A Multivector Teager Filter

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Over the last couple of years the mathematical framework of Geometric Algebra (GA) [1] has gained increasing interest in multidimensional signal processing. Particularly, new important concepts for image analysis like the monogenic signal [2] and the structure multivector [3] have been developed.

Here we want to use GA for the direct design of a new nonlinear discrete filter of Teager type. The 1D Teager filter [4] is a homogenous quadratic Volterra filter

$$y_n = x_n^2 - x_{n-1}x_{n+1}\,. \tag{1}$$

A 2D extension and its use for image processing and analysis is detailed in [6]. Put simply, the complete 2D Teager filter results from overlaying horizontal (1), vertical and diagonal Teager basis filter. The following filter design in GA will follow this principle. Additionally, GA will be used for geometrically motivated signal embedding. Combined this will then give rise to a filter which is more powerful than the classical 2D Teager filter.

A GA is a real associative algebra that arises naturally from any real non–degenerate quadratic space $\mathbb{R}^{p,q}$ (p and q denoting the number of basis vectors of positive and negative square, respectively). Basic vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n \equiv \sigma_1, \ldots, \sigma_p, \sigma_{p+1}, \ldots, \sigma_{p+q}$ and 1 are carried over to the algebra and multiplication (so-called geometric product) is defined by the rules

$$\sigma_i^2 = \begin{cases} +1 & \text{if } 1 \leq i \leq p \\ -1 & \text{else} \end{cases} \tag{2}$$

and $\sigma_k\,\sigma_l = -\sigma_l\,\sigma_k,\ l \neq k$. Note the use of juxtaposition for the geometric product. Elements of a GA are called multivectors. Here we are only interested in $\mathcal{G}_{3,0} \equiv \mathcal{G}_3 = \text{span}\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{123}\}$. We identify $[a\ b\ c]^T$ with $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$.

Let us consider the constant Volterra kernel

$$\boldsymbol{h}_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{3}$$

and the linear Volterra kernel

$$\boldsymbol{h}_1 = \begin{bmatrix} \boldsymbol{h}_1(0,0) & \boldsymbol{h}_1(1,0) \\ \boldsymbol{h}_1(0,1) & \boldsymbol{h}_1(1,1) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \begin{bmatrix} -1 & -1 & 0 \end{bmatrix}^T & \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T \\ \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T & \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \end{bmatrix}. \tag{4}$$

If we look at the signal $x(m,n)$ as $m\,\sigma_1 + n\,\sigma_2 + x(m,n)\,\sigma_3$ and compute the convolution (involving the geometric product)

$$\boldsymbol{X}(\boldsymbol{n}) = \boldsymbol{h}_0 + \sum_{\boldsymbol{i}} \boldsymbol{h}_1(\boldsymbol{i})\,x(\boldsymbol{n} - \boldsymbol{i}) \tag{5}$$

a nice geometric interpretation for the resulting multivector $\boldsymbol{X}(\boldsymbol{n})$ is available. Let us fix a direction

$$\boldsymbol{d} = x \cdot \boldsymbol{e}_x + y \cdot \boldsymbol{e}_y, \qquad x, y \in \{-1, 0, 1\}\,, \tag{6}$$

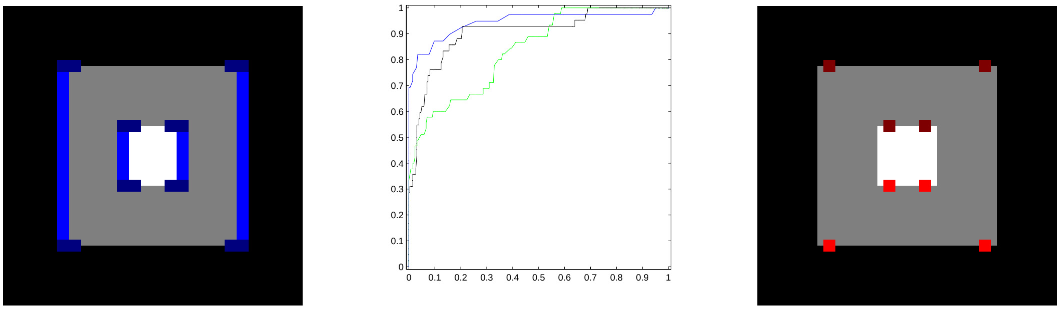

Figure 1: Experimental results for the multivector Teager filter (MVTF). The MVTF as edge detector (left image): signal (gray level) and overlaid scalar response (blue) of the MVTF. Typical ROC-curves of several edge detctors on a test image (middle image): Laplace filter (green), Teager filter (black) and MVTF (blue). The MVTF as corner detector (right image): signal (gray level) and overlaid bivector response. Positive values of the σ_{12} component are indicated by light red squares and negative values by dark red squares, respectively.

to derive multivector Teager basis filter $\boldsymbol{T_d}$. The multivector linear kernel of such directional filter is given by

$$\boldsymbol{g}_1 : \mathbb{Z}^2 \to \mathcal{G}_3, \quad \boldsymbol{g}_1(\boldsymbol{i}) = (\boldsymbol{h}_1(\boldsymbol{i}) - \boldsymbol{h}_{1\mathrm{a}}(\boldsymbol{i}))\,\boldsymbol{h}_0 + \boldsymbol{h}_0(\boldsymbol{h}_1(\boldsymbol{i}) - \boldsymbol{h}_{1\mathrm{b}}(\boldsymbol{i})) \tag{7}$$

and the multivector quadratic kernel is given by

$$\boldsymbol{g}_2 : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathcal{G}_3, \quad \boldsymbol{g}_2(\boldsymbol{i},\boldsymbol{j}) = \boldsymbol{h}_1(\boldsymbol{i})\,\boldsymbol{h}_1(\boldsymbol{j}) - \boldsymbol{h}_1(\boldsymbol{i}-\boldsymbol{d})\,\boldsymbol{h}_1(\boldsymbol{j}+\boldsymbol{d})\,. \tag{8}$$

The above kernels constitute the filter $\boldsymbol{T_d}$, which is a multivector Volterra filter

$$\boldsymbol{T_d}[x(\boldsymbol{n})] = \sum_{\boldsymbol{i}} \boldsymbol{g}_1(\boldsymbol{i})x(\boldsymbol{n}-\boldsymbol{i}) + \sum_{\boldsymbol{i}}\sum_{\boldsymbol{j}} \boldsymbol{g}_2(\boldsymbol{i},\boldsymbol{j})x(\boldsymbol{n}-\boldsymbol{i})x(\boldsymbol{n}-\boldsymbol{j}) \tag{9}$$

$$= \boldsymbol{G}_1[x(\boldsymbol{n})] + \boldsymbol{G}_2[x(\boldsymbol{n})]\,. \tag{10}$$

From these basis filter the multivector Teager filter (MVTF) results by overlaying. The MVTF is more powerful than the original 2D Teager filter, for example it allows for detection of lines and vectorial features. First illustrative examples of image analysis with the MVTF are given in Fig. 1.

References

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