A Spherical Harmonic Expansion of the Hilbert Transform on the Two-Sphere

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Abstract. The classical Hilbert transform on the real line is a valuable tool in signal processing. It constitutes the analytic signal which allows the determination of the instantaneous phase and amplitude of a one dimensional signal. For signals in in the Euclidean plane its analogue is the monogenic signal based on the Riesz transform, a generalization of the Hilbert transform to the plane. In addition to the instantaneous phase and amplitude, the orientation of intrinsically one dimensional structures in the plane can be determined. Various disciplines like geosciences, omnidirectional vision or astrophysics have to deal with signals arising on the two-sphere. A Hilbert transform on the two-sphere is well known from Clifford analysis. Yet it lacks a suitable interpretation from a signal processing viewpoint, especially in the frequency domain. In this paper we derive a series expansion of the Hilbert transform on the two-sphere in terms of spherical harmonics. It provides an intuitive interpretation and turns out to be a gradient-like operator acting only on the angular parts of the signal. This leads to intensity and rotation invariant signal analysis techniques on the two-sphere in analogue to the Euclidean plane.

INTRODUCTION

The classical Hilbert transform on the real line for functions $f \in L^2(\mathbb{R})$ given by the convolution with the Hilbert kernel

$$\mathscr{H}[f](x) = (f * \frac{1}{\pi})(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy \tag{1}$$

has a well known interpretation in the Fourier domain:

$$\mathscr{F}[\mathscr{H}[f]](u) = -i\, sgn(u)\, \mathscr{F}[f](u). \tag{2}$$

It therefore justifies the interpretation as a $\frac{\pi}{2}$ phase shift for negative frequencies and a $\frac{-\pi}{2}$ phase shift for positive frequencies. Assuming a sinusoid signal model $f(x) = A(x)\cos(\phi(x))$, where $\phi(x)$ denotes the local phase and $A(x)$ the local amplitude of f, its Hilbert transform is obtained as $\mathscr{H}[f](x) = A(x)\sin(\phi(x))$. The sinusoid signal model together with its Hilbert transform constitutes the *analytic signal* [1]

$$f_a(x) = f(x) + i\mathscr{H}[f](x) = A(x)e^{i\phi(x)} \tag{3}$$

consisting of strictly positive frequencies in the Fourier domain. The analytic signal is a standard tool in signal processing used to obtain the local phase and local amplitude of sinusoid signals. It arises as the non-tangential boundary value of the *Cauchy transform*. The classical Cauchy integral formula is well known from complex analysis. Its generalization in the sense of Clifford analysis working in the real Clifford algebra $\mathbb{R}_{0,n}$ is given by [2]

$$\mathscr{C}[f](x) = \frac{2}{A_n} \int_{\partial G} E(x-y)n(y)f(y)dS(y) = \frac{2}{A_n} \int_{\partial G} \frac{x-y}{|x-y|^{(n+1)/2}} n(y)f(y)dS(y) \tag{4}$$

with ∂G as the smooth boundary of $G \subseteq \mathbb{R}^n$, dS the surface element of ∂G, A_n the surface area and $n(y)$ the outward pointing unit normal at y. For $G = \mathbb{R}^2_+, \partial G = \mathbb{R}, x \in G$ the non-tangential boundary value of the Cauchy results in the classical analytic signal

$$\lim_{n.t.x \to \xi} \mathscr{C}[f](x) = \lim_{n.t.x \to \xi} (E * f)(x) = f_a(\xi). \tag{5}$$

In the following we investigate an analogue to the analytic signal on $\mathbb{S}^2$. The construction follows the classical concept by considering the non-tangential boundary value of the Cauchy transform on $\mathbb{S}^2$ instead of the Cauchy transform in the upper half space $\mathbb{R}_+^n$. Furthermore we are interested in a spectral characterization in analogue to the Fourier domain interpretation of the classical Hilbert transform. We will provide this characterization in terms of spherical harmonic coefficients. The coefficients are obtained by a series expansion of the Cauchy transform integral.

