8. Non-commutative Hypercomplex Fourier Transforms of Multidimensional Signals*

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8.1 Introduction

Harmonic transforms, and among those especially the Fourier transform, play an essential role in mathematical analysis, in almost any part of modern physics, as well as in electrical engineering. The analysis of the following four chapters is motivated by the use of the Fourier transform in signal processing. It turns out that some powerful concepts of one-dimensional signal theory can hardly be carried over to the theory of n-dimensional signals by using the complex Fourier transform. We start by introducing and studying the hypercomplex Fourier transforms in the following two chapters. In this chapter representations in non-commutative algebras are investigated, while chapter 9 is concerned with representations in commutative hypercomplex algebras. After these rather theoretical investigations we turn towards practice in chapter 10 where fast algorithms for the transforms are presented and in chapter 11 where local quaternion-valued LSI-filters based on the quaternionic Fourier transform are introduced and applied to image processing tasks.

* This work has been supported by German National Merit Foundation and by DFG Grants So-320-2-1, So-320-2-2, and Graduiertenkolleg No. 357.

In this chapter we will introduce hypercomplex Fourier transforms. The main motivation lies in the following two facts:

1. *The basis functions of the complex Fourier transform of arbitrary dimension n are intrinsically one-dimensional.*
2. *The symmetry selectivity of the 1-D complex Fourier transform is not carried forward completely to n-D by a complex transform.*

The first point refers to the fact that the basis functions of the complex Fourier transform look like plane waves, i.e. they vary along one orientation while being constant within the orthogonal $(n-1)$-dimensional hyperplane. This turns out to be a severe restriction in the analysis of the local structure of multidimensional signals. The implications of introducing transforms with intrinsically multidimensional basis functions will be regarded in chapter 11 for the case $n = 2$. The second point is important since the phase concept depends on the symmetry selectivity of the transform. E.g. the local phase of a signal is defined as the angular phase of the complex number made up of a real filter-response of the locally even signal component and the imaginary filter-response of the locally odd component. Extending the phase concept to higher dimensions we need a representation handling more than two symmetry components separately. This second point is directly related to the first one, since the introduction of a transform with higher symmetry selectivity leads directly to intrinsically multidimensional basis functions.

The structure of this chapter is as follows. In section 8.2 we consider several 1-D harmonic transforms. Among these transforms are those which map real-valued functions to real-valued, to complex-valued, and to vector-valued ones. We compare these transforms and thus motivate the introduction of multidimensional transforms with values in hypercomplex algebras. In 2-D the quaternionic Fourier transform (QFT) is such a transform. The QFT will be introduced and compared to real- and complex-valued transforms in section 8.3. In section 8.4 a hierarchy of 2-D transforms is introduced which is based on the symmetry selectivity of the transforms. Furthermore the main theorems for the QFT like the shift-theorem, Rayleigh's theorem, and some more are proven in this section. The definition of n-D transforms with values in Clifford algebras, i.e. in non-commutative hypercomplex algebras, is given in section 8.5 as an extension of the QFT. Before concluding this chapter we give a short overview of the literature on hypercomplex Fourier transforms in section 8.6 which seems of interest because the field is rather disjointed still.

8.2 1-D Harmonic Transforms

Before delving into the theory of hypercomplex transforms, we present some of the well-known harmonic transforms. In this section we restrict ourselves to 1-D signals and the corresponding 1-D transforms. The signals considered are assumed to be square integrable and real-valued: $f \in L^2(\mathbb{R}, \mathbb{R})$. For these signals the transforms considered in the following are guaranteed to exist.

Definition 8.2.1 (Cosine transform and sine transform).
For $f \in L^2(\mathbb{R}, \mathbb{R})$

$$F_c(u) = 2 \int_0^\infty f(x) \cos(2\pi ux) dx$$

is called the cosine transform of f. *Analogously*

$$F_s(u) = 2 \int_0^\infty f(x) \sin(2\pi ux) dx$$

defines the sine transform of f.

The trigonometric transforms as defined above take no account of f to the left of the origin. Thus, since in signal processing we are interested in complete transforms, we modify definition 8.2.1 slightly.

Definition 8.2.2 (C-transform and S-transform). *For* $f \in L^2(\mathbb{R}, \mathbb{R})$ *we define the two transforms* $\mathcal{C} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R})$ *and* $\mathcal{S} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R})$, *where*

$$\mathcal{C}\{f\}(u) = C(u) = \int_{\mathbb{R}} f(x) \cos(2\pi ux) dx$$

is called the C*-transform of* f. *Analogously*

$$\mathcal{S}\{f\}(u) = S(u) = \int_{\mathbb{R}} f(x) \sin(2\pi ux) dx$$

defines the S*-transform of* f.

Since each transform takes account either of the even or the odd part of f neither the C- nor the S-transform is invertible. If a complete transform is desired, both transforms have to be combined. We will show three different combinations of the C- and the S-transform, all of which lead to complete and thus invertible transforms.

Definition 8.2.3 (1-D Hartley transform).
Consider $f \in L^2(\mathbb{R}, \mathbb{R})$. *Then,* $\mathcal{H} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R})$, *with*

$$\mathcal{H}\{f\}(u) = H(u) = \mathcal{C}\{f\}(u) + \mathcal{S}\{f\}(u)$$

is called the Hartley transform of f.

Definition 8.2.4 (1-D Fourier transform). *Let* $f \in L^2(\mathbb{R}, \mathbb{R})$. *Then,* $\mathcal{F} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{C})$, *with*

$$\mathcal{F}\{f\}(u) = F(u) = \mathcal{C}\{f\}(u) - i\mathcal{S}\{f\}(u)$$

is the Fourier transform of f.

Finally, we introduce a transform which results from combining the C- and the S-transform into a vector.

Definition 8.2.5 (Trigonometric vector transform).
For any square-integrable one-dimensional real signal $f \in L^2(\mathbb{R}, \mathbb{R})$ we define a vector-valued transform $\mathcal{V} : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R}^2)$ by

$$\mathcal{V}\{f\}(u) = V(u) = \begin{pmatrix} \mathcal{C}\{f\}(u) \\ \mathcal{S}\{f\}(u) \end{pmatrix} .$$

Theorem 8.2.1. *The Hartley transform, the Fourier transform, and the trigonometric vector transform are invertible.*

The main difference between the Hartley transform on the one hand and the Fourier and the vector-valued transform on the other hand is that the Hartley transform does not separate even signal components from odd ones while the others do.

A question that often arises when talking about hypercomplex spectral transforms is: *Do we really need this complicated mathematics of hypercomplex algebras? Or can we do the same using real numbers or vectors?* The answer is: We can do the same using real numbers or vectors, but in fact using hypercomplex numbers makes things easier and more natural rather than complicated. This is at least true for the applications we have in mind and which we will demonstrate in the following chapters. We can partly explain this on the example of the complex Fourier transform and the vector-valued transform. Both transforms are complete, both transforms separate even from odd signal components. Thus, insofar the transforms are equivalent to each other. However, there are properties of the transforms which can be expressed very naturally only using the complex transform. For demonstration purpose we merely mention the shift theorem and the Hermite symmetry of a real signal's Fourier transform. The shift theorem of the Fourier transform describes how the transform of a signal varies when the signal is shifted. If the signal f is shifted by d, its Fourier transform is multiplied by a phase factor $\exp(-2\pi\, i\, du)$. Thus, a shift in spatial domain corresponds to multiplication of the complex transform by a complex number. Expressing this theorem for the vector-valued transform is of course possible. However, the algebraic frame would have to be extended to include not only vectors but also square matrices. The Hermite symmetry of the Fourier transform of a real signal is expressed by $F^*(u) = F(-u)$ which immediately explains the redundancy of the spectrum. There is no special notation for expressing this in vector algebra. These two simple examples already explain why the complex Fourier transform is more convenient than the vector-valued transform. Similar arguments apply for the introduction of hypercomplex numbers for signals of higher dimension.

