On Averaging in Clifford Groups*

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Abstract. Averaging measured data is an important issue in computer vision and robotics. Integrating the pose of an object measured with multiple cameras into a single mean pose is one such example. In many applications data does not belong to a vector space. Instead, data often belongs to a non–linear group manifold as it is the case for orientation data and the group of three–dimensional rotations SO(3). Averaging on the manifold requires the utilization of the associated Riemannian metric resulting in a rather complicated task. Therefore the Euclidean mean with best orthogonal projection is often used as approximation. In SO(3) this can be done by rotation matrices or quaternions. Clifford algebra as a generalization of quaternions allows a general treatment of such approximated averaging for all classical groups. Results for the two–dimensional Lorentz group $\mathrm{SO}(1,2)$ and the related groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(1,1)$ are presented. The advantage of the proposed Clifford framework lies in its compactness and easiness of use.

1 Introduction

Averaging measured data is one of the most frequently arising problems in many different applications. For example, integrating the pose of an object measured with multiple cameras into a single mean pose is a standard task in computer vision. Feature–based registration of images would be another such example. The original motivation for this paper has been the following. In a neural network where every neuron represents a geometric transformation, say three–dimensional rotation, one has to average over several neurons in order to adjust the network topology to new presented data.

Surely, averaging data belonging to some vector space is rather trivial. For a set of points $\{x\}_i^n$ one only has to calculate the barycentre

$$\mathcal{A} = \frac{1}{n}\sum_{i=1}^{n} x_i \,. \tag{1}$$

In $\mathbb{R}^d$ this also minimizes the sum of the squared distances to the given points. Because of that variational property, (1) is then also called the arithmetic mean. Distance here refers of course to the usual Euclidean metric $d_E(\cdot,\cdot)$ yielding

$$\mathcal{A} = \arg\min_{x\in\mathbb{R}^d} \sum_{i=1}^{n} d_E(x, x_i)^2 \,. \tag{2}$$

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All of the examples given above, however, involve data which does not belong to a vector space. Rather, the data are elements of a group, like SO(3) as for the case of three–dimensional rotations. In fact one therefore has to deal with non–linear manifolds having different geometrical structure than that of "flat" vector spaces. So let M be a matrix group. Then the Riemannian distance between two group elements is given by

$$d_R(G_1, G_2) = \frac{1}{\sqrt{2}} \| \log(G_1^T, G_2) \| \,, \tag{3}$$

where log refers to matrix logarithm and the usual Frobenius norm is applied (see e.g. [5]). The Riemannian metric (3) measures the length of the shortest geodesic connecting G_1 and G_2. Since every group acts transitively on itself there is a closed form solution to that. However, the shortest geodesic may not be unique. The Riemannian mean associated with (3) now reads

$$\mathcal{R} = \arg\min_{G \in M} \sum_{i=1}^{n} \| \log(G_i^T, G) \|^2 \,. \tag{4}$$

For many important groups solving (4) analytically is not possible. Recently, in [11] it was proven that there is no closed form solution of (4) for SO(3). Therein it was also demonstrated that Riemannian averaging is already a very hard problem for one–parameter subgroups of SO(3).

As an alternative to computing the Riemannian mean, approximative embedding techniques are well established (see e.g. [4]). The basic idea is to embed the data into a larger vector space in which operations are then performed, and project the result back onto the manifold. Technically, one thereby performs constrained optimization and usually uses orthogonal projection. Both aspects are treaded extensively in [3]. The natural embedding for a matrix group is of course the covering general linear group GL(n), which in turn is also a vector space. Hence the Frobenius norm induces the following metric on GL(n)

$$d_F(G_1, G_2) = \|G_1 - G_2\| \,. \tag{5}$$

Associated with (5) is the mean

$$\mathcal{E} = \arg\min_{G \in M} \sum_{i=1}^{n} \|G_i - G)\|^2 \,, \tag{6}$$

which will be termed Euclidean mean from now on. Note that without back–projection $\mathcal{E}$ does not have to be an element of M.

Whenever the group M is a differentiable manifold, i.e. a Lie group, there is a further alternative for an approximation of the Riemannian mean $\mathcal{R}$ (4). Roughly speaking, the Riemannian distance in the Lie group can be approximated by the Euclidean distance in the corresponding Lie algebra. This is done by virtue of the famous Baker–Campbell–Hausdorff formula [8]. The whole method is very common in robotics [13]. A recent example for its use for motion estimation is [6].

