we show in figure 1 how the function $f(x, y) = \cos(x)\cos(y)$ is made up of patches of the size $[0, 2\pi[\times [0, \pi[$.

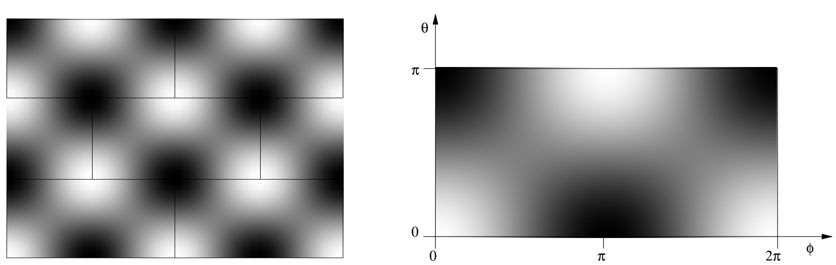

Fig. 1. The function $f(x, y) = \cos(x)\cos(y)$ with $(x, y) \in [0, 4\pi[\times[0, 3\pi[$ (left) and $(x, y) \in [0, 2\pi[\times[0, \pi[$ (right).

According to the definition of the two–dimensional argument function we can define the instantaneous phase of a two–dimensional signal f at (x, y) as

$$\textit{instantaneous phase of } f(x, y) = \arg(f_A(x, y)) \quad . \tag{39}$$

As Granlund [7] states for the one–dimensional case also we have to say that the instantaneous phase in general will not describe the local behavior of f. For this reason we will introduce the concept of *local phase* here.

In one dimension the local phase concept is well known. The local phase can be estimated using a quadrature filter, e.g. a Gabor filter with a central frequency u_0 which is defined by

$$g_{u_0}(x) = e^{-\pi x^2/\sigma^2} e^{i2\pi u_0 x} \quad . \tag{40}$$

The Gabor filter consists of a real part which is even and an odd imaginary part. Convolving the signal with the Gabor filter leads to a complex filter response. The argument of the response at position x is then called the local phase of the signal at x.

Of course the local phase of a signal is dependent of the central frequency of the Gabor filter. In order to demonstrate the local phase concept we borrow a figure from Granlund's book ([7], p. 262) which shows in which way the local phase corresponds to the local form of the signal (figure 4a).

There are several attempts to use the local phase for multi–dimensional signals. One possibility is to extend a Gabor filter to two dimensions in the following way[4]:

$$g_{\boldsymbol{u}_0}(x, y) = e^{-\pi \boldsymbol{x}^2/\sigma^2} e^{i2\pi \boldsymbol{u}_0 \cdot \boldsymbol{x}} \tag{41}$$

An example of such a Gabor filter is shown in figure 2. For arbitrary $\boldsymbol{u}_0$ we can

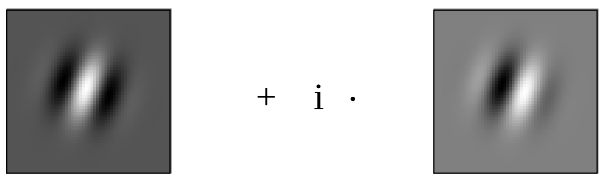

Fig. 2. A two–dimensional complex Gabor filter with an even real part (left) and an odd imaginary part (right).

obtain $g_{\boldsymbol{u}_0}(x, y)$ by rotating the Gabor filter

$$g(x, y) = e^{-\pi \boldsymbol{x}^2/\sigma^2} e^{i2\pi(u_0 x + 0y)} \tag{42}$$

by some angle θ about the origin. The local phase can then be defined along the direction $\boldsymbol{e} = (\cos(\theta), \sin(\theta))$ by evaluating the argument of the filter response of $g_{\boldsymbol{u}_0}(x, y)$. Thus, we find that this generalized filter is in principle the same as a one–dimensional Gabor filter. Therefore, we will define the notion of a quaternionic Gabor filter. As the real part of this Gabor filter we take the function $f(x, y) = \cos(2\pi u_0 x)\cos(2\pi v_0 y)$ windowed with a Gaussian function:

$$g^q_{ee}(x, y) = e^{-\pi(x^2+y^2)/\sigma^2} \cos(2\pi u_0 x)\cos(2\pi v_0 y) \quad . \tag{43}$$

[4] We restrict ourselves to to the usage of isotropic Gaussian windows here. It is also possible to use different values of σ for the directions x and y.