8.3 2-D Harmonic Transforms

8.3.1 Real and Complex Harmonic Transforms

Again, we start with defining real trigonometric transforms from which we will derive the transforms of interest in this chapter.

Definition 8.3.1.
Let f be a real two-dimensional square-integrable signal $f \in L^2(\mathbb{R}^2, \mathbb{R})$. Then we define the transforms $\mathcal{CC}, \mathcal{SC}, \mathcal{CS}, \mathcal{SS} : L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R})$ by

$$\mathcal{CC}\{f\}(\boldsymbol{u}) = CC(\boldsymbol{u}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \cos(2\pi ux) \cos(2\pi vy) d^2\boldsymbol{x} \tag{8.1}$$

$$\mathcal{SC}\{f\}(\boldsymbol{u}) = SC(\boldsymbol{u}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \sin(2\pi ux) \cos(2\pi vy) d^2\boldsymbol{x} \tag{8.2}$$

$$\mathcal{CS}\{f\}(\boldsymbol{u}) = CS(\boldsymbol{u}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \cos(2\pi ux) \sin(2\pi vy) d^2\boldsymbol{x} \tag{8.3}$$

$$\mathcal{SS}\{f\}(\boldsymbol{u}) = SS(\boldsymbol{u}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \sin(2\pi ux) \sin(2\pi vy) d^2\boldsymbol{x}. \tag{8.4}$$

We could have started in def. 8.3.1 with an 2-D C- and S-transform. However, the four transforms allow the construction of more general transforms than the C- and S-transform, which can in fact be constructed from the transforms of def. 8.3.1 by linear combination due to the addition theorem of the sine and cosine function. Actually, the introduction of the four separable transforms is crucial four the following analysis.

As it is possible to construct the 1-D Hartley- and Fourier-transform from the C- and the S-transform, we can combine the separable trigonometric transforms given in def. 8.3.1 in different ways to yield the well-known 2-D spectral transforms.

Definition 8.3.2 (2-D Hartley and Fourier transform). *Let f be a real 2-D signal $f \in L^2(\mathbb{R}^2, \mathbb{R})$. The 2-D Hartley transform of f is then given by*

$$\mathcal{CC}\{f\}(\boldsymbol{u}) + \mathcal{SC}\{f\}(\boldsymbol{u}) + \mathcal{CS}\{f\}(\boldsymbol{u}) + \mathcal{SS}\{f\}(\boldsymbol{u}) = \mathcal{H}\{f\}(\boldsymbol{u}) = H(\boldsymbol{u}).$$

The 2-D Fourier transform of f is

$$\mathcal{CC}\{f\}(\boldsymbol{u}) - \mathcal{SS}\{f\}(\boldsymbol{u}) - i(\mathcal{CS}\{f\}(\boldsymbol{u}) + \mathcal{SC}\{f\}(\boldsymbol{u}) = \mathcal{F}\{f\}(\boldsymbol{u}) = F(\boldsymbol{u}).$$

Definition 8.3.3 (2-D Trigonometric vector transform).
A vector-valued transform of $\mathcal{V} : L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R}^4)$ is given by

$$\mathcal{V}\{f\}(\boldsymbol{u}) = V(\boldsymbol{u}) = \begin{pmatrix} \mathcal{CC}\{f\}(\boldsymbol{u}) \\ \mathcal{SC}\{f\}(\boldsymbol{u}) \\ \mathcal{CS}\{f\}(\boldsymbol{u}) \\ \mathcal{SS}\{f\}(\boldsymbol{u}) \end{pmatrix}.$$

In section 8.2 we saw that it is advantageous to replace the the transform $\mathcal{V}$ with values in $\mathbb{R}^2$ by the Fourier transform with values in $\mathbb{C}$. Actually, $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic as vector spaces. However, $\mathbb{C}$ has an additional algebraic structure. In the following section we will introduce a 2-D transform which adds an algebraic structure to the values of the 2-D $\mathcal{V}$-transform by replacing $\mathbb{R}^4$ by $\mathbb{H}$.

8.3.2 The Quaternionic Fourier Transform (QFT)

Definition 8.3.4 (Quaternionic Fourier transform). *The quaternionic Fourier transform $\mathcal{F}_q : L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{H})$ is given by*

$$\begin{aligned}\mathcal{F}_q\{f\}(\boldsymbol{u}) &= F^q(\boldsymbol{u}) \\ &= \mathcal{CC}\{f\}(\boldsymbol{u}) - i\,\mathcal{SC}\{f\}(\boldsymbol{u}) - j\,\mathcal{CS}\{f\}(\boldsymbol{u}) + k\,\mathcal{SS}\{f\}(\boldsymbol{u}).\end{aligned}$$

The three symbols i, j, and k denote the imaginary units of the algebra of quaternions. The choice of the signs in Def. 8.3.4 will become clear below. We shortly review the quaternion algebra in the following.

The quaternions are a special Clifford algebra, namely $\mathbb{R}_{0,2}$. Historically, the algebra of quaternions is one of the predecessors of Clifford's geometric algebra. In 1843 quaternions were first introduced by Hamilton. To his honor the algebra is commonly denoted by the letter $\mathbb{H}$.

Definition 8.3.5. *The set*

$$\mathbb{H} = \{a + bi + cj + dk |\, a, b, c, d \in \mathbb{R}\}$$

together with the multiplication rules

$$ij = -ji = k \quad \text{and} \quad i^2 = j^2 = k^2 = -1,$$

as well as component-wise addition and multiplications by real numbers form an associative $\mathbb{R}$-algebra, called the quaternions.

The impulse for introducing quaternions was the quest for an algebra which was able to represent rotations in three-dimensional space. Later, when considering the polar representation of a quaternion, we will exploit this relationship between rotations and quaternions.

For later use we present some definitions and properties concerning $\mathbb{H}$. The *conjugate* of a quaternion

$$q = a + i\,b + j\,c + k\,d$$

is given by

$$\bar{q} = a - i\,b - j\,c - k\,d.$$

The *norm* of q is given by $|q| = \sqrt{q\bar{q}}$. It can be shown that $\mathbb{H}$ is a *normed algebra*, i.e. for $q_1, q_2 \in \mathbb{H}$ we have $|q_1||q_2| = |q_1 q_2|$. $\mathbb{H}$ forms a group under multiplication, i.e. there exist a unit element, namely $e = 1 \in \mathbb{H}$, and to each $q \in \mathbb{H}$ there exists a multiplicative inverse q^{-1} with $qq^{-1} = q^{-1}q = e$. The multiplicative inverse is given by $q^{-1} = \bar{q}/|q|^2$. For the components of a quaternion $qa + i\,b + j\,c + k\,d$ we sometimes write

$$a = \mathcal{R}q, \quad b = \mathcal{I}q, \quad c = \mathcal{J}q, \quad d = \mathcal{K}q.$$

There are three non-trivial involutions defined on $\mathbb{H}$:

$$\begin{aligned} \alpha : \mathbb{H} \to \mathbb{H}, q \mapsto \alpha(q) &= -iqi = a + ib - jc - kd, \\ \beta : \mathbb{H} \to \mathbb{H}, q \mapsto \beta(q) &= -jqj = a - ib + jc - kd, \\ \gamma : \mathbb{H} \to \mathbb{H}, q \mapsto \gamma(q) &= -kqk = a - ib - jc + kd, . \end{aligned} \tag{8.5}$$

These involutions will be used in order to extend the notion of Hermite symmetry from complex to quaternion-valued functions. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called *Hermite symmetric* or *hermitian* if $f(\boldsymbol{x}) = f^*(-\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. The notion of Hermite symmetry of a function is useful in the context of Fourier transforms since the Fourier transform of a real function owes this property.