THE CAUCHY TRANSFORM ON $\mathbb{S}^2$

The Cauchy transform on $\mathbb{S}^2$ for functions $f \in L^2(\mathbb{S}^2)$ is defined as [3]

$$\mathscr{C}[f](x) = \frac{2}{A_3} \int_{\mathbb{S}^2} E(x - \omega)\omega f(\omega) dS(\omega) = \frac{2}{A_3} \int_{\mathbb{S}^2} \frac{x - \omega}{|x - \omega|^3} \omega f(\omega) dS(\omega) \tag{6}$$

where A_3 denotes the surface area of the two-sphere and $x = r\xi \in \mathbb{B}^2$. The Cauchy transform allows the splitting into the Poisson transform and the conjugate Poisson transform in the unit ball $\mathbb{B}^2$ respectively as [4]

$$\mathscr{C}[f](x) = \frac{1}{2}(\mathscr{P}[f](x) + \mathscr{Q}[f](x)) \tag{7}$$

$$= \frac{1}{2A_3} \left(\int_{\mathbb{S}^2} \frac{1 - |x|^2}{|x - \omega|^3} f(\omega) dS(\omega) + \int_{\mathbb{S}^2} \frac{1 + |x|^2 + 2x\omega}{|x - \omega|^3} f(\omega) dS(\omega) \right) \tag{8}$$

with non-tangential boundary values

$$\lim_{n.t. \ x \to \xi} \mathscr{P}[f](x) = f(\xi) \quad \text{and} \quad \lim_{n.t. \ x \to \xi} \mathscr{Q}[f](x) = \mathscr{H}[f](\xi). \tag{9}$$

The Poisson transform is the harmonic extension of f from $\mathbb{S}^2$ to $\mathbb{B}^2$. In accordance the conjugate Poisson transform is the harmonic extension of the Hilbert transform $\mathscr{H}[f]$ on $\mathbb{S}^2$ to $\mathbb{B}^2$. They both solve the Laplace equation $\Delta \mathscr{P}[f] = 0$, $\Delta \mathscr{Q}[f] = 0$, where Δ denotes the Laplace operator. In terms of scale-space theory the Cauchy transform naturally embeds f in the Poisson scale-space which constitutes a linear scale space according to the well known Gaussian one.

THE CAUCHY TRANSFORM AS A GROUP CONVOLUTION

In the upper half space $\mathbb{R}_+^n$ the Cauchy transform acts as a convolution with the Cauchy kernel on the target function. Since we want to proceed analogously on the two-sphere, the Cauchy transform is interpreted as a group convolution over $SO(3)$. Let $\mathscr{R}(\rho) = \mathscr{R}(\theta, \varphi, \gamma)$ describe a rotation operator rotating around the $z - y - z$ axis. For every $\omega = (\theta, \varphi) \in \mathbb{S}^2$ we first rotate the function f by $\mathscr{R}^{-1}(\theta, \varphi, 0) = \mathscr{R}^{-1}(\rho)$ to the north pole, i.e. $f(\mathscr{R}^{-1}(\theta, \varphi, 0)\omega)$, and evaluate the Cauchy transform at the north pole η of the rotated function.

$$\mathscr{C}_\rho[f](r\eta) = \int_{\mathbb{S}^2} E(r\eta - \omega)\omega f(\mathscr{R}^{-1}(\rho)\omega) dS(\omega) = \mathscr{R}(\rho)[h] \star f. \tag{10}$$

The modified transform $\mathscr{C}_\rho[f](r\eta)$ is a function defined on $SO(3)$ describing a convolution over $SO(3)$ with the Cauchy kernel centered at the north pole.

SERIES EXPANSION

The interpretation of the Cauchy transform in terms of a group convolution over $SO(3)$ allows a series expansion into $SO(3)$ basis functions [5]:

$$\mathscr{R}(\rho)[h] \star f = \sum_{l \in \mathbb{N}} \sum_{m=-l}^{l} \sum_{n=-l}^{l} [\widehat{\mathscr{R}[h] \star f}]_{m,n}^{l} \overline{D_{m,n}^{l}(\rho)} \quad \text{with} \quad [\widehat{\mathscr{R}[h] \star f}]_{m,n}^{l} = \overline{\hat{h}_{l,n}} \hat{f}_{l,m} \tag{11}$$

where the $SO(3)$ basis functions used in this case are the *Wigner-D* functions defined by

$$D_{m,n}^{l}(\rho) = D_{m,n}^{l}(\theta, \varphi, \psi) = e^{-im\psi} d_{m,n}^{l}(\cos \theta) e^{-in\varphi}. \tag{12}$$