A one–dimensional filter that is an analytic function itself, is called a quadrature filter. We want to apply this notion also in the two–dimensional case and call a filter which is a quaternionic analytic signal a quaternionic quadrature filter. One–dimensional Gabor filters are quadrature filters, so we should require this also in the two–dimensional case. By taking the analytic signal of g_{ee} we get

$$\begin{aligned} g^q(x,y) &= g^q_{ee\,A}(x,y) \\ &= e^{-\pi(x^2+y^2)/\sigma^2}\left(\cos(2\pi u_0 x)\cos(2\pi v_0 y) + i\sin(2\pi u_0 x)\cos(2\pi v_0 y)\right. \\ &\quad \left. + j\cos(2\pi u_0 x)\sin(2\pi v_0 y) + k\sin(2\pi u_0 x)\sin(2\pi v_0 y)\right) \quad . \end{aligned} \tag{44}$$

In the following we will call such a filter a two–dimensional quaternionic Gabor filter. It is depicted for $u_0 = v_0$ in figure 3.

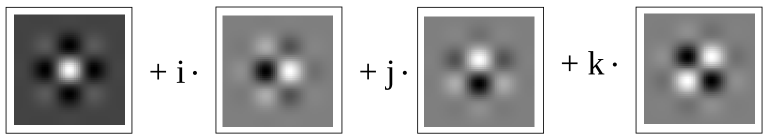

Fig. 3. A quaternionic Gabor filter with $u_0 = v_0$.

In one dimension the local phase gives information about the local symmetry or form of the signal, especially whether there is a peak or a step in the signal at the considered position. Using the quaternionic Gabor filters and evaluating the local signal phase by the two–dimensional *arg*–function we get the analogous information for an image signal, which is more complicated and contains more possible symmetries as the one–dimensional phase. In analogy to figure 4a we show the relation between the two–dimensional phase and the local signal structure in figure 4b and 4c.

As mentioned earlier we can evaluate the two–dimensional phase in a region $[0, 2\pi[\times [0, \pi[$ which can be thought of as a half torus. The circles in figures 4b and 4c result from cutting through the torus for different values of θ.

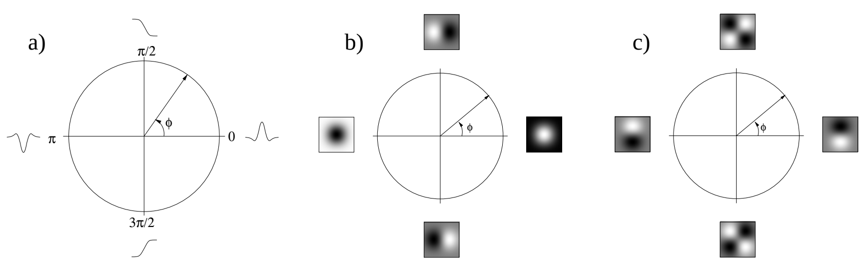

Fig. 4. Relation between the local phase and the local signal strcture: **a.** the one–dimensional case (see [7]), **b.** the two–dimensional case with $\theta = 0$, **c.** the two–dimensional case with $\theta = \pi/2$.

7 Experimental Results

Some experiments have been made which show how the local phase can be estimated from the answer of a quaternionic Gabor filter. We estimate the local phase of the function $f(\phi, \theta) = \cos(\phi)\cos(\theta)$ along some path through its domain in the following way. The signal function $f(\phi, \theta)$ is convolved with the quaternionic Gabor filter shown in figure 3. The filter response at each position in the (ϕ, θ)–plane is given by a quaternion. Along the line s shown in figure 5a the quaternionic argument function which is defined in the appendix A is applied to the quaternion–valued filter response. We denote the estimated local phase by $(\hat{\phi}, \hat{\theta})$ and compare it to the instantaneous phase that can be evaluated analytically for $f(\phi, \theta) = \cos(\phi)\cos(\theta)$ as (ϕ, θ) for $(\phi, \theta) \in [0, 2\pi[\ \times [0, \pi[$.

The central frequency of the used Gabor filters is four times higher than the frequency of the signal f. In Fig. 5b and 5c the estimated values $\hat{\phi}$ and $\hat{\theta}$ are compared to ϕ and θ, respectively. The straight lines are the values of the instantaneous phase (ϕ, θ) while the slightly curved lines represent the estimated local phase $(\hat{\phi}, \hat{\theta})$.

The arguments ϕ and θ of the Gaborian's answers are nearly linear and give a good approximation to the instantaneous phase of the signal.