Definition 8.3.6 (Quaternionic Hermite symmetry). *A function $f : \mathbb{R}^2 \to \mathbb{H}$ is called quaternionic hermitian if:*

$$f(-x, y) = \beta(f(x, y)) \quad \textit{and} \quad f(x, -y) = \alpha(f(x, y)) \quad , \tag{8.6}$$

for each $(x, y) \in \mathbb{R}^2$.

One main subject of chapter 11 is the local quaternionic phase of a signal. In order to define the phase we introduce the angular phase of a quaternion as follows.

Theorem 8.3.1. *Each quaternion q can be represented in the form*

$$q = |q|e^{i\phi}e^{k\psi}e^{j\theta} \textit{ with } (\phi, \theta, \psi) \in [-\pi, \pi[\times[-\pi/2, \pi/2[\times[-\pi/4, \pi/4]. \tag{8.7}$$

The triple (ϕ, θ, ψ) is called the angular phase of q.

The angular phase of a quaternion can be understood in terms of rotations. Any 3-D rotation about the origin can be expressed in terms of quaternions. The set of unit quaternions is the 3D unit hypersphere

$$S^3 = \{q \in \mathbb{H} |\, |q| = 1\}.$$

Let $q \in S^3$ be given by $q = \cos(\phi) + \boldsymbol{n}\sin(\phi)$, where $\boldsymbol{n}$ is a pure unit quaternion. Further let $\boldsymbol{x}$ be a pure quaternion, representing the three-dimensional vector $(x_1, x_2, x_3)^\top$. A rotation about the axis defined by $\boldsymbol{n}$ through the angle 2ϕ takes $\boldsymbol{x}$ to $\boldsymbol{x}' = q\boldsymbol{x}q^{-1}$. Thus, any unit quaternion q represents a rotation in $\mathbb{R}^3$.

In this interpretation the angles $\phi/2$, $\theta/2$ and $\psi/2$ are the Euler angles of the corresponding rotation[1].

[1] Note that the definition of the Euler angles in not unique. The above representation corresponds to a concatenation of rotations about the y-axis, the z-axis, and the x-axis.

Table 8.1. How to calculate the quaternionic phase-angle representation from a quaternion given in Cartesian representation

<table>
<tr><td colspan="4">$q = a + bi + cj + dk, \quad |q| = 1$</td></tr>
<tr><td colspan="4">$\psi = -\frac{arcsin(2(bc-ad))}{2}$</td></tr>
<tr><td rowspan="2">if $\psi \in]-\frac{\pi}{4}, \frac{\pi}{4}[$

$\phi = \frac{arg_i(q\beta(\bar{q}))}{2}$

$\theta = \frac{arg_j(\alpha(\bar{q})q)}{2}$</td><td colspan="3">if $\psi = \pm\frac{\pi}{4}$
choose</td></tr>
<tr><td colspan="2">$\phi = 0$

$\theta = \frac{arg_j(\gamma(\bar{q})q)}{2}$</td><td>or</td><td>$\theta = 0$

$\phi = \frac{arg_i(q\gamma(\bar{q}))}{2}$</td></tr>
<tr><td colspan="2">if $e^{i\phi}e^{k\psi}e^{j\theta} = -q$
and $\phi \geq 0$

$\phi \to \phi - \pi$</td><td colspan="2">if $e^{i\phi}e^{k\psi}e^{j\theta} = -q$
and $\phi < 0$

$\phi \to \phi + \pi$</td></tr>
</table>

8.4 Some Properties of the QFT

8.4.1 The Hierarchy of Harmonic Transforms

Before analyzing some properties of the QFT we present what we call the *hierarchy of harmonic transforms*. The hierarchy of transforms is understood in terms of selectivity of the transforms with respect to the specular symmetry of an analyzed signal: Let $L^2_s(\mathbb{R}^n, \mathbb{R})$ be the set of functions in $L^2(\mathbb{R}^n, \mathbb{R})$ with symmetry $s \in S_n = \{(s_1, \dots, s_n), s_i \in \{e, o\}\}$, where s_i is the symmetry (even or odd) with respect to x_i, $i \in \{1, \dots, n\}$. Furthermore, let $\mathcal{T}$ be an n-D harmonic transform, e.g. the Fourier transform:

$$\mathcal{T} \,:\, L^2(\mathbb{R}^n, \mathbb{R}) \to L^2(\mathbb{R}^n, V) \tag{8.8}$$

$$\mathcal{T} \,:\, L^2_s(\mathbb{R}^n, \mathbb{R}) \to L^2_s(\mathbb{R}^n, V_s). \tag{8.9}$$

Since all the transforms considered here are based on trigonometric integral kernels, the transforms preserve the symmetries of signals (see eq. (8.9)). The values of the transformed signal functions $\mathcal{T}\{f\}$ are supposed to lie in the real vector space V. In case of algebra-valued functions, V is the underlying $\mathbb{R}$-vector-space, e.g. $\mathbb{R}^2$ for complex-valued functions. V_s is supposed to be the smallest possible subspaces of V fulfilling (8.9). If the V_r and V_s, $r, s \in S$ intersect only in the zero-vector $V_r \cap V_s = \{(0, \dots, 0)\}$, $\mathcal{T}$ is said to separate signal components with symmetry s from those of symmetry r. The more symmetry components are separated by a transform, the higher this transform stands in the hierarchy of harmonic transforms.

In the case $n = 1$ we merely have to consider the Hartley transform $\mathcal{H}$, the trigonometric vector transform $\mathcal{V}$, and the Fourier transform $\mathcal{F}$. For the Hartley transform $\mathcal{H}$ we find $V = \mathbb{R}$. The even and odd components of a signal f are mixed in the transform $H(u)$ since $V_e = V_o = \mathbb{R}$. In contrast the 1-D Fourier transform and the trigonometric vector transform separate even from odd components of a real signal: While $V = \mathbb{R}^2$ in these cases, we find $V_e = \{(a,0)|a \in \mathbb{R}\} =: P_1\mathbb{R}^2$ and $V_o = \{(0,b)|b \in \mathbb{R}\} =: P_2\mathbb{R}^2$, thus $V_e \cap V_o = \{(0,0)\}$.

The symmetry selectivity of the Fourier transform, is also expressed by the fact, that the Fourier transform of a real signal is hermitian, i.e. $F(u) = F^*(-u)$. Thus, the real part of F is even, while its imaginary part is odd.

In the case $n = 2$ we consider the four transforms $\mathcal{H}$, $\mathcal{F}$, $\mathcal{F}^q$, and $\mathcal{V}$. For the Hartley and the Fourier transform we get similar results as for $n = 1$: For $\mathcal{H}$, we have $V = \mathbb{R}$ and $V_s = \mathbb{R}$ for all $s \in S_2$. For $\mathcal{F}$ we find $V = \mathbb{R}^2$, while $V_{ee} = V_{oo} = P_1\mathbb{R}^2$ and $V_{oe} = V_{eo} = P_2\mathbb{R}^2$. Thus, the 2-D Fourier transform separates the four symmetry components of a signal into two subspaces. In this case it is more natural to write $V_e = P_1\mathbb{R}^2$ and $V_o = P_2\mathbb{R}^2$. Here the indices e and o mean even and odd with respect to the *argument vector* of an n-D function $f : \mathbb{R}^n \to \mathbb{R}$. I.e. $f_e(\boldsymbol{x}) = f_e(-\boldsymbol{x})$ and $f_o(\boldsymbol{x}) = -f_o(-\boldsymbol{x})$. Finally, for $\mathcal{V}$ and $\mathcal{F}^q$ we get $V = \mathbb{R}^4$ and the four symmetry components are completely separated:

$$V_{ee} = P_1\mathbb{R}^4 \tag{8.10}$$
$$V_{oe} = P_2\mathbb{R}^4 \tag{8.11}$$
$$V_{eo} = P_3\mathbb{R}^4 \tag{8.12}$$
$$V_{oo} = P_4\mathbb{R}^4. \tag{8.13}$$

Thus, we get a three-level hierarchy of 2-D harmonic transforms, on the lowest level of which stands the Hartley transform. On the second level we find the complex Fourier transform while on the highest level the quaternionic Fourier transform and the trigonometric vector transform can be found. This hierarchy is visualized in figure 8.1.