The series expansion depends on the spherical harmonic coefficients $\hat{h}_{l,n}, \hat{f}_{l,m}$ of the filter kernel and the target function respectively. Since the Cauchy transform splits into the Poisson and conjugate Poisson transform we can treat these two transforms separately. Furthermore the conjugate Poisson kernel evaluated at the north pole $x = \eta$ consists of a scalar and a bivector part

$$\mathscr{Q}_r(x, \omega) = \frac{1 + |x|^2 + 2x\omega}{|x - \omega|^3} = \frac{1 + r^2 - 2r\omega_3}{\left(1 + r^2 - 2r\omega_3\right)^{3/2}} - \frac{2r\omega_1 e_{13}}{\left(1 + r^2 - 2r\omega_3\right)^{3/2}} - \frac{2r\omega_2 e_{23}}{\left(1 + r^2 - 2r\omega_3\right)^{3/2}} = \mathscr{Q}_r^{(0)} + \mathscr{Q}_r^{(1)} e_{13} + \mathscr{Q}_r^{(2)} e_{23}. \tag{13}$$

which can be regarded as single filter kernels. We will therefore expand three filter kernels into its spherical harmonic coefficients: the scalar Poisson kernel, which is equal to the scalar conjugate Poisson kernel part, and the two bivector parts of the conjugate Poisson kernel. The Poisson kernel in $\mathbb{B}^2$ has a well known series expansion into Legendre polynomials given by [6]

$$\mathscr{P}_r(x, \omega) = \frac{1 - |x|^2}{|x - \omega|^3} = \sum_{k=0}^{\infty} (2k+1) r^k P_n(\langle \xi, \omega \rangle). \tag{14}$$

such that its spherical harmonic coefficients are equal to

$$[\widehat{\mathscr{P}_r}]_{l,m} = \int_{\mathbb{S}^2} \mathscr{P}_r(r\eta, \omega) \overline{Y_{l,m}(\omega)} dS(\omega) = \begin{cases} r^l & \text{for } m = 0 \\ 0 & \text{else} \end{cases} \tag{15}$$

where $Y_{l,m}(\omega)$ denote the standard spherical harmonics on $\mathbb{S}^2$.
For the conjugate poisson kernel we first expand $\frac{1}{\left(1 + r^2 - 2r\omega_3\right)^{3/2}}$ into a series of Gegenbauer polynomials [7]

$$\frac{1}{\left(1 + |x|^2 - 2r\omega_3\right)^{3/2}} = \sum_{k=0}^{\infty} r^k C_n^{3/2}(\langle \xi, \omega \rangle) = \sum_{k=0}^{\infty} r^k C_n^{3/2}(\cos \theta). \tag{16}$$

such that the spherical harmonic coefficients are obtained by evaluating

$$[\widehat{\mathscr{Q}_r^{(1/2)}}]_{l,m} == \int_{\mathbb{S}^2} \mathscr{Q}_r^{(1/2)}(\omega) \overline{Y_{l,m}(\omega)} dS(\omega) = \int_{\mathbb{S}^2} \omega_{1/2} \sum_{k=0}^{\infty} r^k C_k^{3/2}(\cos \theta) \overline{Y_{l,m}(\omega)} dS(\omega). \tag{17}$$

Its coefficients turn out to be 0 for orders $m \neq \mp 1$ such that

$$[\widehat{\mathscr{Q}_r^{(1)}}]_{l,\pm 1} = \mp 2r 4\pi \sqrt{\frac{2\pi}{3}} \frac{1}{4\pi} \sqrt{\frac{3l(l+1)}{2(2l+1)}} r^{l-1} = \mp \sqrt{\frac{4\pi l(l+1)}{(2l+1)}} r^l \tag{18}$$

$$\widehat{[\mathscr{Q}_r^{(2)}]}_{l,\pm1} = \mp i2r4\pi\sqrt{\frac{2\pi}{3}}\frac{1}{4\pi}\sqrt{\frac{3l(l+1)}{2(2l+1)}}r^{l-1} = \mp i\sqrt{\frac{4\pi\,l(l+1)}{(2l+1)}}r^l. \tag{19}$$

