9. Commutative Hypercomplex Fourier Transforms of Multidimensional Signals*

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9.1 Introduction

In Chap. 8 the approach of the Clifford Fourier transform (CFT) and of the quaternionic Fourier transform (QFT) have been introduced. We have shown that the CFT yields an extended and more efficient multi-dimensional signal theory compared to the theory based on complex numbers. Though the CFT of a *real* signal does not include new information (the complex Fourier transform is a *complete* transform in the mathematical sense), the Clifford spectrum is a richer representation with respect to the *symmetry concepts* of n-D signals than the complex spectrum. Furthermore, the possibility of designing *Clifford-valued filters* represents a fundamental extension in multi-dimensional signal theory. Our future aim is to develop principles for the design of hypercomplex filters. The first method is introduced in Chap. 11, where the quaternionic Gabor filters are explored.

One main property of Clifford algebras is the non-commutativity of the Clifford product. This property is impractical in some cases of analytic and numerical calculations. Some theorems are very complicated to formulate in

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higher dimensions, e.g. the affine theorem (The. 8.4.9) and the convolution theorem (The. 8.4.3). Similar problems occur in the derivation of fast algorithms (Chap. 10), because for the decimation of space method, the exponential functions need to be separated. Due to non-commutativity, the additional exponential terms cannot be sorted, and hence, no closed formulation of the partial spectra is obtained.

Therefore, we have generalized the approach of Davenport [58], who introduces 'a commutative hypercomplex algebra with associated function theory'. Davenport uses the $\mathbb{C}^2$ algebra (commutative ring with unity) in order to extend the classical complex analysis for treating four-dimensional variables, which are similar to quaternions. Ell [69] applies this approach to the quaternionic Fourier transform in order to simplify the convolution theorem.

We have picked up this idea to develop fast algorithms for the CFT (Chap. 10). For the separation of the CFT-kernel, we need a commutative algebra. Therefore, we have designed a new transform which is based on a different algebra, but yields the same spectrum as the CFT for real signals. For hypercomplex valued signals the spectrum differs from the Clifford spectrum.

Though it seems that the commutative hypercomplex Fourier transform (HFT) is no more than a tool for easier or faster calculation of the CFT, we will show in this chapter, that the HFT has the same right to exist as the CFT, because neither transform can be considered to be *the* correct extension of the complex Fourier transform (both yield the complex FT in the 1-D case). Up to now, there is no fundamental reason that determines which transform to use. Therefore, we study the properties of both transforms.

In this chapter we show several important properties of the algebra that generalizes Davenport's approach. After introducing the algebraic framework, we define the HFT and prove several theorems. We do this to motivate the reader to make his own experiments. This chapter together with Chap. 10 should form a base for further analytic and numerical investigations.

9.2 Hypercomplex Algebras

In this section, we define the algebraic framework for the rest of the chapter. The term *hypercomplex algebra* is explained and a specific four-dimensional algebra is introduced.

9.2.1 Basic Definitions

In general, a hypercomplex algebra is generated by a hypercomplex number system and a multiplication which satisfies the algebra axioms (see [129]).

To start with, we define what is meant by the term *hypercomplex number* (see also Cha. 7):

Definition 9.2.1 (Hypercomplex numbers). *A hypercomplex number of dimension n is an expression of the form*

$$\boldsymbol{a} = a_0 + a_1 i_1 + a_2 i_2 + \ldots + a_{n-1} i_{n-1} \tag{9.1}$$

where $a_j \in \mathbb{R}$ for all $j \in \{0, \ldots, n-1\}$ and i_j ($j \in \{1, \ldots, n-1\}$) are formal symbols (often called imaginary units). Two hypercomplex numbers

$$\begin{aligned} \boldsymbol{a} &= a_0 + a_1 i_1 + \ldots + a_{n-1} i_{n-1} \qquad \text{and} \\ \boldsymbol{b} &= b_0 + b_1 i_1 + \ldots + b_{n-1} i_{n-1} \end{aligned}$$

are equal if and only if $a_j = b_j$ for all $j \in \{0, \ldots, n-1\}$.

Take, for example, $n = 2$. In this case we obtain numbers of the form $a_0 + a_1 i_1$ – this could be the complex numbers, dual numbers or double numbers. If $n = 4$ we obtain numbers of the form $a_0 + a_1 i_1 + a_2 i_2 + a_3 i_3$. This could be the quaternions or the commutative algebra which we will introduce at the end of this section.

Definition 9.2.2 (Addition, subtraction, and multiplication). *The addition of two hypercomplex numbers $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by*

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= (a_0 + a_1 i_1 + \ldots + a_{n-1} i_{n-1}) + (b_0 + b_1 i_1 + \ldots + b_{n-1} i_{n-1}) \\ &= (a_0 + b_0) + (a_1 + b_1) i_1 + \ldots + (a_{n-1} + b_{n-1}) i_{n-1} \end{aligned} \tag{9.2}$$

and their subtraction is defined by

$$\begin{aligned} \boldsymbol{a} - \boldsymbol{b} &= (a_0 + a_1 i_1 + \ldots + a_{n-1} i_{n-1}) - (b_0 + b_1 i_1 + \ldots + b_{n-1} i_{n-1}) \\ &= (a_0 - b_0) + (a_1 - b_1) i_1 + \ldots + (a_{n-1} - b_{n-1}) i_{n-1} \ . \end{aligned} \tag{9.3}$$

The multiplication of two hypercomplex numbers is defined by an $(n-1) \times (n-1)$ multiplication table with the entries

$$i_\alpha i_\beta = p_0^{\alpha\beta} + p_1^{\alpha\beta} i_1 + \ldots + p_{n-1}^{\alpha\beta} i_{n-1} \tag{9.4}$$

where $\alpha, \beta \in \{1, \ldots, n-1\}$. The product

$$\boldsymbol{ab} = (a_0 + a_1 i_1 + \ldots + a_{n-1} i_{n-1})(b_0 + b_1 i_1 + \ldots + b_{n-1} i_{n-1}) \tag{9.5}$$

is evaluated by using the distributive law and the multiplication table.

The sum and the difference of two hypercomplex numbers are calculated like in an n-dimensional vectorspace with the base vectors $1, i_1, i_2, \ldots, i_{n-1}$. The product is more general than a vectorspace product: we can embed the commonly used products in this hypercomplex product.

If we, for example, consider the scalar product according to the Euclidean norm, then we have $p_j^{\alpha\beta} = 0$ for $j \neq 0$ and $p_0^{\alpha\beta} = 1$ for $\alpha = \beta$.

Standard algebra products are covered by the hypercomplex product, too. For example, the product of the algebra of complex numbers is obtained for $n = 2$, $p_0^{11} = -1$ and $p_1^{11} = 0$. The quaternion product is obtained by the following table 9.1. According to this table, we have $p_0^{jj} = -1$ ($j = 1, 2, 3$), $p_k^{jj} = 0$ ($j = 1, 2, 3$, $k = 1, 2, 3$), etc..

Table 9.1. Multiplication table of the quaternion algebra

	i_1	i_2	i_3
i_1	-1	i_3	$-i_2$
i_2	$-i_3$	-1	i_1
i_3	i_2	$-i_1$	-1

A *hypercomplex number system* of dimension n consists of all numbers of the form (9.1) of dimension n and the operations which are defined in (9.2),(9.3), and (9.5).

A hypercomplex number system contains even more structure than it seems so far. In the following theorem, we show that a hypercomplex number system forms an associative algebra.

