The Multidimensional Isotropic Generalization of Quadrature Filters in Geometric Algebra*

Michael Felsberg and Gerald Sommer

Christian-Albrechts-University of Kiel
Institute of Computer Science and Applied Mathematics
Cognitive Systems
Preußerstraße 1-9, 24105 Kiel, Germany
Tel: +49 431 560433, Fax: +49 431 560481
{mfe,gs}@ks.informatik.uni-kiel.de

Abstract. In signal processing, the approach of the analytic signal is a capable and often used method. For signals of finite length, quadrature filters yield a band-pass filtered approximation of the analytic signal. In the case of multidimensional signals, the quadrature filters can only be applied with respect to a preference direction. Therefore, the orientation has to be sampled, steered or orientation adaptive filters have to be used. Up to now, there has been no linear approach to obtain an isotropic analytic signal which means that the amplitude is independent of the local orientation. In this paper, we present such an approach using the framework of geometric algebra. Our result is closely related to the Riesz transform and the structure tensor. It is seamless embedded in the framework of Clifford analysis. In a suitable coordinate system, the filter response contains information about local amplitude, local phase and local orientation of intrinsically one-dimensional signals. We have tested our filters on two- and three-dimensional signals.

Keywords: quadrature filter, analytic signal, Riesz transform

1 Introduction

In image and image sequence processing, different paradigms of interpreting the signals exist. Regardless of they are following a constructive or an appearance based strategy, they all need a capable low-level preprocessing scheme.

For one-dimensional signals, the analytic signal and the quadrature filters are capable theoretical and practical methods, respectively. The analytic signal codes the local properties of structure in an optimal way. Using quadrature filters, it is simple to detect steps and spikes in the signal.

Accordingly, in image processing, the detection of edges and lines is a frequently discussed topic, which suffers from the fact that there has been no odd filter with isotropic energy up to now (e.g. [12]). The corresponding problem in the frequency

* This work has been supported by German National Merit Foundation and by DFG Graduiertenkolleg No. 357 (M. Felsberg) and by DFG So-320-2-2 (G. Sommer).

domain is that one cannot define positive and negative frequencies (see [7]), such that it is not possible to create a 'real' 2D quadrature filter.

To overcome this problem, several approaches has been developed in the past, all using the quadrature filters with respect to a preference direction:

1. orientation adaptive filtering using the structure tensor, e.g. [7, 8]
2. sampling of the orientation, e.g. [12, 13]
3. steerable filters, e.g. [5, 16]

The first two approaches are non-linear and the corresponding algorithms have high complexities (compared to convolutions). The steerable filters are linear and fast, but they are not related to a generalized analytic signal and only yield approximative quadrature filters with steered preference direction. In our opinion, the structure tensor is the method which is closest to a generalized quadrature filter. It is isotropic but not linear because the phase is neglected. Actually, the phase contains all information about the characteristic of structure [18]!

Therefore, one should keep the phase, which is automatically fulfilled if a linear approach is developed. Since the preprocessing is only the first link in a long chain of operations, it is also useful to have a linear approach, because otherwise it would be nearly impossible to design the higher-level processing steps. If the preprocessing is linear, one can consider simple cases because the effect in a more complex signal is simply the sum of the parts.

On the other hand, we need a rich representation if we want to treat as much as possible in the preprocessing stage. Furthermore, the representation of the signal during the different operations should be complete, in order to prevent a loss of information. These constraints enforce us to leave the approach of complex analysis and to use the framework of geometric algebra instead which is also advantageous if we combine image processing with neural computing and robotics (see [20]).

In this paper, we introduce a new approach for the 2D analytic signal which enables us to substitute the structure tensor by an entity which is linear, preserves the split of the identity and has a geometrically meaningful representation. We have overcome the problem of odd filters in higher dimensions, the resulting method is of low complexity and is naturally embedded as a generalized analytic signal in the field of Clifford analysis.

2 Fundamentals

Since we work on signals in Euclidean space ($\mathbb{R}^n$), we have to use the geometric algebra $\mathbb{R}_{0,n}$. That is, for 1D signals we use $\mathbb{R}_{0,1}$ (isomorphic to the algebra of complex numbers), for image processing we use $\mathbb{R}_{0,2}$ (isomorphic to the algebra of quaternions $\mathbb{H}$), and for image sequences we use $\mathbb{R}_{0,3}$. The classical complex signal theory naturally embeds in these algebras, since the algebra of complex numbers can be considered as a subalgebra.