This paper, however, concentrates on the study of Clifford groups for approximating the Riemannian mean $\mathcal{R}$. It is well known, that any three–dimensional rotation can be represented by a rotation matrix or a unit quaternion (among other possible representations like Euler angles). Unit quaternions do form a group which acts by a two–sided operation. That way a different Riemannian mean approximation results than the one induced by rotation matrices. Unit quaternions are a particularly example of a Clifford group. In fact all classical groups do have a covering Clifford group. Hence the Clifford algebra framework offers a general alternative for averaging on such groups.

The remainder of this paper is organized as follows. In section 2 we briefly present basic facts about Clifford groups. This is then followed in section 3 by reviewing what is known about averaging on SO(3) using both rotations matrices and unit quaternions. Additionally, an experimental comparison of the two approaches for noisy data is presented. Averaging on the two–dimensional Lorenz group SO(1, 2) and its covering Clifford group $\Gamma_{1,2}$ is discussed in section 4. Therein a closer look on the related groups $\mathrm{SL}(2,\mathbb{R})$ and SU(1, 1) is also provided. The paper finishes with some concluding remarks.

2 Clifford Groups

Associated with every Clifford algebra there is the so–called Clifford group (or Lipschitz group) formed by all of its invertible elements. A Clifford algebra can be constructed from a quadratic space. Here we are particularly interested in real quadratic spaces $\mathbb{R}^{p,q}$, meaning $\mathbb{R}^{p+q}$ equipped with a quadratic form Q of signature (p, q).

Definition 1 [10]. An associative algebra over $\mathbb{R}$ with unity 1 is the Clifford algebra $\mathcal{C}_{p,q}$ of $\mathbb{R}^{p,q}$ if it contains $\mathbb{R}^{p+q}$ and $1 \cdot \mathbb{R} = \mathbb{R}$ as distinct subspaces so that

(a) $\mathbf{x}^2 = Q(x)$ for any $x \in \mathbb{R}^{p+q}$
(b) $\mathbb{R}^{p+q}$ generates $\mathcal{C}_{p,q}$
(c) $\mathcal{C}_{p,q}$ is not generated by any proper subspace of $\mathbb{R}^{p+q}$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^{p,q}$. Then the following relations hold

$$e_i^2 = 1,\ 1 \leq i \leq p, \quad e_i^2 = -1,\ p < i \leq n \quad e_i e_j = -e_j e_i,\ i < j\,. \tag{7}$$

That way an algebra of dimension 2^n is generated (putting $e_0 = 1$). The canonical basis of a Clifford algebra is therefore formed by

$$B_\mu = e_{j_1} e_{j_2} \cdots e_{j_r},\ 1 \leq j_1 < \ldots j_r \leq p + q\,. \tag{8}$$

Those basis vectors which consist of an even number of factors do form a subalgebra. This subalgebra $\mathcal{C}^+_{p,q}$ is called the even part of $\mathcal{C}_{p,q}$.

For Clifford algebras the following isomorphisms hold

$$\mathcal{C}_{0,0} \cong \mathbb{R} \tag{9a}$$

$$\mathcal{C}_{0,1} \cong \mathbb{C} \tag{9b}$$

$$\mathcal{C}_{0,2} \cong \mathbb{H}\,, \tag{9c}$$

with $\mathbb{C}$ denoting complex numbers and $\mathbb{H}$ denoting quaternions as usual. The embedding used in Definition 1 (a) above can be made more explicit as follows. Define

$$\alpha : \mathbb{R}^{p+q} \to \mathcal{C}_{p,q}, x \mapsto \mathbf{x} = \sum_{i=1}^{n} x_i e_i \tag{10}$$

and identify $\mathbb{R}^{p+q}$ with its image under that mapping. The elements $\alpha(\mathbb{R}^{p+q})$ are termed vectors again. All invertible vectors ($\mathbf{x}^2 \neq 0$) already generate the Clifford group. For a more revealing characterization the following two mappings are required. Inversion, which is an automorphism, is defined by $\hat{\mathbf{x}} = -\mathbf{x}$ and $\widehat{ab} = \hat{a}\hat{b}$. Reversion is an anti–automorphism defined by $\tilde{\mathbf{x}} = \mathbf{x}$ and $\widetilde{ab} = \tilde{b}\tilde{a}$. Using (8) these mappings become

$$\widehat{B_\mu} = (-1)^r B_\mu \qquad \widetilde{B_\mu} = (-1)^{\frac{r(r-1)}{2}} B_\mu\,. \tag{11}$$