According to (11) the convolutions over $SO(3)$ for the bivector parts result in

$$\mathscr{R}(\omega)[\mathscr{Q}_r^{(1)}]\star f = 2\sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}r^l\widehat{f}_{l,m}\frac{\partial}{\partial\theta}Y_{l,m}(\omega) \tag{20}$$

$$\mathscr{R}(\omega)[\mathscr{Q}_r^{(2)}]\star f = 2\sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}r^l\widehat{f}_{l,m}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{l,m}(\omega). \tag{21}$$

We compare these two operators with the angular parts of the gradient operator ∇ on $\mathbb{S}^2$ acting on a function $f\in L_2(\mathbb{S}^2)$ as

$$(\nabla f)(\omega) = \sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}\widehat{f}_{l,m}(\nabla Y_{l,m})(\omega) = e_\theta\sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}\widehat{f}_{l,m}\frac{\partial}{\partial\theta}Y_{l,m}(\omega) + e_\varphi\sum_{l\in\mathbb{N}}\sum_{m=-l}^{l}\widehat{f}_{l,m}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{l,m}(\omega). \tag{22}$$

It turns out that the bivector parts of the conjugate Poisson kernel just act like the angular parts of the gradient operator on $\mathbb{S}^2$. Instead of acting on the original function f they act on their harmonic extension $\mathscr{P}[f]$ into $\mathbb{B}^2$. We may therefore analyze signals in terms of their local orientation β, local phase ϕ and local amplitude A in the plane tangent to $\mathbb{S}^2$ at ω as

$$\beta = \arctan 2(\mathscr{R}(\omega)[\mathscr{Q}_r^{(2)}]\star f, \mathscr{R}(\omega)[\mathscr{Q}_r^{(1)}]\star f). \tag{23}$$

$$\phi = \arctan 2(\sqrt{(\mathscr{R}(\omega)[\mathscr{Q}_r^{(1)}]\star f)^2 + (\mathscr{R}(\omega)[\mathscr{Q}_r^{(2)}]\star f)^2}, \mathscr{R}(\omega)[\mathscr{P}_r]\star f) \tag{24}$$

$$A = \sqrt{(\mathscr{R}(\omega)[\mathscr{Q}_r^{(1)}]\star f)^2 + (\mathscr{R}(\omega)[\mathscr{Q}_r^{(2)}]\star f)^2 + (\mathscr{R}(\omega)[\mathscr{P}_r]\star f)^2} \tag{25}$$

CONCLUSION

It turned out that the Hilbert transform on $\mathbb{S}^2$ acts as a multiplier in the domain of $SO(3)$ basis functions. This result was motivated by the series expansion of the Cauchy transform in terms of Wigner-D functions. Compared to the Hilbert transform on the real line the Hilbert transform on $\mathbb{S}^2$ acts like a local gradient operator. It may be used to analyze the local orientation in terms of the local orientation, local phase and local amplitude in the tangent plane.

REFERENCES

1. D. Gabor, *Journal of the IEE (London)* **93**, 429–457 (1946).
2. R. Delanghe, F. Sommen, and V. Soucek, *Clifford Algebra and Spinor-Valued Functions: A Function Theory for the Dirac Operator (Mathematics and Its Applications)*, Springer, 1992, 1 edn., ISBN 079230229X.
3. R. Delanghe, *Bull. Belg. Math. Soc. Simon Stevin* **11**, 163–180 (2004).
4. F. Brackx, B. De Knock, H. De Schepper, and D. Eelbode, *Mathematical Methods in the Applied Sciences* **29**, 1435–1450 (2006).
5. Y. Wiaux, L. Jacques, P. Vielva, , and P. Vandergheynst, *The Astrophysical Journal* **652**, 820–832 (2006).
6. W. Freeden, and U. Windheuser, *Advances in Computational Mathematics* **5**, 51–94 (1996).
7. A. Jeffrey, *Table of Integrals, Series, and Products (Academic Press)*, Elsevier LTD, Oxford, 2000, 6. a. edn., ISBN 0122947576.