Theorem 9.2.1 (Hypercomplex algebra). *All hypercomplex number systems fulfill the following properties and therefore they are associative algebras:*

1. *the product is bilinear, i.e.*

$$(a\boldsymbol{u})\boldsymbol{v} = a(\boldsymbol{u}\boldsymbol{v}) = \boldsymbol{u}(a\boldsymbol{v}) \tag{9.6a}$$

$$(\boldsymbol{v}+\boldsymbol{w})\boldsymbol{u} = \boldsymbol{v}\boldsymbol{u} + \boldsymbol{w}\boldsymbol{u} \tag{9.6b}$$

$$\boldsymbol{u}(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\boldsymbol{v} + \boldsymbol{u}\boldsymbol{w} \quad , \tag{9.6c}$$

2. *and the product is associative, i.e.*

$$\boldsymbol{u}(\boldsymbol{v}\boldsymbol{w}) = (\boldsymbol{u}\boldsymbol{v})\boldsymbol{w} \ , \tag{9.7}$$

where $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ *are hypercomplex numbers and* $a \in \mathbb{R}$.

Proof. The theorem is proved by elementary calculations using Def. 9.2.2. □

Therefore, complex numbers and their product form the algebra of complex numbers, the quaternions and their product form the algebra of quaternions, etc..

9.2.2 The Commutative Algebra $\mathcal{H}_2$

In the following we consider a further, specific four-dimensional hypercomplex algebra, which is commutative and somehow similar to the algebra of quaternions. The new algebra denoted by $\mathcal{H}_2$ is formed by the space

$$\mathrm{span}(1, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4)$$

and the geometric product. Consequently, $\mathcal{H}_2$ is a subalgebra of $\mathbb{R}^+_{4,0}$ and we have the following multiplication table (Tab. 9.2):

The same multiplication table is obtained for the two-fold tensor product of the complex algebra ($\mathbb{C} \otimes \mathbb{C}$). In this case, we have the basis elements $\{1 \otimes 1, i \otimes 1, 1 \otimes i, i \otimes i\}$. Since $\mathcal{H}_2$ and $\mathbb{C} \otimes \mathbb{C}$ have the same dimension

Table 9.2. Multiplication table of $\mathcal{H}_2$

	$\mathbf{e}_1 \wedge \mathbf{e}_3$	$\mathbf{e}_2 \wedge \mathbf{e}_4$	$\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$
$\mathbf{e}_1 \wedge \mathbf{e}_3$	-1	$\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$	$-\mathbf{e}_2 \wedge \mathbf{e}_4$
$\mathbf{e}_2 \wedge \mathbf{e}_4$	$\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$	-1	$-\mathbf{e}_1 \wedge \mathbf{e}_3$
$\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$	$-\mathbf{e}_2 \wedge \mathbf{e}_4$	$-\mathbf{e}_1 \wedge \mathbf{e}_3$	1

and the multiplication tables[1] are the same, $\mathcal{H}_2$ and $\mathbb{C}\otimes\mathbb{C}$ are isomorphic as algebras by the mapping $f^2 : \mathcal{H}_2 \to \mathbb{C}\otimes\mathbb{C}$ and $f(1) = 1\otimes 1$, $f(\mathbf{e}_1\wedge\mathbf{e}_3) = i\otimes 1$, $f(\mathbf{e}_2 \wedge \mathbf{e}_4) = 1 \otimes i$, and $f(\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4) = i \otimes i$.

Since the multiplication table is symmetric with respect to the major diagonal, the algebra $\mathcal{H}_2$ is commutative. Furthermore, Tab. 9.2 is equal to the multiplication table of the quaternion algebra (Tab. 9.1) in the cells $(1,1), (1,2), (1,3), (2,2)$ and $(3,2)$. In particular, we obtain for $(a+ib)(c+jd)$ in the quaternion algebra the same coefficients as for $(a+b\mathbf{e}_1\wedge\mathbf{e}_3)(c+d\mathbf{e}_2\wedge\mathbf{e}_4)$ in the algebra $\mathcal{H}_2$:

$$(a+ib)(c+jd) = ac + ibc + jad + kbd \tag{9.8a}$$

$$\begin{aligned}(a + b\mathbf{e}_1 \wedge \mathbf{e}_3)(c + d\mathbf{e}_2 \wedge \mathbf{e}_4) = {} & ac + bc\mathbf{e}_1 \wedge \mathbf{e}_3 + ad\mathbf{e}_2 \wedge \mathbf{e}_4 \\ & + bd\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 \ .\end{aligned} \tag{9.8b}$$

From this fact we will conclude The. 9.3.1 about the commutative hypercomplex Fourier transform (HFT2) of a real signal in the following section.

9.3 The Two-Dimensional Hypercomplex Fourier Analysis

In this section, we firstly define an integral transform which is based on the commutative algebra $\mathcal{H}_2$ and acts on hypercomplex 2-D signals. This transform which is denoted HFT2 yields the same spectrum as the QFT (8.3.4) for real signals. We reformulate the affine theorem, the convolution theorem, the symmetry theorem, and the shift theorem. Additionally, we prove that the algebra $\mathcal{H}_2$ and the two-fold Cartesian product of the complex numbers are isomorphic as algebras (see also [58]).

9.3.1 The Two-Dimensional Hypercomplex Fourier Transform

We introduce the HFT2 according to Ell [69] in the following. Furthermore, we make some fundamental considerations about this transform.

Definition 9.3.1 (Commutative hypercomplex Fourier transform). *The two-dimensional commutative hypercomplex Fourier transform (HFT2) of a two-dimensional signal $f(x, y)$ is defined by*

[1] Note that both algebras are multilinear.

$$F^h(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi(xu\mathbf{e}_1\wedge\mathbf{e}_3+yv\mathbf{e}_2\wedge\mathbf{e}_4)}\,dx\,dy \ . \tag{9.9}$$

Note that due to the commutativity of $\mathcal{H}_2$ we have the identity

$$e^{-2\pi(xu\mathbf{e}_1\wedge\mathbf{e}_3+yv\mathbf{e}_2\wedge\mathbf{e}_4)} = e^{-2\pi xu\mathbf{e}_1\wedge\mathbf{e}_3}e^{-2\pi yv\mathbf{e}_2\wedge\mathbf{e}_4} \ .$$

The commutativity implies that all commutator terms in the Campbell-Hausdorff formula (see e.g. [211]) vanish.

As mentioned at the end of the last section, the product of two quaternions and the product of two $\mathcal{H}_2$ elements are equal if the multiplication in the algebra $\mathbb{H}$ is ordered wrt. the index set of the basis. That means that no product of the form $\mathbf{e}_i\mathbf{e}_j$ with $i \geq j$ appears. This is the case for the quaternionic Fourier transform of real signals:

Theorem 9.3.1 (Correspondence of HFT2 and QFT). *The 2-D commutative hypercomplex Fourier transform of a real 2-D signal $f(x,y)$ yields the same coefficients as the QFT of $f(x,y)$.*

Proof. The coefficient of the spectra are the same, because all multiplications have the form (9.8a,b). □

In particular, we can decompose both transforms into four real-valued transforms: a cos-cos-transform, a cos-sin-transform, a sin-cos-transform, and a sin-sin-transform. Then, we can take the real valued transforms as coefficients of the QFT and the HFT2 spectrum (see Def. 8.3.1):

$$F^q = \mathcal{CC}\{f\} - \mathcal{SC}\{f\}i - \mathcal{CS}\{f\}j + \mathcal{SS}\{f\}k \tag{9.10a}$$

$$\begin{aligned} F^h = {} & \mathcal{CC}\{f\} - \mathcal{SC}\{f\}\mathbf{e}_1\wedge\mathbf{e}_3 - \mathcal{CS}\{f\}\mathbf{e}_2\wedge\mathbf{e}_4 \\ & + \mathcal{SS}\{f\}\mathbf{e}_1\wedge\mathbf{e}_3\wedge\mathbf{e}_2\wedge\mathbf{e}_4 \ . \end{aligned} \tag{9.10b}$$

The HFT2 (9.9) yields a geometric interpretation concerning the spatial and the frequency domain. If we span the spatial domain by $\mathbf{e}_1$ and $\mathbf{e}_2$, i.e. each point is represented by $x\mathbf{e}_1 + y\mathbf{e}_2 = \boldsymbol{x} + \boldsymbol{y}$, and same with the frequency domain ($u\mathbf{e}_3 + v\mathbf{e}_4 = \boldsymbol{u} + \boldsymbol{v}$), we can rewrite (9.9) as

$$F^h(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})}\,dx\,dy \ . \tag{9.11}$$

Both, the spatial and the frequency domain are 2-D vectorspaces, which are orthogonal with respect to each other (see Fig. 9.1).