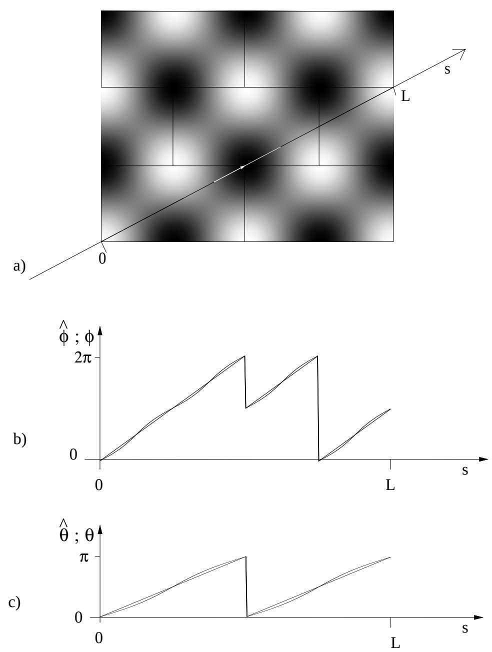

Fig. 5. The function $f(\phi, \theta) = \cos(\phi)\cos(\theta)$ with the path along which the local phase (ϕ, θ) is estimated, **b)** Variation of ϕ and $\hat{\phi}$ along the depicted path, **c)** Variation of θ and $\hat{\theta}$ along the depicted path.

8 Conclusion

In this article we presented the quaternionic Fourier transform (QFT), an integral transform for two–dimensional signals which is based on the Fourier transform but provides a quaternion–valued representation of the signal in the frequency domain.

Based on the QFT we generalized the concepts of the analytic signal, of Gabor filters and the local phase to two dimensions in a novel way.

This generalization could be of interest especially in PAC systems for the following reason. There are recent attempts to embed the different tasks of a PAC system into one mathematical system using Clifford algebras [11]. Clifford algebras in the form of Geometric algebras have already been applied to neural computation [1] and to computer vision [2]. Since quaternions are a special Clifford algebra, it should be possible to integrate the QFT approach into a Geometric algebra PAC system.

A The *arg*–function

Definition: For every quaternion q which can be given in the form $q = |q|e^{i\phi}e^{j\theta}$ the angles ϕ and θ within a range $(\phi, \theta) \in [0, 2\pi[\, \times [0, \pi[$ are called the *argument* of q. We define the function $arg : \mathbb{H}\backslash\{0\} \mapsto \mathbb{R}^2$ that recovers for quaternions q of the mentioned form the argument of q. Let $q = a + bi + cj + dk$, $q \neq 0$.

$$\arg(q) = (\phi, \theta) \quad , \tag{45}$$

with

$$\phi = \begin{cases} \pi - d' \frac{\pi}{2} & \text{for } a = b = c = 0 \\ \pi - \operatorname{sign}(b) \operatorname{sign}(bd)\ \frac{\pi}{2} & \text{for } a = c = 0, b \neq 0 \\ \operatorname{sign}(c)\ \arcsin(d') + \operatorname{step}(-c)\ \pi & \\ +2\pi \operatorname{step}(c) \operatorname{step}(-d) & \text{for } a = 0, c \neq 0 \\ \arctan(b/a) + \pi \operatorname{step}(-a) & \\ +2\pi \operatorname{step}(a) \operatorname{step}(-ab) & \text{for } a \neq 0, c = d = 0 \\ \arctan(b/a) + \pi \operatorname{step}(-c) & \\ +2\pi \operatorname{step}(-d) \operatorname{step}(c) & \text{for } a \neq 0 \wedge (c \neq 0 \vee d \neq 0) \end{cases} \tag{46}$$

and

$$\theta = \begin{cases} \frac{\pi}{2} & \text{for } a = b = c = 0 \\ \arcsin(\operatorname{sign}(b)\ d') + \pi \operatorname{step}(-bd) & \text{for } a = c = 0, b \neq 0 \\ \frac{\pi}{2} & \text{for } a = 0, c \neq 0 \\ 0 & \text{for } a \neq 0, c = d = 0 \\ \arctan(c/a) + \pi \operatorname{step}(-c/a) & \text{for } a \neq 0 \wedge (c \neq 0 \vee d \neq 0) \end{cases} \quad , \tag{47}$$

with $d' = d/|q|$. Here we used the *step*– and the *sign*–function which are defined by

$$\mathrm{step}(x) = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ 1 & \text{for} \quad x > 0 \end{cases} \tag{48}$$

and

$$\mathrm{sign}(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x \geq 0 \end{cases} \quad . \tag{49}$$

Acknowledgment

We would like to thank Dr. Kostas Daniilidis for his interest in this work and for valuable discussions on the subject of this article.

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