Definition 2. The Clifford group associated with the Clifford algebra $\mathcal{C}_{p,q}$ is defined by

$$\Gamma_{p,q} = \{ s \in \mathcal{C}_{p,q} \mid \forall x \in \mathbb{R}^{p,q}, s\mathbf{x}\hat{s}^{-1} \in \mathbb{R}^{p,q} \}\,.$$

Hence the Clifford group is determined by its two–sided action on vectors. Furthermore the map $\mathbf{x} \mapsto s\mathbf{x}\hat{s}^{-1}$ is an orthogonal automorphism of $\mathbb{R}^{p,q}$ [12].

Normalizing the Clifford group $\Gamma_{p,q}$ yields

$$\mathrm{Pin(p,q)} = \{ \mathrm{s} \in \Gamma_{\mathrm{p,q}} \mid \mathrm{s\tilde{s}} = \pm 1 \}\,. \tag{12}$$

The group $\mathrm{Pin(p,q)}$ is a two–fold covering of the orthogonal group $\mathrm{O(p,q)}$. Further subgroups of $\mathrm{Pin(p,q)}$ are

$$\mathrm{Spin(p,q)} = \mathrm{Pin(p,q)} \cap \mathcal{C}^+_{\mathrm{p,q}} \tag{13}$$

and

$$\mathrm{Spin^+(p,q)} = \{ \mathrm{s} \in \mathrm{Spin(p,q)} \mid \mathrm{s\tilde{s}} = 1 \}\,. \tag{14}$$

Both groups are again two–fold covers of their classical counterparts. The whole situation can be summarized as

$$\mathrm{Pin(p,q)} \backslash \{\pm 1\} \cong \mathrm{O(p,q)} \tag{15a}$$

$$\mathrm{Spin(p,q)} \backslash \{\pm 1\} \cong \mathrm{SO(p,q)} \tag{15b}$$

$$\mathrm{Spin^+(p,q)} \backslash \{\pm 1\} \cong \mathrm{SO^+(p,q)}\,. \tag{15c}$$

Thereby $\mathrm{SO^+(p,q)}$ is formed by those elements which are connected with the identity. This does not carry over to the covering groups, i.e. $\mathrm{Spin^+(p,q)}$ does not have to be connected [10]. In case of $q = 0$ one has $\mathrm{SO^+(p,0)} = \mathrm{SO(p,0)}$ and $\mathrm{Spin^+(p,0)} = \mathrm{Spin(p,0)}$ (analogously for $q = 0$). Further on we simply write $\mathrm{SO(p)}$ and $\mathrm{Spin(p)}$ then. Finally note that every Lie group can be represented as spin group [2]. Hence averaging using Lie methods is also possible inside the Clifford framework.

3 Averaging on Spin(3) and SO(3)

The already mentioned group of unit quaternions is isomorphic to the three–sphere S^3. The latter, in turn, being isomorphic to the group Spin(3). Additionally $\mathcal{C}^+_{0,3} \cong \mathcal{C}_{0,2} \cong \mathbb{H}$ yields, and therefore averaging rotations using Spin(3) is the same as if using quaternions. To remain consistent everything in the following will be denoted in terms of Spin(3).

Before actually turning to the problem of averaging rotations, a more theoretical remark may be in order. The group Spin(3) can also be used for averaging on the three-sphere, which of course can not be done by using rotation matrices. On the other hand no disadvantage results from the fact that Spin(3) is a two–fold cover of SO(3). Formally, some care has to be taken due to the existence of antipodal points ($s, -s \in$ Spin(3) induce the same rotation). A consistent set of group elements, however, can always be chosen easily if necessary.