The hypercomplex spectrum includes a scalar part, two bivector parts ($\mathbf{e}_1\wedge\mathbf{e}_3$ and $\mathbf{e}_2\wedge\mathbf{e}_4$) and a four-vector part. Therefore, the spectral values are denoted in the same algebra as the coordinates! This is an obvious advantage of Def. 9.11 and therefore, we use this definition for proving theorems in the following. Nevertheless, all results can easily be transferred to the Def. 9.9.

One property which is important in signal theory is the uniqueness of a transform and the existence of an inverse transform. Otherwise, the identification and manipulation of the signal in frequency representation would not be possible.

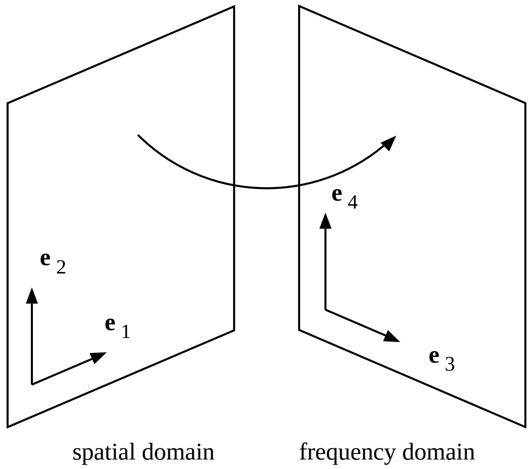

Fig. 9.1. The HFT2 visualized

Theorem 9.3.2 (The HFT2 is unique and invertible). *The HFT2 of a signal f is unique and invertible.*

Proof. In order to show the uniqueness, we prove that the kernel of the transform consists of orthogonal functions. We do so by reducing the exponential function to sine and cosine functions:

$$\begin{aligned} e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})} &= e^{2\pi\boldsymbol{u}\wedge\boldsymbol{x}}e^{2\pi\boldsymbol{v}\wedge\boldsymbol{y}} \\ &= \big(\cos(2\pi ux) - \mathbf{e}_1\wedge\mathbf{e}_3\sin(2\pi ux)\big)\big(\cos(2\pi vy) - \mathbf{e}_2\wedge\mathbf{e}_4\sin(2\pi vy)\big) \end{aligned}$$

Since the sine and cosine functions are orthogonal, the HFT2 is unique and furthermore, the inverse transform reads

$$f(x,y) = \int_{\mathbb{R}^2} F^h(u,v)e^{2\pi(\boldsymbol{x}\wedge\boldsymbol{u}+\boldsymbol{y}\wedge\boldsymbol{v})}\,du\,dv \ ,$$

which can be verified by a straightforward calculation. □

One nice property of the HFT2 is the fact that both, the transform and the inverse transform, are formulated in the same way[2]. The minus sign which we have for the complex Fourier transform and for the QFT can be omitted for the HFT2.

If we recall the isomorphism between $\mathcal{H}_2$ and $\mathbb{C}\otimes\mathbb{C}$, we can rewrite the HFT2-kernel as

$$e^{-2\pi(xu\mathbf{e}_1\wedge\mathbf{e}_3+yv\mathbf{e}_2\wedge\mathbf{e}_4)} \cong e^{-i2\pi xu}\otimes e^{-i2\pi yv} \ . \tag{9.12}$$

Consider now a real-valued, separable signal $f(x,y) = f^x(x)f^y(y)$. Then, due to multilinearity, the HFT2 of $f(x,y)$ itself can be written as

[2] We know such property from the Hartley transform.

$$f(x,y) \circ\!\!-\!\!\bullet\, F^h(u,v) \cong F^x(u) \otimes F^y(v) \tag{9.13}$$

where $F^x(u)$ is the 1-D Fourier transform of the signal $f(x,y)$ wrt. the x-coordinate and $F^y(v)$ accordingly to the y-coordinate.

This notation introduces another interpretation of the HFT2: we obtain the HFT2 of a real, separable signal by the tensor product of the complex 1-D spectra. Note that this is not valid for hypercomplex signals, because in that case we cannot exchange the tensor product and the product between signal and kernel[3]. Nevertheless, since the coefficients of the quaternionic spectrum and the $\mathcal{H}_2$ spectrum of a real signal are the same, the QFT of a separable signal can be interpreted as the tensor product of complex 1-D Fourier transforms wrt. to x and y as well.

9.3.2 Main Theorems of the HFT2

In this section, we consider some theorems for the HFT2. For the QFT, some of the main theorems are more complicated compared to those of the complex FT. We will show that this drawback is less crucial for the HFT2.

In the shift theorem of the QFT (Eq. 8.29) one exponential factor moves to the left (the i term) and one moves to the right (the j term). This is necessary since the algebra of quaternions is not commutative. But what can we do for Clifford transforms of higher dimension? There are only two ways of multiplication: one from the left and one from the right. The great advantage of commutative algebras is the fact that neither the order nor the direction of multiplication is relevant. Hence, the shift theorem for the HFT yields two exponential factors which can be placed arbitrarily or even composed in one exponential factor:

Theorem 9.3.3 (Shift theorem). *Let $F^h(u,v)$ be the HFT2 of a signal $f(x,y)$. Then, the HFT2 of the signal $f'(x,y) = f(x-\xi, y-\eta)$ reads $F^{h'}(u,v) = e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{\xi}+\boldsymbol{v}\wedge\boldsymbol{\eta})}F^h(u,v)$ where $\boldsymbol{\xi} = \xi\mathbf{e}_1$ and $\boldsymbol{\eta} = \eta\mathbf{e}_2$.*

Proof. We prove this theorem by straightforward calculation:

$$\begin{aligned}
&F^{h'}(u,v)\\
&= \int_{\mathbb{R}^2} f(x-\xi, y-\eta)\, e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})}\, dx\, dy\\
&\overset{\substack{\boldsymbol{x}-\boldsymbol{\xi}=\boldsymbol{x}'\\ \boldsymbol{y}-\boldsymbol{\eta}=\boldsymbol{y}'}}{=} \int_{\mathbb{R}^2} f(x', y')\, e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}'+\boldsymbol{v}\wedge\boldsymbol{y}')} e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{\xi}+\boldsymbol{v}\wedge\boldsymbol{\eta})}\, dx\, dy\\
&= e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{\xi}+\boldsymbol{v}\wedge\boldsymbol{\eta})} F^h(u,v)
\end{aligned}$$

Hence, the theorem is proved. □

[3] Note that, since we use the field $\mathbb{R}$, the multilinearity is only valid for real factors.