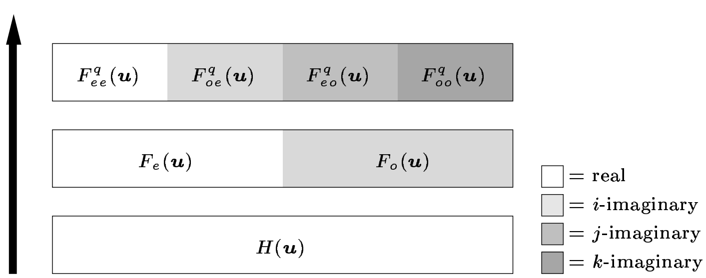

Fig. 8.1. The hierarchy of 2-D harmonic transforms

8.4.2 The Main QFT-Theorems

All harmonic transforms share some important properties. In his famous book on the Fourier transform Bracewell states that for every theorem about the Fourier transform there is a corresponding Hartley transform theorem ([28], p. 391). In order to put the QFT on a theoretically firm basis we derive the most important QFT-analogies to Fourier theorems in the following. First of all we rewrite the definition of the QFT given in def. 8.3.4.

Theorem 8.4.1. *The QFT of a 2-D signal is given by*

$$F^q(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-i2\pi u_1 x_1} f(\boldsymbol{x}) e^{-j2\pi u_2 x_2} d^2\boldsymbol{x}. \tag{8.14}$$

Proof. Euler's equation $exp(\boldsymbol{n}\phi) = \cos(\phi) + \boldsymbol{n}\sin(\phi)$ holds for any *pure unit* quaternion $\boldsymbol{n}$. Thus, it applies to the two exponentials in theorem 8.4.1 where we have $\boldsymbol{n} = i$ and $\boldsymbol{n} = j$, respectively. Expanding the product of the two exponentials expressed as sums via Euler's equality gives the expression in def. 8.3.4. □

For clarity we depict the basis functions of the complex Fourier transform and the QFT in figures 8.4.2 and 8.4.2, respectively. The small images show the real part of the basis functions in the spatial domain for fixed frequency $\boldsymbol{u}$. The frequency-parameter varies from image to image. Since only the real component is shown, in case of the complex Fourier transform the imaginary component is missing while in case of the quaternionic Fourier transform three imaginary components exist which are not shown. It can be seen that the basis functions of the complex Fourier transform are intrinsically 1-D. They resemble plane waves. In contrast, the basis functions of the quaternionic Fourier transform are intrinsically 2-D. As the complex Fourier transform the quaternionic Fourier transform is an invertible transform.

Theorem 8.4.2 (Inverse QFT). *The QFT is invertible. The transform $\mathcal{G}$ given by*

$$\mathcal{G}\{F^q\}(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{i2\pi ux} F^q(\boldsymbol{u}) e^{j2\pi vy} d^2\boldsymbol{u} \tag{8.15}$$

is the inverse of the QFT.

Proof. By inserting (8.14) into the right hand side of (8.15) we get

$$\mathcal{G}\{F^q\}(\boldsymbol{x}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i2\pi ux} e^{-i2\pi ux'} f(\boldsymbol{x}') e^{-j2\pi vy'} e^{j2\pi vy} d^2\boldsymbol{u}\, d^2\boldsymbol{x}'. \tag{8.16}$$

Integrating with respect to $\boldsymbol{u}$ and taking into account the orthogonality of harmonic exponential functions this simplifies to

$$\begin{aligned}\mathcal{G}\{F^q\}(\boldsymbol{u}) &= \int_{\mathbb{R}^2} \delta(x - x') f(\boldsymbol{x}') \delta(y - y') d^2\boldsymbol{x}' \\ &= f(\boldsymbol{x}),\end{aligned} \tag{8.17}$$

thus $\mathcal{G} = \mathcal{F}_q^{-1}$. □

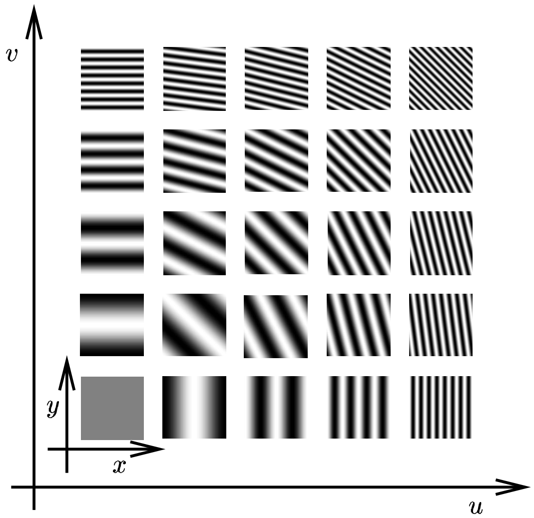

Fig. 8.2. The basis functions of the complex 2-D Fourier transform

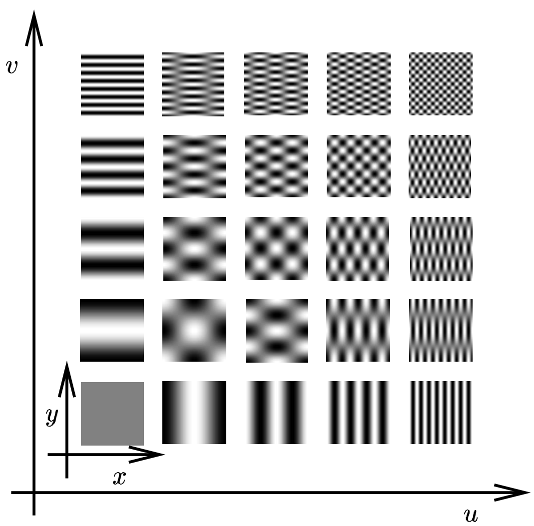

Fig. 8.3. The basis functions of the quaternionic Fourier transform

The convolution theorem of the Fourier transform states that convolution of two signals in the spatial domain corresponds to their pointwise multiplication in the frequency domain, i.e.

$$f(\boldsymbol{x}) = (g * h)(\boldsymbol{x}) \quad \Leftrightarrow \quad F(\boldsymbol{u}) = G(\boldsymbol{u})H(\boldsymbol{u}) \tag{8.18}$$

where f, g and h are two-dimensional signals and F, G and H are their Fourier transforms. We now give the corresponding QFT theorem.

Theorem 8.4.3 (Convolution theorem (QFT)). *Let f, g and h be two-dimensional signals and F^q, G^q and H^q their QFT's. In the following g is assumed to be real-valued, while h and consequently f may be quaternion-valued. Then,*

$$f(\boldsymbol{x}) = (g * h)(\boldsymbol{x}) \iff F^q(\boldsymbol{u}) = G^q_{\cdot e}(\boldsymbol{u})H^q(\boldsymbol{u}) + G^q_{\cdot o}(\boldsymbol{u})\beta(H^q(\boldsymbol{u})).$$

Here β denotes one of the three non-trivial automorphisms of the quaternion algebra as defined in (8.5). $G_{\cdot e}$ and $G_{\cdot o}$ are the components of G which are even or odd with respect to the second argument.