The base vectors of $\mathbb{R}^n$ are denoted

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \qquad \text{where } \mathbf{e}_k\mathbf{e}_k = -1,\ k \in \{1, \ldots, n\}$$

and the base elements of $\mathbb{R}_{0,n}$ are denoted

$$\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{e}_{12}, \mathbf{e}_{13}, \ldots, \mathbf{e}_{(n-1)n}, \mathbf{e}_{123}, \ldots, \mathbf{e}_{1\ldots n}$$

where $\mathbf{e}_0$ is the base element for the scalar part, i.e. it commutes with all base elements and squares to 1. The subspace of $\mathbb{R}_{0,n}$ consisting of k-vectors is denoted $\mathbb{R}_{0,n}^k$. The conjugation (inversion and reversion) of $a \in \mathbb{R}_{0,n}$ is denoted by $\bar{a}$. For a complete introduction to geometric algebra see e.g. [10].

All signals are considered to be defined on vector spaces, hence a real 1D signal is not any longer a function $\mathbb{R} \to \mathbb{R}$ but a function $\mathbf{e}_1\mathbb{R} \to \mathbf{e}_0\mathbb{R}$ or a curve in $\mathbb{R}_{0,1}^1 \oplus \mathbb{R}_{0,1}^0$. Accordingly, an image is a surface in $\mathbb{R}_{0,2}^1 \oplus \mathbb{R}_{0,2}^0$ (i.e. in a 3D space), and an image sequence is a 3D subspace in $\mathbb{R}_{0,3}^1 \oplus \mathbb{R}_{0,3}^0$. Vectors (elements of $\mathbb{R}_{0,n}^1 \cong \mathbb{R}^n$) are denoted in bold face: $\boldsymbol{x} = \sum_{k=1}^n x_k \mathbf{e}_k$. Elements of $\mathbb{R}_{0,n}^1 \oplus \mathbb{R}_{0,n}^0$ (*paravectors*, [19]) are denoted in normal face: $x = \sum_{k=0}^n x_k \mathbf{e}_k = x_0 \mathbf{e}_0 + \boldsymbol{x}$.

The Fourier transform of the nD signal $f(\boldsymbol{x})$ is denoted

$$f(\boldsymbol{x}) \circ\!\!-\!\!\bullet\, F(\boldsymbol{u}) = \int_{\mathbb{R}_{0,n}^1} f(\boldsymbol{x}) \exp(\mathbf{e}_1 2\pi \boldsymbol{u} \cdot \boldsymbol{x})\, d\boldsymbol{x} \quad .$$

Since we want to extend the analytic signal, we briefly introduce the 1D approach (see e.g. [11]). The Hilbert transform of a 1D signal f is denoted f_H and it is obtained by the transfer function $H(\boldsymbol{u}) = \mathbf{e}_1 \operatorname{sign}(\boldsymbol{u})$ in the frequency domain or by the convolution kernel $-\frac{1}{\pi x_1} = \frac{\boldsymbol{x}\mathbf{e}_1}{\pi|\boldsymbol{x}|^2}$. The analytic signal is obtained by the sum $f_A = f - f_H \mathbf{e}_1$.

The typical property of the analytic signal is the *split of identity* which means that it contains local phase and local amplitude. While the local phase represents a *qualitative* measure of a structure, the local amplitude represents a *quantitative* measure of a structure (see e.g. [18, 12]). For higher dimensions, a consequent generalization of the analytic signal should keep the property of splitting the identity.

Local structures in multi-dimensional signals can be classified in different categories according to the intrinsic dimensionality (see [14]). In our approach, we keep with a single structure phase. Therefore, we can only classify intrinsically 1D signals. What is left for a complete description of structure is the local orientation. Obviously, local orientation is not independent of the phase because the local direction of a signal fixes both properties at the same time (see [6]). The resulting constraint is fulfilled by our approach which we will introduce in the next section.