The shift theorem of the complex FT is closely related to the modulation theorem. The relation is even more general: we have a so-called symmetry theorem, which yields the Fourier transform of a signal simply by the inverse Fourier transform of the signal. We can formulate this theorem for the HFT2 as well:

Theorem 9.3.4 (Symmetry of the HFT2). *Let $f(x,y)$ be a $\mathcal{H}_2$-valued signal and $F^h(u,v)$ its HFT2. Then, the HFT2 of ${F^h}^\dagger(x,y)$ reads $f^\dagger(u,v)$ (where $\cdot^\dagger$ indicates the reversion of the underlying geometric algebra).*

Proof. In this proof we notate the exponents in the form $ux\mathbf{e}_3 \wedge \mathbf{e}_1$ instead of $\boldsymbol{u} \wedge \boldsymbol{x}$. The HFT2 of ${F^h}^\dagger(x,y)$ reads

$$\begin{aligned}
&\int_{\mathbb{R}^2} {F^h}^\dagger(x,y)\, e^{2\pi(ux\mathbf{e}_3\wedge\mathbf{e}_1+vy\mathbf{e}_4\wedge\mathbf{e}_2)}\, dx\, dy \\
&= \int_{\mathbb{R}^4} (f(u',v')\, e^{2\pi(u'x\mathbf{e}_3\wedge\mathbf{e}_1+v'y\mathbf{e}_4\wedge\mathbf{e}_2)})^\dagger\, du'dv'\, e^{2\pi(ux\mathbf{e}_3\wedge\mathbf{e}_1+vy\mathbf{e}_4\wedge\mathbf{e}_2)}\, dx\, dy \\
&= \int_{\mathbb{R}^2} f^\dagger(u',v') \int_{\mathbb{R}^2} e^{2\pi(u'x\mathbf{e}_1\wedge\mathbf{e}_3+v'y\mathbf{e}_2\wedge\mathbf{e}_4)}\, e^{2\pi(ux\mathbf{e}_3\wedge\mathbf{e}_1+vy\mathbf{e}_4\wedge\mathbf{e}_2)}\, dx\, dy\, du'dv' \\
&= \int_{\mathbb{R}^2} f^\dagger(u',v')\delta(u-u')\delta(v-v')\, du'dv' = f^\dagger(u,v)
\end{aligned}$$

Note that in the commutative algebra $\mathcal{H}_2$ the order of the factors is not inverted by the reversion $\cdot^\dagger$ (so the reversion is an automorphism in $\mathcal{H}_2$). □

The shift theorem together with the symmetry theorem yield the modulation theorem of the HFT2 which we do not formulate explicitly.

Up to now there is no significant improvement in the formulation of the theorems, although some formulation might be more elegant. However, the next theorem shows that in the commutative algebra a closed formulation of the convolution theorem is possible. The 2-D convolution is defined as follows.

Definition 9.3.2 (2-D convolution). *Let $f(x,y), g(x,y)$ be two 2-D signals. The 2-D convolution $f * g$ is then defined by*

$$(f * g)(x,y) = \int_{\mathbb{R}^2} f(\xi,\eta)g(x-\xi,y-\eta)\, d\xi\, d\eta \quad . \tag{9.14}$$

In contrast to the convolution theorem of the QFT, the convolution theorem of the HFT2 can be formulated similarly to the convolution theorem of the complex Fourier transform.

Theorem 9.3.5 (Convolution theorem of the FHT2). *Let $f(x,y)$ and $g(x,y)$ be two 2-D signals and let $F^h(u,v)$ and $G^h(u,v)$ be their HFT2s, respectively. Then, the HFT2 of $f * g$ is equivalent to the pointwise product of F^h and G^h, i.e.*

$$\int_{\mathbb{R}^2} (f * g)(x,y) e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})}\, dx\, dy = F^h(u,v)G^h(u,v) \quad . \tag{9.15}$$

Proof. We obtain by straightforward calculation

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (f * g)(x, y) e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})}\, dx\, dy \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\xi, \eta) g(x-\xi, y-\eta)\, d\xi\, d\eta\, e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}+\boldsymbol{v}\wedge\boldsymbol{y})}\, dx\, dy \\
&\overset{\substack{x-\xi=x'\\ y-\eta=y'}}{=} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\xi, \eta) g(x', y') e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{\xi}+\boldsymbol{v}\wedge\boldsymbol{\eta})} e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}'+\boldsymbol{v}\wedge\boldsymbol{y}')}\, d\xi\, d\eta\, dx'\, dy' \\
&= F^h(u, v) G^h(u, v)
\end{aligned}
$$

and therefore, the theorem is proved. □

Of course, the convolution defined in Def. 9.3.2 can be formulated for discrete signals as well. Note that we obtain the *cyclic* convolution by the pointwise product in the frequency domain and not the *linear* convolution. If the latter is needed, the signal must be filled up by zeroes.

9.3.3 The Affine Theorem of the HFT2

The next theorem states an isomorphism between $\mathcal{H}_2$ and the two-fold (Cartesian) product of the complex algebra ($\mathbb{C}^2$, see also [58]). Though this theorem seems to be a pure mathematical result, it will be important for the subsequent theorems.

Theorem 9.3.6 **($\mathcal{H}_2 \cong \mathbb{C}^2$).** *The commutative hypercomplex algebra $\mathcal{H}_2$ is isomorphic to the two-fold (Cartesian) product of the complex algebra $\mathbb{C}^2$. For an arbitrary element $Z = a + b\mathbf{e}_1 \wedge \mathbf{e}_3 + c\mathbf{e}_2 \wedge \mathbf{e}_4 + d\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$ we obtain the representation $(\xi, \eta) = \big((a-d) + i(b+c)\,,\,(a+d) + i(b-c)\big) \in \mathbb{C}^2$.*

Proof. Consider the matrix representations of $Z \in \mathcal{H}_2$ and $z = (x + iy) \in \mathbb{C}$

$$
Z \cong a \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} + b \begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{bmatrix} + c \begin{bmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix} + d \begin{bmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&-1&0&0\\ 1&0&0&0 \end{bmatrix}
$$

$$
z \cong x \begin{bmatrix} 1&0\\ 0&1 \end{bmatrix} + y \begin{bmatrix} 0&-1\\ 1&0 \end{bmatrix}
$$

which can both easily be verified to be isomorphic.

The eigenvectors of the matrix representation of Z read

$$
e_1 = \begin{bmatrix} 1\\ -i\\ -i\\ -1 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 1\\ i\\ i\\ -1 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 1\\ -i\\ i\\ 1 \end{bmatrix} \qquad e_4 = \begin{bmatrix} 1\\ i\\ -i\\ 1 \end{bmatrix} \tag{9.16}
$$

and since these eigenvectors are independent of the coefficients a, b, c, d, they yield an eigenvalue transform which turns the matrix representation of any Z into diagonal form. The eigenvalues read

$$\begin{aligned} \xi &= (a-d)+i(b+c) & \xi^* &= (a-d)-i(b+c) \\ \eta &= (a+d)+i(b-c) & \eta^* &= (a+d)-i(b-c) \end{aligned}$$

and therefore, the matrix multiplication yields a pointwise product on (ξ, η). □

Note 9.3.1. The proof of The. 9.3.6 is only sketched because it is a special case of The. 9.4.1. In the following, we use this theorem in a less formal way, since we replace the i of the complex numbers by $\mathbf{e}_1 \wedge \mathbf{e}_3$ in the following. The reason for this is that we can write $Z = \xi b_1 + \eta b_2$ now, where $b_1 = (1 - \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4)/2$ and $b_2 = (1 + \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4)/2$.

Theorem 9.3.6 can be used to state a relation between the complex Fourier transform and the HFT2. Due to the isomorphism we can map the signal and the kernel to $\mathbb{C}^2$ and can perform all calculations in this representation. Afterwards, the HFT2-spectrum can be obtained from the two complex spectra. In this context, it is not surprising that the representation of the HFT2 kernel in $\mathbb{C}^2$ consists of two Fourier kernels.