Proof. We prove the convolution theorem by directly calculating the QFT of the convolution integral:

$$\begin{aligned}
F^q(\boldsymbol{u}) &= \int_{\mathbb{R}^2} e^{-2\pi i x u} \left[\int_{\mathbb{R}^2} (g(\boldsymbol{x}')h(\boldsymbol{x}-\boldsymbol{x}'))d^2\boldsymbol{x}' \right] e^{-2\pi j y v} d^2\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} e^{-2\pi i x' u} g(\boldsymbol{x}') H^q(\boldsymbol{u}) e^{-2\pi j y' v} d^2\boldsymbol{x}' \\
&= \int_{\mathbb{R}^2} e^{-2\pi i x' u} g(\boldsymbol{x}') \cos(-2\pi y' v) H^q(\boldsymbol{u}) d^2\boldsymbol{x}' \\
&\quad + \int_{\mathbb{R}^2} e^{-2\pi i x' u} g(\boldsymbol{x}') j \sin(-2\pi y' v) \beta(H^q(\boldsymbol{u})) d^2\boldsymbol{x}' \\
&= G^q_{\cdot e}(\boldsymbol{u})H^q(\boldsymbol{u}) + G^q_{\cdot o}(\boldsymbol{u})\beta(H^q(\boldsymbol{u})),
\end{aligned} \tag{8.19}$$

which completes the proof. □

Analogously it can be shown, that

$$F^q(\boldsymbol{u}) = H^q(\boldsymbol{u})G^q_{e\cdot}(\boldsymbol{u}) + \alpha(H^q(\boldsymbol{u}))G^q_{o\cdot}(\boldsymbol{u}).$$

If h is a quaternion-valued function which QFT is real-valued the convolution theorem simplifies to

$$f(\boldsymbol{x}) = (g * h)(\boldsymbol{x}) \;\Rightarrow\; F^q(\boldsymbol{u}) = G^q(\boldsymbol{u})H^q(\boldsymbol{u}), \tag{8.20}$$

which is of the same form as the convolution theorem of the two-dimensional Fourier transform. This is an important fact, since later we will convolve real-valued signals with quaternionic Gabor filters, which QFT's are real-valued.

According to (8.20) in this case the convolution theorem can be applied as usually.

The energy of a signal is defined as the integral (or sum in the case of discrete signals) over the squared magnitude of the signal. Rayleigh's theorem states that the signal energy is preserved by the Fourier transform:

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d^2\boldsymbol{x} = \int_{\mathbb{R}^2} |F(\boldsymbol{u})|^2 d^2\boldsymbol{u} \quad , \tag{8.21}$$

where $F(\boldsymbol{u})$ is the Fourier transform of $f(\boldsymbol{x})$. Rayleigh's theorem is valid for arbitrary integer dimension of the signal. In mathematical terms Rayleigh's theorem states that the L^2-norm of a signal is invariant under the Fourier transform. We will show that the analogous statement for the QFT is true.

Theorem 8.4.4 (Rayleigh's theorem (QFT)).
The quaternionic Fourier transform preserves the L^2-norm of any real two-dimensional signal $f(\boldsymbol{x})$:

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d^2\boldsymbol{x} = \int_{\mathbb{R}^2} |F^q(\boldsymbol{u})|^2 d^2\boldsymbol{u}, \tag{8.22}$$

where $F^q(\boldsymbol{u})$ is the QFT of $f(\boldsymbol{x})$.

Proof. We make use of Rayleigh's theorem for the two-dimensional Fourier transform. Thus, we only have to prove that

$$\int_{\mathbb{R}^2} |F(\boldsymbol{u})|^2 d^2\boldsymbol{u} - \int_{\mathbb{R}^2} |F^q(\boldsymbol{u})|^2 d^2\boldsymbol{u} = 0 \quad . \tag{8.23}$$

Regarding the integrands and using the addition theorems of the sine and the cosine function we find out that

$$|F(\boldsymbol{u})|^2 = (\mathcal{CC}\{f\}(\boldsymbol{u}) - \mathcal{SS}\{f\}(\boldsymbol{u}))^2 + (\mathcal{SC}\{f\}(\boldsymbol{u}) + \mathcal{CS}\{f\}(\boldsymbol{u}))^2 \,, \tag{8.24}$$

while

$$|F^q(\boldsymbol{u})|^2 = \mathcal{CC}\{f\}^2(\boldsymbol{u}) + \mathcal{SC}\{f\}^2(\boldsymbol{u}) + \mathcal{CS}\{f\}^2(\boldsymbol{u}) + \mathcal{SS}\{f\}^2(\boldsymbol{u}). \tag{8.25}$$

Thus, the left hand side of (8.23) can be evaluated as follows:

$$\begin{aligned} &\int_{\mathbb{R}^2} |F(\boldsymbol{u})|^2 d^2\boldsymbol{u} - \int_{\mathbb{R}^2} |F^q(\boldsymbol{u})|^2 d^2\boldsymbol{u} \\ &= 2\int_{\mathbb{R}^2} (\mathcal{SC}\{f\}(\boldsymbol{u})\mathcal{CS}\{f\}(\boldsymbol{u}) - \mathcal{CC}\{f\}(\boldsymbol{u})\mathcal{SS}\{f\}(\boldsymbol{u}))\, d^2\boldsymbol{u}. \end{aligned} \tag{8.26}$$

The integrand in (8.26) is odd with respect to both arguments (since $\mathcal{S}$-terms are odd). Thus, the integral is zero which completes the proof. □

The shift theorem of the Fourier transform describes how the transform of a signal varies when the signal is shifted. If the signal f is shifted by $\boldsymbol{d}$, it is known that its Fourier transform is multiplied by a phase factor $\exp(-2\pi\, i\, \boldsymbol{d} \cdot \boldsymbol{x})$. How the QFT of f is affected by the shift is described by the following theorem.

Theorem 8.4.5 (Shift theorem (QFT)). *Let*

$$F^q(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-i2\pi ux} f(\boldsymbol{x}) e^{-j2\pi vy} d^2\boldsymbol{x} \tag{8.27}$$

and

$$F^q_s(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-i2\pi ux} f(\boldsymbol{x} - \boldsymbol{d}) e^{-j2\pi vy} d^2\boldsymbol{x} \tag{8.28}$$

be the QFT's of a 2-D signal f and a shifted version of f, respectively. Then, $F^q(\mathbf{u})$ and $F^q_s(\boldsymbol{u})$ are related by

$$F^q_s(\boldsymbol{u}) = e^{-i2\pi ud_1} F^q(\boldsymbol{u}) e^{-j2\pi vd_2} \quad . \tag{8.29}$$

If we denote the phase of $F^q(\boldsymbol{u})$ by $(\phi(\boldsymbol{u}), \theta(\boldsymbol{u}), \psi(\boldsymbol{u}))^\top$ then, as a result of the shift, the first and the second component of the phase undergo a phase-shift

$$\begin{pmatrix} \phi(\boldsymbol{u}) \\ \theta(\boldsymbol{u}) \\ \psi(\boldsymbol{u}) \end{pmatrix} \quad \to \quad \begin{pmatrix} \phi(\boldsymbol{u}) - 2\pi ud_1 \\ \theta(\boldsymbol{u}) - 2\pi vd_2 \\ \psi(\boldsymbol{u}) \end{pmatrix} . \tag{8.30}$$