3 The Monogenic Signal

The constraint which must be fulfilled by the multidimensional extension of the analytic signal is the following: If the signal is rotated such that it is reflected wrt. the origin[1] (e.g. 2D: rotation of π), the change of the orientation phase must yield a negation of the

[1] Note that for most dimensions, it is not possible to find a rotation which is identical to a reflection through the origin for arbitrary objects. In our case, we only consider intrinsically 1D signals which means that it is sufficient to reflect wrt. the direction in which the signal changes (normal vector). This is always done by a rotation around an axis orthogonal to the normal vector through the origin.

structure phase. This is fulfilled if we use the standard spherical coordinates and assign the first angle to the structure phase:

$$
\begin{aligned}
x_0 &= r\cos\theta_1 \\
x_1 &= r\sin\theta_1\cos\theta_2 \\
&\vdots \\
x_{n-1} &= r\sin\theta_1\ldots\sin\theta_{n-1}\cos\theta_n \\
x_n &= r\sin\theta_1\ldots\sin\theta_{n-1}\sin\theta_n
\end{aligned}
$$

where $r=\sqrt{x\bar{x}}$, $\theta_2,\ldots,\theta_n\in[0;\pi)$ and $\theta_1\in[0;2\pi)$.
For the 2D case this coordinate system is illustrated in Fig. 1. The reflection of a point

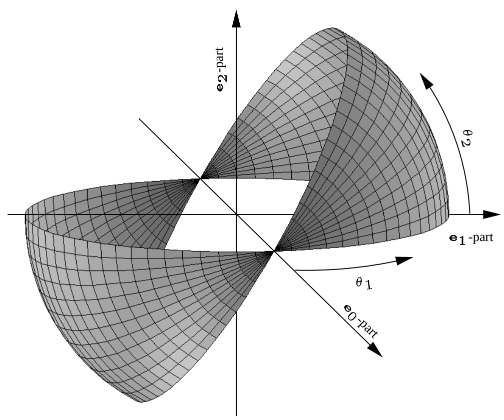

Fig. 1. Spherical coordinate system for 2D

wrt. the origin in vector space corresponds to a rotation of the angular coordinate θ_2 by π. This yields a negation of $\boldsymbol{x}$ (identical to the conjugation of $\boldsymbol{x}$) and therefore, the structure phase is negated as well. In other words, we have to find an operator $\mathcal{O}$ such that

$$\mathcal{O}\{f(\boldsymbol{x})\}\in\mathbb{R}^1_{0,n}\oplus\mathbb{R}^0_{0,n} \qquad \text{and} \tag{1}$$

$$\mathcal{O}\{f(-\boldsymbol{x})\}=\overline{\mathcal{O}\{f(\boldsymbol{x})\}} \tag{2}$$

which is a multidimensional generalization of the Hermite symmetry of the analytic signal. Consequently, the $\mathbf{e}_0$-part (real part) of $\mathcal{O}\{f(\boldsymbol{x})\}$ must be even and the vector part of $\mathcal{O}\{f(\boldsymbol{x})\}$ must be odd. Up to now, it was a common opinion that no odd operator with isotropic energy exists (e.g. [12]). Actually, this statement is true if the operator is required to consist of only one scalar valued component. If we extend our operator space to vector valued operators, the statement is not true any more.

The operator $\mathcal{O}$ that fulfills the constraints (1) and (2) would be the multidimensional generalization of the analytic signal we are looking for. The even part of $\mathcal{O}$ can be adopted from the 1D analytic signal, since the Dirac impulse $\delta_0(\boldsymbol{x})$ is even and isotropic. The harder task is to find the odd part of $\mathcal{O}$. If we have an intrinsically 1D signal with normal vector $\boldsymbol{n}$ (i.e. $f(\boldsymbol{x}) = g((\boldsymbol{x}\cdot\boldsymbol{n})\mathbf{e}_1)$), a good choice for the odd part of $\mathcal{O}$ would transform $f(\boldsymbol{x})$ such that

$$\mathcal{O}_{\mathrm{odd}}\{f(\boldsymbol{x})\} = \pm\boldsymbol{n} g_H((\boldsymbol{x}\cdot\boldsymbol{n})\mathbf{e}_1) \tag{3}$$

where g_H is the 1D Hilbert transform of g. The factor $\boldsymbol{n}$ yields the odd symmetry we need.
In order to obtain $\mathcal{O}_{\mathrm{odd}}$ we have to look at the Fourier transform of $f(\boldsymbol{x})$:

$$f(\boldsymbol{x}) \circ\!\!-\!\!\bullet\; F(\boldsymbol{u}) = \delta_0(\boldsymbol{u}\wedge\boldsymbol{n})G((\boldsymbol{u}\cdot\boldsymbol{n})\mathbf{e}_1) \ ,$$