As for representing rotations itself, the Euclidean mean can be defined both in terms of SO(3)

$$\mathcal{E}_{\mathrm{SO}(3)} = \arg\min_{R\in\mathrm{SO}(3)} \sum_{i=1}^{n} \|R_i - R)\|^2 \tag{16}$$

and Spin(3)

$$\mathcal{E}_{\mathrm{Spin}(3)} = \arg\min_{s\in\mathrm{Spin}(3)} \sum_{i=1}^{n} \|s_i - s)\|^2 \,. \tag{17}$$

For the latter note that every Clifford algebra is of course also a real vector space. The following was derived (using quaternions) in [7] for the Euclidean mean on Spin(3)

$$\begin{aligned}\mathcal{E}_{\mathrm{Spin}(3)} &= \arg\max_{s\in\mathrm{Spin}(3)} \sum_{i=1}^{n} s s_i \\ &= \arg\max_{s\in\mathrm{Spin}(3)} s \sum_{i=1}^{n} s_i \\ &= \frac{\sum_{i=1}^{n} s_i}{\|\sum_{i=1}^{n} s_i\|^2}\,,\end{aligned} \tag{18}$$

which is the ordinary arithmetic mean with normalization. Solving for the matrix mean (16) is a special case of the famous Procrustes problem yielding

$$\begin{aligned}\mathcal{E}_{\mathrm{SO}(3)} &= \arg\max_{R\in\mathrm{SO}(3)} \sum_{i=1}^{n} tr(R^T R_i) \\ &= \arg\max_{R\in\mathrm{SO}(3)} tr(R^T \sum_{i=1}^{n} R_i) \\ &= \arg\max_{R\in\mathrm{SO}(3)} tr\big(R^T \frac{\sum_{i=1}^{n} R_i}{n}\big)\,,\end{aligned} \tag{19}$$

which is the orthogonal projection of the arithmetic mean onto SO(3). The actual solution can then be obtained by using Singular Value Decomposition (SVD). This, although not too complicated, is somehow more costly than simple normalization as in (18). Moreover the two discussed approximation methods for the Riemannian mean are indeed based on different linearizations.

A rotation is represented in the algebra $\mathcal{C}_{0,2}$ as

$$\cos(\frac{\phi}{2})e_0 + \sin(\frac{\phi}{2})(xe_1 + ye_2 + ze_1e_2)\,, \tag{20}$$

with (x, y, z) being the rotation axis and θ being the angle of rotation. Since every Clifford group operates by a two–sided action (see Definition 2 again) the approximation (18) is based on half the angle θ. Contrary, every matrix group acts by ordinary matrix multiplication and the approximation (16) is therefore based on the whole angle θ. An experimental comparison of the two methods have been already provided in [7] using a Gaussian sampling for the angle and a uniform one for the axis. Both methods have been reported as equally very good. In our opinion a Gaussian sampling for both parameter types seems to be at least as realistic.

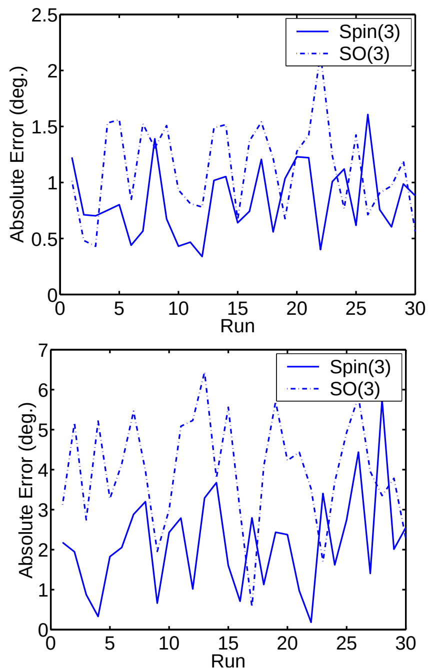

Fig. 1. Abbreviation of the approximated mean from the Riemannian mean for standard deviation of 0.2 (top) and 0.5 (bottom). See text for details.

For demonstration a little experiment on synthetic data has been carried out. The sample size is set to 20, the mean is set to $[1,2,3,4]/\|[1,2,3,4]\|$ with standard deviation of 0.2 (first setup) and 0.5 (second setup). Each setup is repeated 30 times. The true Riemannian mean is computed by using non–linear optimization from `MATLAB`. The angle of the rotation transforming the approximated mean into the Riemannian mean is used as error measure. The obtained results are shown in Fig. 1.

The average error in the small standard deviation setup is 0.83 deg. for Spin(3) averaging versus 1.13 deg. for SO(3) averaging. Both values are very good and the difference is rather insignificant. For the larger standard deviation setup the value for Spin(3) is 2.17 deg. compared to 3.98 deg. for SO(3). Here the difference is obviously noticeable. All in all in our chosen setup averaging using Spin(3) seems therefore preferable.