Proof. Equation (8.29) follows from (8.27) and (8.28) by substituting $(\boldsymbol{x} - \boldsymbol{d})$ with $\boldsymbol{x}'$. If $F^q(\boldsymbol{u})$ has the polar representation

$$F^q(\boldsymbol{u}) = |F^q(\boldsymbol{u})| e^{i\phi(\boldsymbol{u})} e^{k\psi(\boldsymbol{u})} e^{j\theta(\boldsymbol{u})},$$

we find for the polar representation of $F^q_s(\boldsymbol{u})$

$$\begin{aligned} F^q_s(\boldsymbol{u}) &= e^{-i2\pi ud_1} F^q(\boldsymbol{u}) e^{-j2\pi vd_2} \\ &= e^{-i2\pi ud_1} |F^q(\boldsymbol{u})| e^{i\phi(\boldsymbol{u})} e^{k\psi(\boldsymbol{u})} e^{j\theta(\boldsymbol{u})} e^{-j2\pi vd_2} \\ &= |F^q(\boldsymbol{u})| e^{i(\phi(\boldsymbol{u}) - 2\pi ud_1)} e^{k\psi(\boldsymbol{u})} e^{j(\theta(\boldsymbol{u}) - 2\pi vd_2)}. \end{aligned}$$

This proves (8.30). □

In the shift theorem a shift of the signal in the spatial domain is considered. The effect of such a shift are the modulation factors shown in (8.29). In the following theorem we regard the converse situation: the signal is modulated in the spatial domain, and we ask for the effect in the quaternionic frequency domain.

Theorem 8.4.6 (Modulation theorem (QFT)).
Let $f(\boldsymbol{x})$ be a quaternion-valued signal and $F^q(\boldsymbol{u})$ its QFT. Further, let $f_m(\boldsymbol{x})$ be the following modulated version of $f(\boldsymbol{x})$:

$$f_m(\boldsymbol{x}) = e^{i2\pi u_0 x} f(\boldsymbol{x}) e^{j2\pi v_0 y}. \tag{8.31}$$

The QFT of $f_m(\boldsymbol{x})$ is then given by

$$\mathcal{F}_q\{f_m\}(\boldsymbol{u}) = F^q(\boldsymbol{u} - \boldsymbol{u}_0). \tag{8.32}$$

If $f_m(\boldsymbol{x})$ is a real modulated version of $f(\boldsymbol{x})$, i.e.

$$f_m(\boldsymbol{x}) = f(\boldsymbol{x}) \cos(2\pi x u_0) \cos(2\pi y v_0), \tag{8.33}$$

the QFT of $f_m(\boldsymbol{x})$ is given by

$$\begin{aligned}\mathcal{F}_q\{f_m\}(\boldsymbol{u}) = \frac{1}{4}(F^q(\boldsymbol{u} + \boldsymbol{u}_0) + F^q(u - u_0, v + v_0) \\ + F^q(u + u_0, v - v_0) + F^q(\boldsymbol{u} - \boldsymbol{u}_0)).\end{aligned} \tag{8.34}$$

Proof. First, we consider the QFT of

$$f_m(\boldsymbol{x}) = e^{i2\pi u_0 x} f(\boldsymbol{x}) e^{j2\pi v_0 y}.$$

By inserting f_m into the definition of the QFT we obtain

$$\begin{aligned} F_m^q(\boldsymbol{u}) &= \int_{\mathbb{R}^2} e^{-i2\pi u x} f_m(\boldsymbol{x}) e^{-j2\pi v y} d^2\boldsymbol{x} \\ &= \int_{\mathbb{R}^2} e^{-i2\pi (u-u_0) x} f(\boldsymbol{x}) e^{-j2\pi (v-v_0) y} d^2\boldsymbol{x} \\ &= F^q(\boldsymbol{u} - \boldsymbol{u}_0). \end{aligned}$$

For the second part of the proof we introduce the abbreviation $\mathfrak{f}(\boldsymbol{x}) = e^{-i2\pi u x} f(\boldsymbol{x}) e^{-j2\pi v y}$. Further, we use the notation

$$\begin{aligned} I_{ee}(\boldsymbol{u}_0) &= \int_{\mathbb{R}^2} \cos(2\pi u_0 x) \mathfrak{f}(\boldsymbol{x}) \cos(2\pi v_0 y) d^2\boldsymbol{x} \\ I_{oe}(\boldsymbol{u}_0) &= i \int_{\mathbb{R}^2} \sin(2\pi u_0 x) \mathfrak{f}(\boldsymbol{x}) \cos(2\pi v_0 y) d^2\boldsymbol{x} \\ I_{eo}(\boldsymbol{u}_0) &= j \int_{\mathbb{R}^2} \cos(2\pi u_0 x) \mathfrak{f}(\boldsymbol{x}) \sin(2\pi v_0 y) d^2\boldsymbol{x} \\ I_{oo}(\boldsymbol{u}_0) &= k \int_{\mathbb{R}^2} \sin(2\pi u_0 x) \mathfrak{f}(\boldsymbol{x}) \sin(2\pi v_0 y) d^2\boldsymbol{x}, \end{aligned}$$

where $I_{ee}(\boldsymbol{u}_0)$ is even with respect to both u_0 and to v_0, $I_{oe}(\boldsymbol{u}_0)$ is odd with respect to u_0 and even with respect to v_0 and so on. We can then write

$$\begin{aligned}
&\frac{1}{4}(F^q(\boldsymbol{u}+\boldsymbol{u}_0)+F^q(u-u_0,v+v_0) && (8.35)\\
&+F^q(u+u_0,v-v_0)+F^q(\boldsymbol{u}-\boldsymbol{u}_0))\\
=&\frac{1}{4}(I_{ee}(\boldsymbol{u}_0)+I_{oe}(\boldsymbol{u}_0)+I_{eo}(\boldsymbol{u}_0)+I_{oo}(\boldsymbol{u}_0))\\
&+\frac{1}{4}(I_{ee}(-u_0,v_0)+I_{oe}(-u_0,v_0)+I_{eo}(-u_0,v_0)+I_{oo}(-u_0,v_0))\\
&+\frac{1}{4}(I_{ee}(u_0,-v_0)+I_{oe}(u_0,-v_0)+I_{eo}(u_0,-v_0)+I_{oo}(u_0,-v_0))\\
&+\frac{1}{4}(I_{ee}(-\boldsymbol{u}_0)+I_{oe}(-\boldsymbol{u}_0)+I_{eo}(-\boldsymbol{u}_0)+I_{oo}(-\boldsymbol{u}_0))\\
=&I_{ee}(\boldsymbol{u}_0)=\mathcal{F}_q\{\cos(2\pi u_0 x)f(\boldsymbol{x})\cos(2\pi v_0 y)\}, && (8.36)
\end{aligned}$$

which completes the proof. □

Theorem 8.4.7 (Derivative theorem (QFT)).
Let f be a real two-dimensional signal, F^q its QFT, and $n = p + r, p, r \in \mathbb{N}$. Then

$$\mathcal{F}_q\left\{\frac{\partial^n}{\partial x^p \partial y^r} f\right\}(\boldsymbol{u}) = (2\pi)^n (iu)^p F^q(\boldsymbol{u})(jv)^r.$$

Proof. We prove the theorem for $(p,r) = (1,0)$ and $(p,r) = (0,1)$ starting with the first case. We have

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{i2\pi ux} F^q(\boldsymbol{u}) e^{j2\pi vy}. \tag{8.37}$$

Thus, it follows that

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \frac{\partial}{\partial x} \int_{\mathbb{R}^2} e^{i2\pi ux} F^q(\boldsymbol{u}) e^{j2\pi vy} = \int_{\mathbb{R}^2} e^{i2\pi ux} (i2\pi u F^q(\boldsymbol{u})) e^{j2\pi vy}.$$

Therefore, we have

$$\mathcal{F}_q\left\{\frac{\partial}{\partial x} f\right\}(\boldsymbol{u}) = i2\pi u F^q(\boldsymbol{u}). \tag{8.38}$$