Theorem 9.3.7 (Relation between FT2 and HFT2). *Let $f(x,y)$ be a two-dimensional $\mathcal{H}_2$-valued signal and let $(f_\xi(x,y), f_\eta(x,y))$ be its representation in $\mathbb{C}^2$. Furthermore, let $F_\xi(u,v)$ and $F_\eta(u,v)$ be the complex Fourier transforms of $f_\xi(x,y)$ and $f_\eta(x,y)$, respectively. Then, the HFT2 of $f(x,y)$ reads*

$$F^h(u,v) = F_\xi(u,v)b_1 + F_\eta(u,-v)b_2 \ . \tag{9.17}$$

Proof. We can rewrite the kernel of the HFT2 by

$$e^{2\pi(xu\mathbf{e}_3\wedge\mathbf{e}_1 + yv\mathbf{e}_4\wedge\mathbf{e}_2)} = e^{2\pi(xu+yv)\mathbf{e}_3\wedge\mathbf{e}_1} b_1 + e^{2\pi(xu-yv)\mathbf{e}_3\wedge\mathbf{e}_1} b_2 \ . \tag{9.18}$$

Therefore, we obtain for the HFT2 of $f(x,y) = f_\xi(x,y)b_1 + f_\eta(x,y)b_2$ by

$$\begin{aligned} F^h(u,v) &= \int_{\mathbb{R}^2} f(x,y)\, e^{2\pi(xu\mathbf{e}_3\wedge\mathbf{e}_1 + yv\mathbf{e}_4\wedge\mathbf{e}_2)}\, dx\, dy \\ &= \int_{\mathbb{R}^2} (f_\xi(x,y)b_1 + f_\eta(x,y)b_2) \\ &\qquad (e^{2\pi(xu+yv)\mathbf{e}_3\wedge\mathbf{e}_1} b_1 + e^{2\pi(xu-yv)\mathbf{e}_3\wedge\mathbf{e}_1} b_2)\, dx\, dy \\ &= \int_{\mathbb{R}^2} f_\xi(x,y)\, e^{2\pi(xu+yv)\mathbf{e}_3\wedge\mathbf{e}_1}\, dx\, dy\, b_1 \\ &\quad + \int_{\mathbb{R}^2} f_\eta(x,y)\, e^{2\pi(xu-yv)\mathbf{e}_3\wedge\mathbf{e}_1}\, dx\, dy\, b_2 \\ &= F_\xi(u,v)b_1 + F_\eta(u,-v)b_2 \end{aligned}$$

(note that the product in $\mathbb{C}^2$ is evaluated pointwise ($b_1 b_2 = 0$) and that the kernels in (9.18) are the kernels of the complex 2-D FTs $\mathcal{F}\{\cdot\}(u,v)$ and $\mathcal{F}\{\cdot\}(u,-v)$). □

Note that we can obtain the HFT2 of a real signal by the complex spectrum because the spectra F_ξ and F_η are equal in that case. Therefore, the extended (i.e. hypercomplex) representation of a real signal is calculated without increased computational effort!

The simple calculation of the HFT2 spectrum is not the only result of the relation between the complex FT and the HFT2. Using the last theorem, we can state the affine theorem in a straightforward way:

Theorem 9.3.8 (Affine theorem). *Let $F^h(u,v)$ be the HFT2 of a signal $f(x,y)$. Then, the HFT2 of the signal $f'(x,y) = f(x',y')$ reads*

$$F^{h'}(u,v) = \frac{1}{|\det A|}\left(F^h(u',v')b_1 + F^h(u'',v'')b_2\right) \tag{9.19}$$

where $(x',y')^T = A(x,y)^T$, $(u',v')^T = A^{-1}(u,v)^T$, and $(u'',v'')^T = A^{-1}(u,-v)^T$.

Proof. First, we decompose $f(x,y) = f_\xi(x,y)b_1 + f_\eta(x,y)b_2$. According to The. 9.3.7 we have

$$F^h(u,v) = F_\xi(u,v)b_1 + F_\eta(u,-v)b_2$$

where $F_\xi(u,v) = \mathcal{F}\{f_\xi(x,y)\}(u,v)$ and $F_\eta(u,v) = \mathcal{F}\{f_\eta(x,y)\}(u,-v)$. Consider now $f'(x,y) = f(x',y') = f_\xi(x',y')b_1 + f_\eta(x',y')b_2$. The HFT2 of f' is obtained by

$$F^{h'}(u,v) = \mathcal{F}\{f_\xi(x',y')\}(u,v)b_1 + \mathcal{F}\{f_\eta(x',y')\}(u,-v)b_2 \ .$$

According to the affine theorem of the complex FT, we have

$$\mathcal{F}\{f_\xi(x',y')\}(u,v) = \frac{1}{|\det A|}F_\xi(u',v') \qquad \text{and}$$
$$\mathcal{F}\{f_\eta(x',y')\}(u,-v) = \frac{1}{|\det A|}F_\eta(u'',v'')$$

and since $F^h(u,v)b_1 = F_\xi(u,v)b_1$ and $F^h(u,v)b_2 = F_\eta(u,v)b_2$ we obtain (9.19). □

Obviously, the affine theorem is more complicated for the HFT2 than for the complex FT. This results from the fact that the spatial coordinates and the frequency coordinates are not combined by a scalar product. This is some kind of drawback since the recognition of a rotated signal in frequency domain is more complicated. On the other hand, filtering can be performed nearly isotropic (rotation invariant), e.g. the concept of hypercomplex Gabor filters yields a lower energy dependency on the orientation in contrast to complex Gabor filters (see 11).

Last but not least, let us consider the energy of an $\mathcal{H}_2$-valued signal. The magnitude of a multi-vector M is obtained by $|M| = \sqrt{MM^\dagger}$. Consequently, the magnitude of an $\mathcal{H}_2$-valued number

$$h = a + b\mathbf{e}_1 \wedge \mathbf{e}_3 + c\mathbf{e}_2 \wedge \mathbf{e}_4 + d\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$$

also reads

$$|h| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{hh^\dagger} \ .$$

The energy of the HFT2 of a signal $f(x, y)$ is then obtained by

$$\int_{\mathbb{R}^2} F^h(u,v) {F^h}^\dagger(u,v)\, du\, dv \tag{9.20a}$$

$$= \int_{\mathbb{R}^6} f(x,y) e^{2\pi(\boldsymbol{x}\wedge\boldsymbol{u}+\boldsymbol{y}\wedge\boldsymbol{v})} e^{2\pi(\boldsymbol{u}\wedge\boldsymbol{x}'+\boldsymbol{v}\wedge\boldsymbol{y}')} f^\dagger(x',y')\, du\, dv\, dx\, dy\, dx'\, dy'$$

$$= \int_{\mathbb{R}^4} f(x,y)\delta(x-x')\delta(y-y') f^\dagger(x',y')\, dx\, dy\, dx'\, dy' \tag{9.20b}$$

$$= \int_{\mathbb{R}^2} f(x,y) f^\dagger(x,y)\, dx\, dy \tag{9.20c}$$

and therefore, the Parseval equation is satisfied by the FHT2. The energy of the HFT2 spectrum is equal to the energy of the signal.

The last subject of this section is the derivative theorem for the HFT2. It reads analogously to the derivative theorem of the QFT (The. 8.4.7)

$$\frac{\partial}{\partial x} f(x,y) \ \circ\!\!-\!\!\bullet \ 2\pi u \mathbf{e}_1 \wedge \mathbf{e}_3 F^h(u,v) \tag{9.21a}$$

$$\frac{\partial}{\partial y} f(x,y) \ \circ\!\!-\!\!\bullet \ 2\pi v \mathbf{e}_2 \wedge \mathbf{e}_4 F^h(u,v) \ . \tag{9.21b}$$

Finally, we have transferred all global concepts of the QFT to the HFT2.

9.4 The n-Dimensional Hypercomplex Fourier Analysis

In this section we generalize the commutative hypercomplex Fourier transform for arbitrary dimensions (HFTn). Firstly, we have to introduce an algebraic framework which is the systematic extension of $\mathcal{H}_2$: $\mathcal{H}_n$.

9.4.1 The Isomorphism between $\mathcal{H}_n$ and the 2^{n-1}-Fold Cartesian Product of $\mathbb{C}$

Consider the Clifford algebra $\mathbb{R}^+_{2n,0}$ and define a 2^n-dimensional hypercomplex number system based on the space which is induced by

$$i_j = \mathbf{e}_j \wedge \mathbf{e}_{n+j} \ , \quad j = 1, \ldots , n \ .$$

The basis elements of the number system are created by the following rules ($j = 1, \ldots , n,\ s \subseteq \{1, \ldots , n\}$[4])

$$i_j i_j = -1 = -i_\emptyset \tag{9.22a}$$
$$i_j i_s = i_s i_j = i_{\{j\}\cup s} \qquad j \notin s \tag{9.22b}$$
$$i_j i_s = i_s i_j = -i_{s\setminus\{j\}} \qquad j \in s \ . \tag{9.22c}$$

Obviously, span($i_s| \ s \subseteq \{1, \ldots , n\}$) and the Clifford product form a commutative 2^n-dimensional hypercomplex algebra which is denoted $\mathcal{H}_n$ in the sequel. The following lemma identifies this algebra.