($\delta_0(\boldsymbol{u}\wedge\boldsymbol{n}) = \delta_0(-(\boldsymbol{u}\wedge\boldsymbol{n})\cdot\mathbf{e}_{12})\ldots\delta_0(-(\boldsymbol{u}\wedge\boldsymbol{n})\cdot\mathbf{e}_{(n-1)n})$) and therefore,

$$\mathcal{O}_{\mathrm{odd}}\{f(\boldsymbol{x})\} \circ\!\!-\!\!\bullet\; \pm\boldsymbol{n}\delta_0(\boldsymbol{u}\wedge\boldsymbol{n})\,\mathrm{sign}(-\boldsymbol{u}\cdot\boldsymbol{n})G((\boldsymbol{u}\cdot\boldsymbol{n})\mathbf{e}_1)\mathbf{e}_1 \ .$$

Since $\boldsymbol{n}$ is equal to $\pm\frac{\boldsymbol{u}}{|\boldsymbol{u}|}$, we obtain

$$\mathcal{O}_{\mathrm{odd}}\{f(\boldsymbol{x})\} \circ\!\!-\!\!\bullet\; \pm\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\delta_0(\boldsymbol{u}\wedge\boldsymbol{n})G((\boldsymbol{u}\cdot\boldsymbol{n})\mathbf{e}_1)\mathbf{e}_1 \ .$$

The transfer function of the generalized Hilbert transform reads

$$H(\boldsymbol{u}) = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} \ , \tag{4}$$

($\mathcal{O}_{\mathrm{odd}}\{f(\boldsymbol{x})\} \circ\!\!-\!\!\bullet\; -H(\boldsymbol{u})F(\boldsymbol{u})\mathbf{e}_1$) where we chose the sign according to the 1D case. The transfer function of the Hilbert transform reads $\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \mathrm{sign}(\boldsymbol{u})\mathbf{e}_1$.

The spatial representation of (4) which can be obtained using the Hankel transform (see [1, 4])

$$h(\boldsymbol{x}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\frac{\bar{\boldsymbol{x}}}{|\boldsymbol{x}|^{n+1}}\mathbf{e}_1 \ , \tag{5}$$

is the kernel of the *Riesz transform* which is the multidimensional generalization of the Hilbert transform from a mathematician's point of view (see e.g. [21] and also [17]). The factor in (5) is two times one over the surface area of the n-dimensional unit-sphere (Γ: Gamma function). Actually, the kernel of the Riesz transform is closely related to the Cauchy kernel

$$E(x) = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\frac{\bar{x}}{|x|^{n+1}} = \frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}}\frac{x_0\mathbf{e}_0 - \boldsymbol{x}}{|x_0\mathbf{e}_0 + \boldsymbol{x}|^{n+1}} \tag{6}$$

(we have $E(x)|_{x_0=0} = -\frac{1}{2}h(\boldsymbol{x})\mathbf{e}_1$) which is the fundamental solution of the generalized Cauchy-Riemann differential equations:

$$\sum_{k=0}^{n}\mathbf{e}_k\frac{\partial}{\partial x_k}f(x) = 0 \ . \tag{7}$$

If a (Clifford valued) function $f(x)$ solves this system of differential equations, it is called a (left) *monogenic* function. Therefore, the monogenic function is the multidimensional generalization of the analytic function.

The analytic signal got its name from the analytic function because the Hilbert transform is identical to a convolution with the 1D Cauchy kernel for $x_0 = 0$ (up to a factor of two) and therefore, the Hilbert transform of an analytic signal reads

$$f_A(\boldsymbol{x}) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{f_A(\boldsymbol{t})}{\boldsymbol{x} - \boldsymbol{t}} \, d\boldsymbol{t} \tag{8}$$

which is quite similar to the Cauchy formula for analytic functions (for details see [9]).