Analogously we derive

$$\frac{\partial}{\partial y} f(\boldsymbol{x}) = \frac{\partial}{\partial y} \int_{\mathbb{R}^2} e^{i2\pi ux} F^q(\boldsymbol{u}) e^{j2\pi vy} = \int_{\mathbb{R}^2} e^{i2\pi ux} (F^q(\boldsymbol{u}) j2\pi v) e^{j2\pi vy},$$

which shows that

$$\mathcal{F}_q\left\{\frac{\partial}{\partial y} f\right\}(\boldsymbol{u}) = 2\pi F^q(\boldsymbol{u})(jv). \tag{8.39}$$

For general derivatives the theorem follows from successive application of first order derivatives. □

Theorem 8.4.8. *The QFT of a real two-dimensional signal f is quaternionic hermitian.*

Proof. We have shown before that the QFT of a real signal has the form

$$F^q(\boldsymbol{u}) = F^q_{ee}(\boldsymbol{u}) + iF^q_{oe}(\boldsymbol{u}) + jF^q_{eo}(\boldsymbol{u}) + kF^q_{oo}(\boldsymbol{u}).$$

Applying the automorphisms α and β yields

$$\begin{aligned}
\alpha(F^q(\boldsymbol{u})) &= F^q_{ee}(\boldsymbol{u}) + iF^q_{oe}(\boldsymbol{u}) - jF^q_{eo}(\boldsymbol{u}) - kF^q_{oo}(\boldsymbol{u}) \\
&= F^q_{ee}(u,-v) + iF^q_{oe}(u,-v) + jF^q_{eo}(u,-v) + kF^q_{oo}(u,-v) \\
&= F^q(u,-v) \qquad (8.40) \\
\beta(F^q(\boldsymbol{u})) &= F^q_{ee}(\boldsymbol{u}) - iF^q_{oe}(\boldsymbol{u}) + jF^q_{eo}(\boldsymbol{u}) - kF^q_{oo}(\boldsymbol{u}) \\
&= F^q_{ee}(-u,v) + iF^q_{oe}(-u,v) + jF^q_{eo}(-u,v) + kF^q_{oo}(-u,v) \\
&= F^q(-u,v), \qquad (8.41)
\end{aligned}$$

which proves the theorem according to definition 8.3.6. □

It can often happen that a signal undergoes an affine transformation in the spatial domain, which can be written as $f(\boldsymbol{x}) \to f(A\boldsymbol{x}+\boldsymbol{b})$, where $\boldsymbol{b} \in \mathbb{R}^2$ and $A \in Gl(2,\mathbb{R})$. In these cases it is desirable to know how this transformation affects the frequency representation F^q of f. The effect of the shift by $\boldsymbol{b}$ is already known from the shift theorem. It remains to work out how the frequency representation is transformed under a linear transformation of the spatial domain: $f(\boldsymbol{x}) \to f(A\boldsymbol{x})$. This is done by the following theorem.

Theorem 8.4.9 (Affine theorem (QFT)).
Let $f(\boldsymbol{x})$ be a real 2D signal and $F^q(\boldsymbol{u}) = \mathcal{F}_q\{f(\boldsymbol{x})\}(\boldsymbol{u})$ its QFT. Further, let A be the real regular 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \textit{with} \quad \det(A) = ad - bc \neq 0.$$

The QFT of $f(A\boldsymbol{x})$ is then given by

$$\begin{aligned}
\mathcal{F}_q\{f(A\boldsymbol{x})\}(\boldsymbol{u}) = \frac{1}{2\det(A)} \Big(& F^q(\det(B)B^{-1\top}\boldsymbol{u}) + F^q(B^{\top t}\boldsymbol{u}) \qquad (8.42) \\
& + i(F^q(\det(B)B^{-1\top}\boldsymbol{u}) - F^q(B^{\top t}\boldsymbol{u}))j \Big).
\end{aligned}$$

where we introduced the matrix

$$B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} =: \frac{A}{\det(A)}.$$

Furthermore, $A^\top$ denotes the transpose of A and A^t the transpose of A according to the minor diagonal:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad A^t = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Proof. The inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For the transformed coordinates we introduce the notation

$$A\boldsymbol{x} = \boldsymbol{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} dx' - by' \\ -cx' + ay' \end{pmatrix}.$$

Now we can express $\mathcal{F}_q\{f(A\boldsymbol{x})\}(\boldsymbol{u})$ using the coordinates $\boldsymbol{x}'$ in the following way:

$$\begin{aligned}
\mathcal{F}_q\{f(A\boldsymbol{x})\}(\boldsymbol{u}) &= \int_{\mathbb{R}^2} e^{-i2\pi ux} f(A\boldsymbol{x}) e^{-j2\pi vy} d^2\boldsymbol{x} \\
&= \frac{1}{\det(A)} \int_{\mathbb{R}^2} e^{-i2\pi u(d'x'-b'y')} f(\boldsymbol{x}') e^{-j2\pi v(-c'x'+a'y')} d^2\boldsymbol{x}' \\
&= \frac{1}{\det(A)} \int_{\mathbb{R}^2} e^{-i2\pi u(d'x-b'y)} f(\boldsymbol{x}) e^{-j2\pi v(-c'x+a'y)} d^2\boldsymbol{y}.
\end{aligned}$$

In order to complete the proof we still have to show that

$$\begin{aligned}
e^{-i2\pi u(d'x-b'y)} e^{-j2\pi v(-c'x+a'y)} = \frac{1}{2} \big(& e^{-i2\pi x(d'u+c'v)} e^{-j2\pi y(b'u+a'v)} \\
& + e^{-i2\pi x(d'u-c'v)} e^{-j2\pi y(-b'u+a'v)} \\
& - i e^{i2\pi x(-d'u+c'v)} e^{-j2\pi y(-b'u+a'v)} j \\
& + i e^{i2\pi x(-d'u-c'v)} e^{-j2\pi y(b'u+a'v)} j \big).
\end{aligned}$$

For a more compact form of (8.43) we introduce the abbriviations

$$\alpha = 2\pi vya', \quad \beta = 2\pi uyb', \quad \gamma = 2\pi vxc', \quad \delta = 2\pi uxd'$$

and we get the following expression:

$$\begin{aligned}
e^{-i(\delta-\beta)} e^{-j(-\gamma+\alpha)} = \frac{1}{2} \Big(& e^{i(-\delta-\gamma)} e^{j(-\beta-\alpha)} + e^{i(-\delta+\gamma)} e^{j(\beta-\alpha)} \\
& - k e^{i(\delta-\gamma)} e^{j(\beta-\alpha)} + k e^{i(\delta+\gamma)} e^{j(-\beta-\alpha)} \Big).
\end{aligned} \tag{8.43}$$

We evaluate the right-hand side:

$$\begin{aligned}
& \frac{1}{2} \Big(e^{i(-\delta-\gamma)} e^{j(-\beta-\alpha)} + e^{i(-\delta+\gamma)} e^{j(\beta-\alpha)} \\
& - i e^{-i(\delta-\gamma)} e^{j(\beta-\alpha)} j + i e^{-i(\delta+\gamma)} e^{j(-\beta-\alpha)} j \Big) \\
&= \frac{1}{2} e^{-i\delta} \left(e^{-i\gamma} e^{-j\beta} + e^{i\gamma} e^{j\beta} - i e^{i\gamma} e^{j\beta} j + i e^{-i\gamma} e^{-j\beta} j \right) e^{-j\alpha} \\
&= e^{-i\delta} e^{i\beta} e^{j\gamma} e^{-j\alpha}.
\end{aligned} \tag{8.44}$$

Obviously, this final result equals the left-hand side of 8.43 which completes the proof. □