Lemma 9.4.1 ($\mathbb{C}^{\otimes n} \cong \mathcal{H}_n$). *The algebra $\mathcal{H}_n \subset \mathbb{R}^+_{2n,0}$ which is formed by (9.22a,b,c) is isomorphic to the n-fold tensor product of the complex algebra.*

Proof. The basis vectors i_j in (9.22a) can be written as

$$i_j = 1 \otimes \ldots \otimes 1 \otimes \underset{\substack{\uparrow \\ j\text{th position}}}{i} \otimes 1 \otimes \ldots \otimes 1 \quad . \tag{9.23}$$

Obviously, the basis vectors satisfy (9.22a, b and c) which can be verified by straightforward calculations. □

Consider, for example, $n = 3$. Then, the isomorphism yields the following correspondences of the basis elements:

$$\begin{aligned}
i_\emptyset &= 1 \cong 1 \otimes 1 \otimes 1 \\
i_1 &= \mathbf{e}_1 \wedge \mathbf{e}_4 \cong i \otimes 1 \otimes 1 \\
i_2 &= \mathbf{e}_2 \wedge \mathbf{e}_5 \cong 1 \otimes i \otimes 1 \\
i_3 &= \mathbf{e}_3 \wedge \mathbf{e}_6 \cong 1 \otimes 1 \otimes i \\
i_{12} &= \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_5 \cong i \otimes i \otimes 1 \\
i_{13} &= \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_6 \cong i \otimes 1 \otimes i \\
i_{23} &= \mathbf{e}_2 \wedge \mathbf{e}_5 \wedge \mathbf{e}_3 \wedge \mathbf{e}_6 \cong 1 \otimes i \otimes i \\
i_{123} &= \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_5 \wedge \mathbf{e}_3 \wedge \mathbf{e}_6 \cong i \otimes i \otimes i
\end{aligned}$$

Lemma 9.4.2 (Matrix representation of $\mathbb{C}^{\otimes n}$ and $\mathcal{H}_n$). *The matrices I^n_s, $s \in \mathcal{P}(\{1, \ldots , n\})$ span the matrix representation of $\mathbb{C}^{\otimes n}$ (and therefore for $\mathcal{H}_n$, too). The matrices I^n_s are defined by*

$$I^0_\emptyset = 1 \tag{9.24a}$$
$$I^m_s = \begin{bmatrix} I^{m-1}_s & 0 \\ 0 & I^{m-1}_s \end{bmatrix} \tag{9.24b}$$
$$I^m_{s\cup\{m\}} = \begin{bmatrix} 0 & -I^{m-1}_s \\ I^{m-1}_s & 0 \end{bmatrix} \tag{9.24c}$$

[4] For the sake of short writing, the indices are sometimes denoted as sets in the sequel (e.g., $i_{123} = i_{\{1,2,3\}}$).

with $s \in \mathcal{P}(\{1, \dots, m-1\})$ *and* $1 \le m \le n$.

Proof. Let $\boldsymbol{v} = (v_1, \dots, v_n)$ be a vector of dimension n and $\mathrm{diag}(\boldsymbol{v})$ denotes the diagonal matrix $\begin{bmatrix} v_1 & 0 & \dots & & \\ 0 & v_2 & 0 & \dots & \\ & & \ddots & & \\ \dots & & 0 & v_n & \end{bmatrix}$.

Then, for $s \in \mathcal{P}(\{1, \dots, n\})$ we have the following identities:

$$I_\emptyset^n = \mathrm{diag}(1, \dots, 1) \tag{9.25a}$$

$$I_j^n I_j^n = -I_\emptyset^n \qquad \text{with } j \in \{1, \dots, n\} \tag{9.25b}$$

$$I_s^n I_j^n = I_j^n I_s^n = I_{s \cup \{j\}}^n \qquad \text{with } j \notin s \tag{9.25c}$$

$$I_s^n I_j^n = I_j^n I_s^n = -I_{s \setminus \{j\}}^n \qquad \text{with } j \in s \ . \tag{9.25d}$$

Hence, this matrix algebra follows the same multiplication rules as $\mathcal{H}_n$ and $\mathbb{C}^{\otimes n}$. Since all three algebras are of the same dimension, they are isomorphic as algebras. □

Consider again the case $n = 3$. The matrix representation of $\mathcal{H}_3$ is obtained by:

$$i_\emptyset \cong \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \qquad i_{123} \cong \begin{bmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}$$

$$i_1 \cong \begin{bmatrix} 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{bmatrix} \qquad i_{23} \cong \begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}$$

$$i_2 \cong \begin{bmatrix} 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix} \qquad i_{13} \cong \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{bmatrix}$$

$$i_3 \cong \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad i_{12} \cong \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The matrix representation of $\mathcal{H}_n$ has eigenvectors which are *independent* of the coefficients. Therefore, every element of $\mathcal{H}_n$ can be expressed by a diagonal matrix, which is obtained by a fixed eigenvalue transform.

Lemma 9.4.3 (Eigenvectors and eigenvalues of $\mathbb{C}^{\otimes n}$). *Let*

$$A^n = \sum_{s \in \mathcal{P}(\{1,\dots,n\})} k_s I_s^n$$

be the matrix representation of an arbitrary element of $\mathbb{C}^{\otimes n}$. Then, the matrix of eigenvectors (row vectors) of A^n is inductively constructed by

$$E^1 = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \qquad \textit{and} \tag{9.26a}$$

$$E^m = \begin{bmatrix} E^{m-1} & -iE^{m-1} \\ E^{m-1} & iE^{m-1} \end{bmatrix} \qquad \textit{with } 2 \le m \le n \ . \tag{9.26b}$$

The corresponding vector of eigenvalues reads $\boldsymbol{\eta}^n$ where $\operatorname{diag}(\boldsymbol{\eta}^n) = \boldsymbol{\eta}^n_\emptyset$ *and $\boldsymbol{\eta}^n_t$ is inductively defined by*

$$\boldsymbol{\eta}^0_t = k_t \qquad \textit{with } t \in \mathcal{P}(\{1,\dots,n\}) \textit{ and} \tag{9.27a}$$

$$\boldsymbol{\eta}^m_t = \begin{bmatrix} \boldsymbol{\eta}^{m-1}_t - i\boldsymbol{\eta}^{m-1}_{\{m\}\cup t} & 0 \\ 0 & \boldsymbol{\eta}^{m-1}_t + i\boldsymbol{\eta}^{m-1}_{\{m\}\cup t} \end{bmatrix} \ , \tag{9.27b}$$

where $t \in \mathcal{P}(\{m+1,\dots,n\})$ and $1 \le m \le n$.