Since the Riesz transform is a convolution with the nD Cauchy kernel for $x_0 = 0$ (up to a factor of two) and the Riesz transform of the signal

$$f_M(\boldsymbol{x}) = f(\boldsymbol{x}) - f_H(\boldsymbol{x})\mathbf{e}_1 \tag{9}$$

fulfills

$$f_M(\boldsymbol{x}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} \frac{\overline{(\boldsymbol{x} - \boldsymbol{t})} f_M(\boldsymbol{t})}{|\boldsymbol{x} - \boldsymbol{t}|^{n+1}} \, d\boldsymbol{t} \tag{10}$$

which can be obtained from the generalized Cauchy formula for monogenic functions, the signal $f_M(\boldsymbol{x})$ is called the *monogenic signal*.

We conclude this section with a last remark. The monogenic signal can be obtained in three ways:

1. by the transfer function $1 - \frac{\boldsymbol{u}}{|\boldsymbol{u}|}\mathbf{e}_1$
2. by the convolution kernel $\delta_0 + \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{\bar{\boldsymbol{x}}}{|\boldsymbol{x}|^{n+1}}\mathbf{e}_1$, and
3. by a modified inverse Fourier transform (see [4] for the 2D case)

Finally, the proofs of the relations from Clifford analysis can be found in [2] and some proofs of the relations of the monogenic signal can be found in [4].

4 Spherical Quadrature Filters

In practical cases of signal processing, signals are of finite length. Therefore, the Hilbert transform is calculated for a bandpass filtered version of the signal. The Hilbert transform of the bandpass filter and the bandpass filter itself form a pair of quadrature filters. This approach can also be applied to the Riesz transform in order to obtain the multidimensional generalization of quadrature filters: the *spherical quadrature filters* (SQF).

The SQF are an $(n + 1)$-tuple of filters which are created by a radial bandpass filter and the convolution of the Riesz kernel (n components) with this bandpass filter. The energy of the filter is isotropic (if the effect of the cubic filter mask can be neglected, see e.g. [3, 15]) and it estimates the local amplitude, local phase and local orientation with only $n + 1$ convolutions. Hence, it is quite fast and should be real time capable.

The SQF are somehow related to the steerable quadrature filters (e.g. [5]), since the vector part is steerable. But there are some differences: firstly, the steered quadrature pair is an approximation to a *classical* quadrature filter with arbitrary preference orientation.

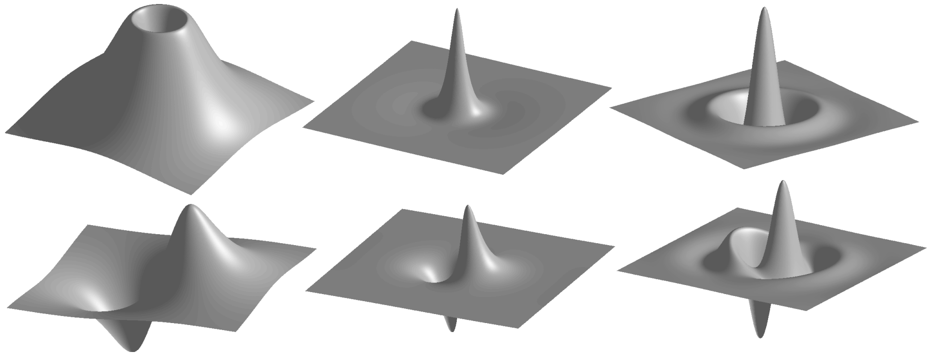

Fig. 2. Spherical quadrature filters (lognormal radial bandpass). Upper row: even filters, bottom row: odd filters. Left column: frequency domain, middle and right column: spatial domain with different c/k-ratio

Secondly, the orientation parameter is directly obtained from the vector part of the SQF in contrast to the steered quadrature filters. The reason for this is the degree of the polynomials which has to be at least two in order to obtain a sufficient approximation of the Hilbert transform. The SQF correspond to polynomials of degree zero and one, such that the filter responses are constant and linear with the orientation vector.

Thirdly, the radial bandpass is derived from a Gaussian function in [5], whereas for the spherical quadrature filters any bandpass can be chosen. In our approach, we use the lognormal bandpass because it has some fundamental advantages wrt. the Gaussian function (see also [13]): it allows arbitrary large bandwidth while always being DC-free.

Therefore, our bandpass filter is represented in the frequency domain by

$$B(\boldsymbol{u}) = \exp\left(-\frac{(\log(|\boldsymbol{u}|/2^k))^2}{2(\log(c))^2}\right) \tag{11}$$

where k is a constant indicating the center frequency and c is a constant indicating the bandwidth of the bandpass (e.g. $c = 0.55$ corresponds to two octaves). The transfer functions and impulse responses of some filters are illustrated in Fig. 2.