Example 1. As an example we will demonstrate the effect of a rotation of the original signal. The transformation matrix A is then given by

$$A = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \Rightarrow \det(A) = 1, \quad B = A^t = A, \tag{8.45}$$

$$A^\top = A^{-1} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}. \tag{8.46}$$

$$\mathcal{F}_q\{f(A\boldsymbol{x})\}(\boldsymbol{u}) = \frac{1}{2}\Big(F^q(A\boldsymbol{u}) + F^q(A^{-1}\boldsymbol{u}) + i(F^q(A\boldsymbol{u}) - F^q(A^{-1}\boldsymbol{u}))j\Big). \tag{8.47}$$

Example 2. Here we regard a pure dilation of the original signal with different scaling factors for the x-axis and the y-axis. In this case the transformation matrix takes the form:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow \det(A) = ab, \tag{8.48}$$

$$B = B^\top = \begin{pmatrix} 1/b & 0 \\ 0 & 1/a \end{pmatrix}, \quad B^t = \frac{1}{ab}B^{-1} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}. \tag{8.49}$$

$$\mathcal{F}_q\{f(A\boldsymbol{x})\}(\boldsymbol{u}) = \frac{1}{2ab}\left(F^q\left(\frac{u}{a}, \frac{v}{b}\right) + F^q\left(\frac{u}{a}, \frac{v}{b}\right)\right. \tag{8.50}$$

$$\left. + i\left(F^q\left(\frac{u}{a}, \frac{v}{b}\right) - F^q\left(\frac{u}{a}, \frac{v}{b}\right)\right) j\right) \tag{8.51}$$

$$= \frac{1}{ab}F^q\left(\frac{u}{a}, \frac{v}{b}\right). \tag{8.52}$$

This result has the same form as the analogue result for the 2-D Fourier transform. The affine theorem of the Hartley transform [29] is like the version for the QFT more complicated than the affine theorem of the Fourier transform.

8.5 The Clifford Fourier Transform

Above we developed the QFT which applies to images or other 2-D signals. When one wants to deal with volumetric data, image sequences or any other signals of higher dimensions, the QFT has to be extended. For this reason we introduce the Clifford Fourier transform for signals of arbitrary dimension n. Which Clifford algebra has to be used depends on the signal's dimension n.

We recall the QFT in the form given in theorem 8.4.1:

$$F^q(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-i2\pi\, u_1 x_1} f(\boldsymbol{x}) e^{-j2\pi\, u_2 x_2}.$$

As mentioned earlier, the position of the signal f between the exponential functions is of no importance as long as f is real-valued.

Definition 8.5.1 (Clifford Fourier transform).
The Clifford Fourier transform $\mathcal{F}_c : L^2(\mathbb{R}^n, \mathbb{R}_{0,n}) \rightarrow L^2(\mathbb{R}^n, \mathbb{R}_{0,n})$ *of an* n*-dimensional signal* $f(\boldsymbol{x})$ *is defined by*

$$F^c(\boldsymbol{u}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \prod_{k=1}^{n} \exp(-e_k 2\pi u_k x_k) d^n\boldsymbol{x} \quad . \tag{8.53}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $e_1, e_2, \ldots, e_n$ *are the basis vectors of the Clifford algebra* $\mathbb{R}_{0,n}$ *as defined in chapter 1. The product is meant to be performed in a fixed order:* $\prod_{j=1}^{n} a_j = a_1 a_2 \cdots a_n$.

For real signals and $n = 2$ the Clifford Fourier transform is identical to the QFT. For $n = 1$ it is the complex Fourier transform.

Theorem 8.5.1 (Inverse Clifford Fourier transform). *The inverse Clifford Fourier transform is obtained by*

$$\mathcal{F}_c^{-1}\{F^c\}(\boldsymbol{x}) = \int_{\mathbb{R}^n} F^c(\boldsymbol{u}) \prod_{k=0}^{n-1} \exp(e_{n-k} 2\pi u_{n-k} x_{n-k}) d^n\boldsymbol{u}. \tag{8.54}$$

Proof. Inserting term (8.53) into the formula (8.54) yields

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} f(\boldsymbol{x}') \prod_{j=1}^{n} \exp(-e_j 2\pi u_j x_j') d^n\boldsymbol{x}' \prod_{k=0}^{n-1} \exp(e_{n-k} 2\pi u_{n-k} x_{n-k}) d^n\boldsymbol{u} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x}') \delta^n(\boldsymbol{x} - \boldsymbol{x}') d^n\boldsymbol{x}' \\ &= f(\boldsymbol{x}), \end{aligned}$$

where the orthogonality of the harmonic exponential functions is used. □

In chapter 9 we will introduce a corresponding transform using an n-D commutative hypercomplex algebra.

8.6 Historical Remarks

Although hypercomplex spectral signal representations are of special interest for image processing tasks, the Clifford Fourier transform does not seem to have attracted a lot of attention yet. The reason may lie in the fact that articles on the subject are spread in the literature of many different fields and are not easily accessible. For this reason it is not surprising that the authors of this chapter first thought to have "invented" the QFT and the Clifford Fourier transform in [35]. Since the literature on the QFT is rather disjointed the following review may be of interest to researchers in this field.

The first appearance we could trace the QFT back to is an article by the Nobel laureate R.R. Ernst et al. which appeared in 1976 [72]. The scope of this work is 2-D NMR spectroscopy. In the analysis of molecular systems transfer functions of perturbed systems are recorded, which leads to 2-D spectra. Ernst shows that for the analysis of so called *quadruple phase 2-D Fourier transform spectroscopy* the introduction of a hypercomplex Fourier transform is necessary. The transform introduced in [72] could be the same as the QFT. However, the algebra involved is not completely defined: The elements i and j are given as imaginary units $i^2 = j^1 = -1$, and a new element ji is introduced. There is nothing said about $(ji)^2$ and on whether ji equals ij or not. This work has again been reported in [73] and [59] where the used algebra is specified to a commutative algebra with $ij = ji$.

In mathematical literature the Clifford Fourier transform was introduced by Brackx et al. [30] in the context of Clifford analysis. This branch of mathematics is concerned with the extension of results of the theory of complex functions to Clifford-valued functions.

In 1992 the QFT was reinvented by Ell for the analysis of 2-D partial-differential systems [69, 70]. This work was taken up and adapted to the use in color image processing by Sangwine [203, 204, 205]. Sangwine represents an RGB color image as a pure quaternion-valued function

$$f(\boldsymbol{x}) = i\, r(\boldsymbol{x}) + j\, g(\boldsymbol{x}) + k\, b(\boldsymbol{x})$$

which can be transformed into the frequency domain by the QFT. This allows to transform color images holistically instead of transforming each color component separately using a complex Fourier transform. A more extensive discussion of algebraic embeddings of color images can be found in Chap. 7 of this book.

The discrete QFT or DQFT has been used by Chernov in order to develop fast algorithms for the 2-D discrete complex Fourier transform [40]. Chernov reduces the size of a real image by assigning to each pixel a quaternion made up from four real pixel-values of the original image. This method is called overlapping. The shrunk image is transformed by the DQFT. The result is expanded to the DFT of the input signal using simple automorphisms of the quaternion algebra.

8.7 Conclusion

The quaternionic Fourier transform (QFT) has been introduced as an alternative to the 2-D complex Fourier transform. It has been shown that the main theorems of the complex Fourier transform have their analogues in case of the QFT. An n-D Clifford Fourier transform has been introduced as an alternative to the complex Fourier transform. It has been shown that there is a hierarchy of harmonic transforms. Actually, all lower level transforms can be easily derived from the higher level transforms. Whereas here mainly theoretical considerations were made, we will demonstrate the impact of the quaternionic Fourier transform on image processing in chapter 11.