Proof. We prove the lemma by induction over n. Firstly, let $n = 1$. Then we have

$$(E^1)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

and hence,

$$\begin{aligned} E^1 A^1 (E^1)^{-1} &= E^1(k_\emptyset I^1_\emptyset + k_1 I^1_1)(E^1)^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} k_\emptyset & -k_1 \\ k_1 & k_\emptyset \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \\ &= \begin{bmatrix} k_\emptyset - ik_1 & 0 \\ 0 & k_\emptyset + ik_1 \end{bmatrix} = \operatorname{diag}(\boldsymbol{\eta}^1) \end{aligned}$$

Now provide $n > 1$. The induction assumption reads

$$E^{n-1}A^{n-1}(E^{n-1})^{-1} = \mathrm{diag}(\boldsymbol{\eta}^{n-1}) \ , \tag{9.28}$$

for any A^{n-1}. Furthermore,

$$(E^n)^{-1} = \frac{1}{2}\begin{bmatrix} (E^{n-1})^{-1} & (E^{n-1})^{-1} \\ i(E^{n-1})^{-1} & -i(E^{n-1})^{-1} \end{bmatrix} \tag{9.29}$$

and according to Lemma 9.4.2, we obtain

$$A^n = \begin{bmatrix} A_\emptyset^{n-1} & -A_{\{n\}}^{n-1} \\ A_{\{n\}}^{n-1} & A_\emptyset^{n-1} \end{bmatrix} \ , \tag{9.30}$$

where $A_\emptyset^{n-1} = \sum_{s \in \mathcal{P}(\{1,\dots,n-1\})} k_s I_s^{n-1}$ and
$A_{\{n\}}^{n-1} = \sum_{s \in \mathcal{P}(\{1,\dots,n-1\})} k_{s\cup\{n\}} I_s^{n-1}$. Therefore, we have

$$\begin{aligned}
&E^n A^n (E^n)^{-1} \\
&= \tfrac{1}{2}\begin{bmatrix} E^{n-1} & -iE^{n-1} \\ E^{n-1} & iE^{n-1} \end{bmatrix}\begin{bmatrix} A_\emptyset^{n-1} & -A_n^{n-1} \\ A_n^{n-1} & A_\emptyset^{n-1} \end{bmatrix}\begin{bmatrix} (E^{n-1})^{-1} & (E^{n-1})^{-1} \\ i(E^{n-1})^{-1} & -i(E^{n-1})^{-1} \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} E^{n-1}(A_\emptyset^{n-1} - iA_n^{n-1}) & E^{n-1}(-A_n^{n-1} - iA_\emptyset^{n-1}) \\ E^{n-1}(A_\emptyset^{n-1} + iA_n^{n-1}) & E^{n-1}(-A_n^{n-1} + iA_\emptyset^{n-1}) \end{bmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\begin{bmatrix} (E^{n-1})^{-1} & (E^{n-1})^{-1} \\ i(E^{n-1})^{-1} & -i(E^{n-1})^{-1} \end{bmatrix} \\
&= \tfrac{1}{2}\begin{bmatrix} 2E^{n-1}(A_\emptyset^{n-1} - iA_n^{n-1})(E^{n-1})^{-1} & 0 \\ 0 & 2E^{n-1}(A_\emptyset^{n-1} + iA_n^{n-1})(E^{n-1})^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{\eta}_\emptyset^{n-1} - i\boldsymbol{\eta}_{\{n\}}^{n-1} & 0 \\ 0 & \boldsymbol{\eta}_\emptyset^{n-1} + i\boldsymbol{\eta}_{\{n\}}^{n-1} \end{bmatrix} \\
&= \mathrm{diag}(\boldsymbol{\eta}^n) \ ,
\end{aligned}$$

so the induction step is proved and therefore, the lemma is proved, too. □

Again, we present the explicit results for the case $n = 3$:

$$\begin{aligned}
\eta_1 = \eta_8^* &= (k_\emptyset - k_{12} - k_{13} - k_{23}) - i(k_1 + k_2 + k_3 - k_{123}) \\
\eta_2 = \eta_7^* &= (k_\emptyset + k_{12} + k_{13} - k_{23}) - i(-k_1 + k_2 + k_3 + k_{123}) \\
\eta_3 = \eta_6^* &= (k_\emptyset + k_{12} - k_{13} + k_{23}) - i(k_1 - k_2 + k_3 + k_{123}) \\
\eta_4 = \eta_5^* &= (k_\emptyset - k_{12} + k_{13} + k_{23}) - i(-k_1 - k_2 + k_3 - k_{123})
\end{aligned}$$

are the eigenvalues of the matrix representation of an arbitrary $\mathcal{H}_3$ element. The matrix of eigenvectors (row-vectors) is given by:

$$E^3 = \begin{bmatrix} 1 & -i & -i & -1 & -i & -1 & -1 & i \\ 1 & i & -i & 1 & -i & 1 & -1 & -i \\ 1 & -i & i & 1 & -i & -1 & 1 & -i \\ 1 & i & i & -1 & -i & 1 & 1 & i \\ 1 & -i & -i & -1 & i & 1 & 1 & -i \\ 1 & i & -i & 1 & i & -1 & 1 & i \\ 1 & -i & i & 1 & i & 1 & -1 & i \\ 1 & i & i & -1 & i & -1 & -1 & -i \end{bmatrix}$$

Theorem 9.4.1 **($\mathbb{C}^{\otimes n} \cong \mathbb{C}^{2^{n-1}}$).** *The n-fold tensor product of $\mathbb{C}$ is isomorphic to the 2^{n-1}-fold Cartesian product of $\mathbb{C}$.*

Proof. The proof follows from Lemma 9.4.3, since the eigenvalues of any matrix representation A^n are complex-valued and the eigenvectors do not depend on the coefficients of A^n. The vector of eigenvalues $\boldsymbol{\eta}^n = (\eta_1, \dots, \eta_{2^n})$ is Hermite symmetric (i.e. $\eta_i = \eta^*_{2^n-i}$, $i \in \{1, \dots, 2^n\}$). Hence, $\boldsymbol{\eta}^n$ is uniquely represented by $2^n/2 = 2^{n-1}$ complex values.

From the eigenvectors we obtain the mapping f^n which maps the matrix representation A^n of an arbitrary element of $\mathbb{C}^{\otimes n}$ onto

$$\mathrm{diag}(\boldsymbol{\eta}^n) = f^n(A^n) = E^n A^n (E^n)^{-1}$$

which is the matrix representation of an element of $\mathbb{C}^{2^{n-1}}$.

In order to show that f^n is a vector space isomorphism, we have to show that the kernel of f^n is $\{0^n\}$. Therefore, we must solve

$$f^n(A^n) = E^n A^n (E^n)^{-1} = 0^n \tag{9.31}$$

By multiplication of E^n from the right and $(E^n)^{-1}$ from the left we get the kernel

$$\mathrm{kern}(f^n) = (E^n)^{-1} 0^n E^n = 0^n \ . \tag{9.32}$$

This result already follows from $\mathrm{rank}(E^n) = 2^n$.

In order to show that f^n is an algebra isomorphism, we have to show that $f^n(A^n B^n) = f^n(A^n) f^n(B^n)$:

$$\begin{aligned} f^n(A^n B^n) &= E^n A^n B^n (E^n)^{-1} \\ &= E^n A^n (E^n)^{-1} E^n B^n (E^n)^{-1} \\ &= \mathrm{diag}(\boldsymbol{\eta}^n_A)\mathrm{diag}(\boldsymbol{\eta}^n_A) \\ &= \mathrm{diag}(\eta^A_1 \eta^B_1, \dots, \eta^A_{2^n-1}\eta^B_{2^n-1}, (\eta^A_{2^n-1}\eta^B_{2^n-1})^*, \dots, (\eta^A_1\eta^B_1)^*) \\ &= f^n(A^n) f^n(B^n) \ , \end{aligned}$$

where $\boldsymbol{\eta}^n_A = (\eta^A_1, \dots, \eta^A_{2^n})$ and $\boldsymbol{\eta}^n_B = (\eta^B_1, \dots, \eta^B_{2^n})$ are the vectors of the eigenvalues of A^n and B^n, respectively. □

Hence, we have identified the algebra $\mathcal{H}_n$ and additionally we have obtained an isomorphism to $\mathbb{C}^{2^{n-1}}$ which will be very useful later on. Using this algebraic framework we introduce now the HFTn.