We applied the 2D SQF to some natural and synthetic images and the 3D filters to a synthetic image sequence[2]. The results of the 2D experiments (synthetic images) can be found in Fig. 3. The amplitude of the filter responses of the Siemens star and the modulated ring show the isotropy of the spherical quadrature filters. The local orientation and the local phase are represented as grey values which vary linearly in angular and radial direction, respectively.

The experiments with real images[3] are shown in Fig. 4. The local amplitude indi-

[2] The 3D results can be found as mpeg-movies at the homepage of the first author (URL: http://www.ks.informatik.uni-kiel.de/~mfe). The normal vector of the plane is estimated with an error of less than $0.1°$.

[3] URL: http://www-syntim.inria.fr/syntim/analyse/images-eng.html

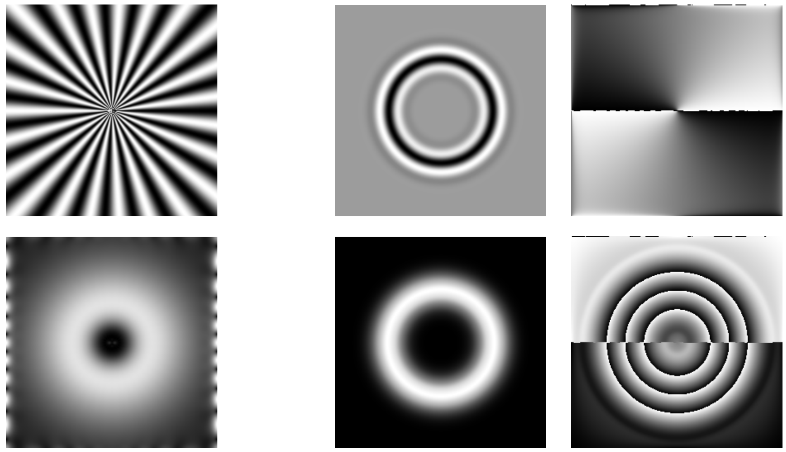

Fig. 3. Experiments with synthetic 2D data. Siemens star (left column, top and bottom), signal and amplitude of filter response, modulated ring (middle column, top) and filter response (amplitude: bottom middle, local orientation: upper right, and local phase: bottom right)

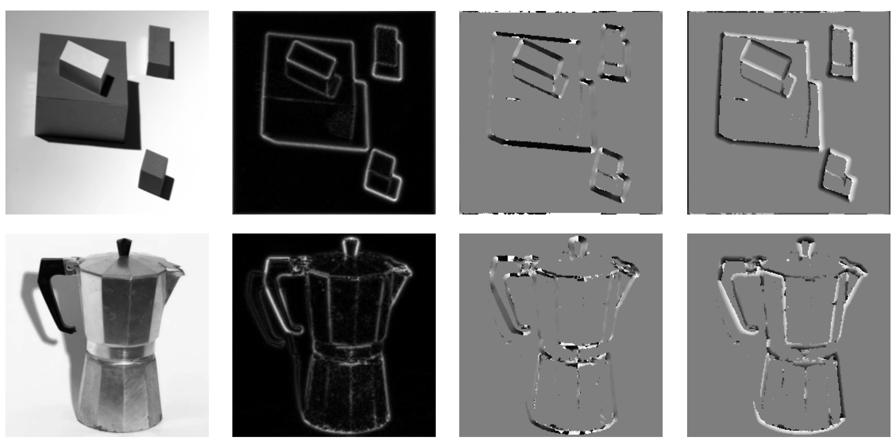

Fig. 4. Experiments with real 2D data. From left to right: original image, local amplitude, local orientation and local phase, images from INRIA-Syntim ©

cates where to find edges. The absolute value depends on both, the markedness of the structure and the local contrast. The images of the local orientation and the local phase are masked by a threshold of the local amplitude. In the areas of very low local amplitude the phase and orientation is irrelevant due to noise. Note that the phase and the orientation are cyclic. Therefore, white indicates nearly the same angle as black.