Approximating the Riemannian mean using the aforementioned projected Euclidean means works quite nice for three–dimensional rotations in reasonable setups (taken into consideration both [7] and the above experiment). One prerequisite for any approximation to make sense is of course that the entity being approximated makes sense itself. The Riemannian metric (shortest geodesic) for three–dimensional rotation does. For example, it is bi–invariant [11]. For the Special Euclidean group SE(3), of which SO(3) is a subgroup, however, no bi–invariant metric does exist [1].

4 Averaging on Spin(1,2) and SO(1, 2)

When not familiar with Clifford algebra everything about quaternions seems to be quite exceptional at first sight. As we have seen this is not true. All Clifford groups do operate in the same way by a two–sided action yielding an orthogonal automorphism. Hence there is also a general treatment of averaging on other orthogonal (sub–)groups in terms of Clifford algebra. Quaternions being just one example. Another such example will be studied in this section. In oder to simplify notations we will use the canonical ordered basis derivable from (8) do denote elements of a Clifford algebra. That is we just write $(a,b,c,d) \in \mathcal{C}_{0,2}$ instead of $ae_0 + be_1 + ce_2 + de_1e_2$, for example.

In the following we want to study the two–dimensional Lorentz group SO(1,2). As shown in [9] this group comes into play whenever measurements with respect to motion are (realistically) considered as taking their own time. The group SO(1,2) has time dimension (t) and spatial dimension (x, y) leaving invariant the scalar product

$$< (t,x,y),(t',x'y') >= tt' - xx' - yy' \,. \tag{21}$$

More precisely, it is formed by those 3×3 matrices with determinant one which preserve (21). Geometrically, everything about SO(1,2) is related to cones. An important example being the future cone $\{(t,x,y) \mid t^2 - x^2 - y^2 \geq 0, t \geq 0\}$.

The group SO(1,2) has the two well known covering groups SU(1,1) and $\mathrm{SL}(2, \mathbb{R})$, which are defined by

$$\left\{ \begin{pmatrix} y_1 & y_2 \\ \overline{y_2} & \overline{y_1} \end{pmatrix} \mid y_1, y_2 \in \mathbb{C}, |y_1|^2 - |y_2|^2 = 1 \right\} , \tag{22}$$

and

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} , \tag{23}$$

respectively. Using the Euclidean mean as approximation for the Riemannian mean requires to project the arithmetic mean onto the manifold. This, yet, is a rather complicated problem for SO(1,2) since the orthogonality condition is now $RR^T = diag(1, -1, -1)$. Of course things are easier using $\mathrm{SL}(2, \mathbb{R})$ instead. In that case, however, there is no need to use a matrix group at all since the following relations hold

$$\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1) \cong \mathrm{Spin}(1, 2) , \tag{24}$$

the latter being a two–fold cover of SO(1,2) by definition. Hence as abstract groups all groups in question are isomorphic. Moreover, the different representations as elements of the Clifford algebra $\mathcal{C}_{1,2}$ do only differ by permutation. That can be easily checked using the fact $\mathcal{C}_{1,2} \cong \mathbb{C}(2)$, where $\mathbb{C}(2)$ denotes the space of all complex 2×2 matrices. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ is represented in $\mathcal{C}_{1,2}$ by

$$\frac{1}{2}(a + d, b + c, 0, c - b, 0, a - d, 0, 0) . \tag{25}$$

Setting $y_1 = \frac{1}{2}((a + d) + i(b - c))$ and $y_2 = \frac{1}{2}((b + c) - i(d - a))$ yields the corresponding SU(1,1) matrix, which in turn is represented by

$$\frac{1}{2}(a + d, b + c, b + c, -(c - b), 0, d - a, 0, 0) . \tag{26}$$

Hence all groups have essentially the same representation in the Clifford algebra $\mathcal{C}_{1,2}$. Furthermore the Euclidean means

$$\mathcal{E}_{\mathrm{Spin}(1,2)} = \arg \min_{s \in \mathrm{Spin}(1,2)} \sum_{i=1}^{n} \|s_i - s)\|^2 \tag{27a}$$

$$\mathcal{E}_{\mathrm{SL}(2,\mathbb{R}} = \arg \min_{R \in \mathrm{SL}(2,\mathbb{R})} \sum_{i=1}^{n} \|R_i - R)\|^2 \tag{27b}$$