9.4.2 The n-Dimensional Hypercomplex Fourier Transform

In this section we introduce the HFTn and transfer some theorems from the two-dimensional case. Actually, all but the relation theorem and the affine theorem are formulated for the n-D case. The relation theorem can be formulated for arbitrary but fixed n. A formulation for all n would be *very* technical and therefore hard to understand. The same situation holds with the affine theorem. Additionally, due to their structure these theorems have little practical relevance for high dimensions.

Definition 9.4.1 (n-dimensional HFT). *The n-dimensional commutative hypercomplex Fourier transform $F^h(\boldsymbol{u})$ of an n-dimensional signal $f(\boldsymbol{x})$ is defined by ($\boldsymbol{u} = (u_1, \ldots, u_n)^T, \boldsymbol{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$)*

$$F^h(\boldsymbol{u}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{2\pi \sum_{j=1}^n u_j x_j \mathbf{e}_{n+j} \wedge \mathbf{e}_j}\, d^n\boldsymbol{x} \ . \tag{9.33}$$

Note that due to the commutativity of $\mathcal{H}_n$ we can factorize the kernel to n exponential functions.

We do not prove every theorem of Sec. 9.3 for the n-dimensional case, since most proofs are straightforward extensions to the HFT2. Nevertheless, we state the most important ones informally:

- The HFTn is unique and the inverse transform reads

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^n} F^h(\boldsymbol{u})\, e^{2\pi \sum_{j=1}^n u_j x_j \mathbf{e}_j \wedge \mathbf{e}_{n+j}}\, d^n\boldsymbol{u} \ . \tag{9.34}$$

- As in the two-dimensional case, the HFTn of a real signal has the same coefficients as the n-D Clifford spectrum. The reason for this lies in the fact, that all multiplications in the CFT are ordered with respect to the index set. Consequently, no n-blade occurring in the kernel is inverted due to permutations. For non-permuted blades, the multiplication tables of $\mathbb{R}_{0,n}$ and $\mathcal{H}_n$ are identical. If the signal is not real-valued, the spectra differ in general, because there are products of the form $\mathbf{e}'_j\mathbf{e}'_k$ with $j > k$ in the CFT (where $\mathbf{e}'_j$ are the basis one-vectors of $\mathbb{R}_{0,n}$). The result is then of course $-\mathbf{e}'_{kj}$ instead of i_{kj} in the algebra $\mathcal{H}_n$.
- The shift theorem and the symmetry theorem read according to theorems 9.3.3 and 9.3.4, respectively

$$f(\boldsymbol{x} - \boldsymbol{\xi}) \circ\!\!-\!\!\bullet\ e^{2\pi \sum_{j=1}^n u_j \xi_j \mathbf{e}_{n+j} \wedge \mathbf{e}_j} F^h(\boldsymbol{u}) \tag{9.35}$$

and

$$F^{h\dagger}(\boldsymbol{x}) \circ\!\!-\!\!\bullet\ f^\dagger(\boldsymbol{u}) \ , \tag{9.36}$$

(where $f(\boldsymbol{x}) \circ\!\!-\!\!\bullet\ F^h(\boldsymbol{u})$, i.e. $F^h(\boldsymbol{u})$ is the HFTn of $f(\boldsymbol{x})$). Note that there is no such factorized version of the shift theorem possible for the CFT and $n \geq 3$.

– The n-dimensional convolution is defined by

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^n} f(\boldsymbol{\xi}) g(\boldsymbol{x} - \boldsymbol{\xi}) \, d^n \boldsymbol{\xi} \tag{9.37}$$

and the convolution theorem reads

$$(f * g)(\boldsymbol{x}) \circ\!\!-\!\!\bullet \; F^h(\boldsymbol{u}) \, G^h(\boldsymbol{u}) \ . \tag{9.38}$$

– The Parseval equation is satisfied by the HFTn which can be verified by a straightforward calculation like in the two-dimensional case.
– The derivative theorem reads

$$\frac{\partial}{\partial x_i} f(\boldsymbol{x}) \circ\!\!-\!\!\bullet \; 2\pi u_i \mathbf{e}_i \wedge \mathbf{e}_{n+i} F^h(\boldsymbol{u}) \ .$$

The only theorems for which we do not have the n-D extensions yet, are theorems 9.3.7 and 9.3.8 (relation FT and HFT and affine theorem). Though we can state the two theorems for any fixed dimension n, we cannot formulate them explicitly for arbitrary n. Nevertheless, we describe how to design the theorems for any n.

Due to The. 9.4.1 we can decompose the (hypercomplex) signal f into 2^{n-1} complex signals. We can do the same with the kernel of the FHTn. The kernel is not only decomposed into 2^{n-1} complex functions but into 2^{n-1} complex *exponential* functions (complex Fourier kernels). In order to understand why this is true, consider the coefficients of the hypercomplex kernel. We obtain

$$k_s = (-1)^{|s|} \prod_{j \in \{1,\dots,n\} \setminus s} \cos(2\pi u_j x_j) \prod_{l \in s} \sin(2\pi u_l x_l) \tag{9.39}$$

for $s \in \mathcal{P}(\{1, \dots, n\})$.

By calculating the eigenvalues (9.27b), one factor changes into an exponential function in each step. Consider, for example, the first step. For $t \in \mathcal{P}(\{2, \dots, n\})$ we obtain

$$\boldsymbol{\eta}_t^1 = \begin{bmatrix} e^{2\pi u_1 x_1} c_t(\boldsymbol{x}, \boldsymbol{u}) & 0 \\ 0 & e^{-2\pi u_1 x_1} c_t(\boldsymbol{x}, \boldsymbol{u}) \end{bmatrix} \tag{9.40}$$

where $c_t(\boldsymbol{x}, \boldsymbol{u}) = (-1)^{|t|} \prod_{j \in \{2,\dots,n\} \setminus t} \cos(2\pi u_j x_j) \prod_{l \in t} \sin(2\pi u_l x_l)$.

The eigenvalues of the kernel of the HFT are the *Fourier kernels for all possible sign-permutations*, i.e. $e^{-i2\pi(\pm x_1 u_1 \pm \dots \pm x_n u_n)}$. Since there are always two exponential functions pairwise conjugated, we have 2^{n-1} different Fourier kernels left.

Now, we have 2^{n-1} signals and 2^{n-1} Fourier transforms. According to the isomorphism we can calculate the HFT by calculating the 2^{n-1} complex transforms and applying the inverse mapping $(f^n)^{-1}$.

Using this knowledge, we can also state an n-dimensional affine theorem by applying the affine theorem for the complex FT to each of the 2^{n-1} transforms. This would result in a sum of 2^{n-1} HFT which would be applied in 2^{n-1} different coordinate systems. Since there is no practical relevance for such a complicated theorem we omit it.

9.5 Conclusion

We have shown in this chapter that the CFT and the QFT can be replaced by the HFTn and the HFT2, respectively. The commutative hypercomplex transforms yield a spectral representation which is as rich as the Clifford spectrum. We have stated several theorems, among those the isomorphism between the n-fold tensor product of $\mathbb{C}$ and the 2^{n-1}-fold Cartesian product of $\mathbb{C}$ is the theoretically most important result. This theorem makes it possible to calculate the hypercomplex spectrum of a signal from the complex spectrum.

The commutative algebra $\mathcal{H}_n$ makes the analytic and numerical calculations easier. We can extend complex 1-D filters to hypercomplex n-D filters by the tensor product. We do not take care of the order of the operation as for the CFT: the shift theorem, the convolution theorem and the affine theorem are easier to formulate. Furthermore, we will be able to state a simple fast algorithm in Chap. 10.

Additionally, the two domains of the HFTn can be visualized by two orthogonal n-D subspaces in a common $2n$-D space. This point of view can lead to further concepts, e.g. a decayed Fourier transform (or Laplace transform), fractional Fourier transforms (similar to [142]), and so on.

Furthermore, the design of hypercomplex filters has to be considered more closely. Our present and future aim is to develop new multi-dimensional concepts which are not only a 'blow-up' of 1-D concepts, but an intrinsic n-D extension, which includes a *new quality* of filter properties.