5 Conclusion

In our opinion, the monogenic signal is the consequent multidimensional generalization of the analytic signal. It is seamless embedded into the theory of Clifford analysis. The Riesz transform is an elegant way to overcome the problem of odd isotropic multidimensional filters and therefore, it is the best way to generalize the Hilbert transform, not only from a mathematician's point of view but also from the perspective of a signal-theorist.

The spherical quadrature filters are more capable than the classical quadrature filters for higher dimensions. Besides the complete representation of local structure, they are linear and of lower complexity than classical approaches (e.g. structure tensor). Due to linearity, they form a good basis for the design of linear and non-linear second level filters.

The algebraic representation of the filter response is geometrically insightful. The interpretation of the data is directly given by the involved geometry, all calculations can be vividly designed. This proves once more the power of geometric algebra.

Acknowledgments

The authors would like to thank Thomas Bülow for many useful discussions and especially for giving some references concerning the Riesz transform.

References

[1] BRACEWELL, R. N. *Two-Dimensional Imaging*. Prentice Hall signal processing series. Prentice Hall, Englewood Cliffs, 1995.

[2] BRACKX, F., DELANGHE, R., AND SOMMEN, F. *Clifford Analysis*. Pitman, Boston, 1982.

[3] BRADY, J. M., AND HORN, B. M. P. Rotationally symmetric operators for surface interpolation. *Computer Vision, Graphics, and Image Processing 22*, 1 (April 1983), 70–94.

[4] FELSBERG, M., AND SOMMER, G. Structure multivector for local analysis of images. Tech. Rep. 2001, Institute of Computer Science and Applied Mathematics, Christian-Albrechts-University of Kiel, Germany, February 2000.

[5] FREEMAN, W. T., AND ADELSON, E. H. The design and use of steerable filters. *IEEE Transactions on Pattern Analysis and Machine Intelligence 13*, 9 (September 1991), 891–906.

[6] GRANLUND, G. H. Hierarchical computer vision. In *Proc. of EUSIPCO-90, Fifth European Signal Processing Conference, Barcelona* (1990), L. Torres, E. Masgrau, and M. A. Lagunas, Eds., pp. 73–84.

[7] GRANLUND, G. H., AND KNUTSSON, H. *Signal Processing for Computer Vision*. Kluwer Academic Publishers, Dordrecht, 1995.

[8] HAGLUND, L. *Adaptive Multidimensional Filtering*. PhD thesis, Linköping University, 1992.

[9] HAHN, S. L. *Hilbert Transforms in Signal Processing*. Artech House, Boston, London, 1996.

[10] HESTENES, D., AND SOBCZYK, G. *Clifford algebra to geometric calculus, A Unified Language for Mathematics and Physics*. Reidel, Dordrecht, 1984.

[11] JÄHNE, B. *Digitale Bildverarbeitung*. Springer, Berlin, 1997.

[12] KOVESI, P. *Invariant Measures of Image Features from Phase Information*. PhD thesis, University of Western Australia, 1996.

[13] KOVESI, P. Image features from phase information. *Videre: Journal of Computer Vision Research 1*, 3 (1999).

[14] KRIEGER, G., AND ZETZSCHE, C. Nonlinear image operators for the evaluation of local intrinsic dimensionality. *IEEE Transactions on Image Processing 5*, 6 (June 1996), 1026–1041.

[15] MERRON, J., AND BRADY, M. Isotropic gradient estimation. In *IEEE Computer Vision and Pattern Recognition* (1996), pp. 652–659.

[16] MICHAELIS, M. *Low Level Image Processing Using Steerable Filters*. PhD thesis, Christian-Albrechts-University of Kiel, 1995.

[17] NABIGHIAN, M. N. Toward a three-dimensional automatic interpretation of potential field data via generalized Hilbert transforms: Fundamental relations. *Geophysics 49*, 6 (June 1984), 780–786.

[18] OPPENHEIM, A., AND LIM, J. The importance of phase in signals. *Proc. of the IEEE 69*, 5 (May 1981), 529–541.

[19] PORTEOUS, I. R. *Clifford Algebras and the Classical Groups*. Cambridge University Press, 1995.

[20] SOMMER, G. The global algebraic frame of the perception-action cycle. In *Handbook of Computer Vision and Applications* (1999), B. Jähne, H. Haußecker, and P. Geissler, Eds., vol. 3, Academic Press, San Diego, pp. 221–264.

[21] STEIN, E., AND WEISS, G. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, New Jersey, 1971.