$$\mathcal{E}_{\mathrm{SU}(1,1)} = \arg \min_{U \in \mathrm{SU}(1,1)} \sum_{i=1}^{n} \|U_i - U)\|^2 \tag{27c}$$

are then also identical and can be computed all inside the algebra $\mathcal{C}_{1,2}$ just by simple normalization

$$\frac{\sum_{i=1}^{n} m_i}{\| \sum_{i=1}^{n} m_i\|^2} , \tag{28}$$

with $m_i = s_i$, $m_i = R_i$, or $m_i = U_i$ accordingly to the cases in (27). Moreover, everything could also been carried out in the algebra $\mathcal{C}_{3,0}$, which is isomorphic to $\mathcal{C}_{1,2}$. For example,

$$\frac{1}{2}(a+d,0,b+c,d-a,0,0,b-c,0) \tag{29}$$

corresponds to (25). In the following we will only consider the group Spin(1,2). In oder to evaluate the quality of approximation by (28) the Riemannian mean has to be studied first a little bit closer. Formally, (4) does apply again. So we are rather looking for a parameterization of SO(1,2). One such parameterization is the Cartan decomposition SO(1,2)= KAK having factors

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \cosh\psi & 0 & \sinh\psi \\ 0 & 1 & 0 \\ \sinh\psi & 0 & \cosh\psi \end{pmatrix}. \tag{30}$$

Again an experiment on synthetic data has been performed to compare the approximated Euclidean mean on Spin(1,2) with the true Riemannian mean. Both angles arising from the Cartan decomposition (30) have been sampled using a Gaussian distribution. In the first experiment we used $\phi = 30$ deg. and $\psi = 20$ deg. as mean values and a standard deviation of 2 deg. in both cases for a sample of size 30. The obtained results are separately reported for both angles in Fig. 2.

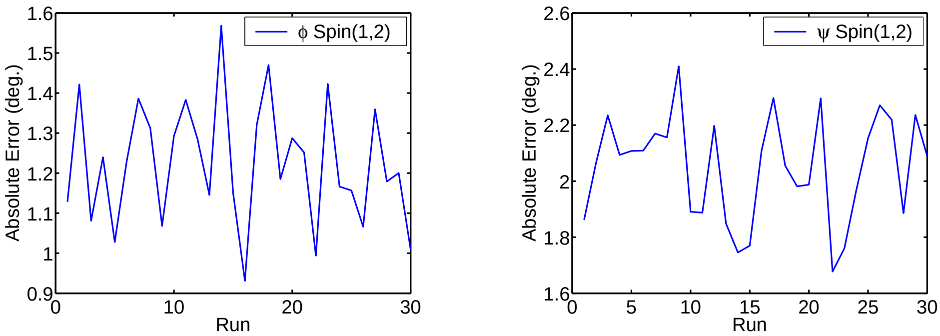

Fig. 2. Abbreviation of the approximated mean from the Riemannian mean for standard deviation of 2 deg. for ϕ (left) and ψ (right). See text for details.

The results are quite good. The average error is 1.27 deg. for ϕ and 2.03 for ψ. Using a standard deviation of both 5 deg. resulted in average errors of 2.56 and 3.87, respectively. As before the error was measured as angles of the transformation needed to carry over the approximated mean into the true Riemannian one. The latter was computed using non–linear optimization from `Matlab` again. From the obtained results averaging in Spin(1,2) seems to be a useful approximation method for practical applications.

5 Conclusions

In this paper we studied averaging in Clifford groups. More precisely, the approximation of the Riemannian means by Euclidean means of such groups have been discussed. The Clifford algebra framework allows a general and elegant treatment of averaging problems. The particular case of three–dimensional rotations has been reviewed comparing averaging in SO(3) (rotation matrices) with averaging in Spin(3) (unit quaternions). In the chosen setup the latter performed slightly better. More important, the Euclidean mean is always easy to compute for a Clifford group, namely by just performing normalization in the associated algebras. This was further demonstrated on $SO(1,2)$, where it has been also demonstrated how related groups can be handled in the same manner. The obtained results suggest that Clifford algebra is a useful and flexible tool for averaging. Future work will be on testing the proposed methods for particular neural networks in practical applications. Also a comparison with common Lie algebra averaging seems to be interesting. Studying the influence of embeddings like the conformal model of Clifford algebra for averaging in SE(3) might also be worthwhile.

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