Analysis of the Curvature Tensor from the Viewpoint of Signal Processing

Lennart Wietzke*, Gerald Sommer*, Christian Schmaltz† and Joachim Weickert†

*Institute of Computer Science, Chair of Cognitive Systems, Christian-Albrechts-University of Kiel, Germany
†Faculty of Mathematics and Computer Science, Mathematical Image Analysis Group, Saarland University, Saarbrücken, Germany

Abstract. So far the recently introduced monogenic curvature tensor has only been known in Fourier domain. In this paper the monogenic curvature tensor will be formulated in spatial domain as a concatenation of two and three Riesz transforms respectively. Furthermore it will be shown that the Riesz transform of any order can be defined by a concatenation of one dimensional Hilbert transforms in Radon space, the Radon transform of the signal and its inverse. The Riesz, Hilbert and Radon transforms provide a connection between differential geometry and signal processing so that already known results from differential geometry can be used to solve problems from phase based local image analysis of intrinsic dimension two and to interpret the monogenic curvature tensor exactly.

Keywords: Differential Geometry, Signal Processing, Geometric (Clifford) Algebra, Monogenic Curvature Tensor, Monogenic Signal, Radon Transform, Hough Transform, Riesz Transform, Hilbert Transform, Intrinsic Dimension, Analytic Signal, Local Phase Based Image Analysis
PACS: 02.10.Xm,02.40.Dr

INTRODUCTION

In the following 2D signals $f \in \mathbb{R}^\Omega$ will be analyzed in the Monge patch embedding $I_f = \{x\mathbf{e}_1 + y\mathbf{e}_2 + f(x,y)\mathbf{e}_3 | (x,y) \in \Omega \subset \mathbb{R}^2\}$ which is well known from differential geometry with $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}$ as the set of basis vectors of the Clifford algebra $\mathbb{R}_3$. A 2D signal f will be classified into local regions $N \subseteq \Omega$ of different intrinsic dimension:

$$f \in \begin{cases} \mathrm{i0D}_N, & f(\mathbf{x}_i) = f(\mathbf{x}_j) \quad \forall \mathbf{x}_i, \mathbf{x}_j \in N \\ \mathrm{i1D}_N, & f(x,y) = g(x\cos\theta + y\sin\theta) \quad \forall (x,y) \in N \text{ with } g \in \mathbb{R}^\mathbb{R} \text{ and } f \notin \mathrm{i0D}_N \\ \mathrm{i2D}_N, & f(x,y) = \sum_{i\in I} f_i(x,y) \text{ with } f_i \in \mathrm{i1D}_N \text{ and the finite index set } I \text{ with } |I| > 1 \end{cases} \tag{1}$$

The set of signals being analyzed in this work can be written as: $\mathscr{L}_2(\Omega,\mathbb{R}) \cap \bigcup_{N\subseteq\Omega}(\mathrm{i0D}_N \cup \mathrm{i1D}_N \cup \mathrm{i2D}_N)$. The set of i1D signals can be completely analyzed by the monogenic signal [Felsberg] which splits the signal locally into phase, orientation and amplitude information. In this paper the monogenic signal and the monogenic curvature tensor [Zang] will be interpreted in Radon space which gives beautiful access to analyzing Riesz transforms of any order. All odd order Riesz transforms apply a one dimensional Hilbert transform to multidimensional signals in a certain orientation which is determined by the Radon transform [Beyerer]. The Radon transform is defined as:

$$r := r(t,\theta) := \mathscr{R}\{f\}(t,\theta) := \int_{(x,y)\in\Omega} f(x,y)\delta_0(x\cos\theta + y\sin\theta - t)d(x,y) \tag{2}$$

with $\theta \in [0..\pi)$ as the orientation, $t \in \mathbb{R}$ as the minimal distance of the line from the origin and δ_0 as the delta distribution. The inverse Radon transform exists and is defined by:

$$\mathscr{R}^{-1}\{r(t,\theta)\}(x,y) := \frac{1}{2\pi^2}\int_{\theta=0}^{\pi}\int_{t\in\mathbb{R}} \frac{1}{x\cos\theta + y\sin\theta - t}\frac{\partial}{\partial t}r(t,\theta)dtd\theta \tag{3}$$

The point of origin where the local phase and orientation information should be obtained within the signal will be translated to position $(0,0)$ for each point $(x,y) \in \Omega$ so that the inverse Radon transform can be simplified to:

$$\mathscr{R}^{-1}\{r\}(0,0) = -\frac{1}{2\pi^2}\sum_{i\in I}\int_{t\in\mathbb{R}} \frac{1}{t}\frac{\partial}{\partial t}r(t,\theta_i)dt \tag{4}$$

because $r(t_1,\theta) = r(t_2,\theta)\ \forall t_1,t_2 \in \mathbb{R}\ \forall\theta \in [0..\pi) - \bigcup_{i\in I}\{\theta_i\}$ implies $\frac{\partial}{\partial t} r(t,\theta) = 0\ \forall t \in \mathbb{R}\ \forall\theta \in [0..\pi) - \bigcup_{i\in I}\{\theta_i\}$ for a finite number $|I| \in \mathbb{N}$ of superimposed i1D signals which construct the i2D signal $f = \sum_{i\in I} f_i$ where each single i1D signal f_i has its own orientation θ_i.

INTERPRETATION OF THE FIRST ORDER RIESZ TRANSFORM

The Riesz transform of a signal f can be written in terms of the Radon transform:

$$\begin{bmatrix} R_x\{f\}(x,y) \\ R_y\{f\}(x,y) \end{bmatrix} = R\{f\}(x,y) = \mathscr{R}^{-1}\{h_1(t) * r(t,\theta) n_\theta\}(x,y) \tag{5}$$

with $n_\theta = [\cos\theta, \sin\theta]^T$ and h_1 as the one dimensional Hilbert kernel in spatial domain. Proof: Central slice theorem [Felsberg]. Applying the Riesz transform to an i1D signal with orientation θ_m results in:

$$\begin{bmatrix} R_x\{f\}(0,0) \\ R_y\{f\}(0,0) \end{bmatrix} = \underbrace{\left[-\frac{1}{2\pi^2}\int_{t\in\mathbb{R}} \frac{1}{t} h_1(t) * \frac{\partial}{\partial t} r(t,\theta_m) dt\right]}_{=:s(\theta_m)} n_{\theta_m} \tag{6}$$

Note that $\frac{\partial}{\partial t}(h_1(t) * r(t,\theta)) = h_1(t) * \frac{\partial}{\partial t} r(t,\theta)$. The orientation of the signal can therefore be derived by $\arctan\frac{R_y\{f\}(0,0)}{R_x\{f\}(0,0)} = \arctan\frac{s(\theta_m)\sin\theta_m}{s(\theta_m)\cos\theta_m} = \theta_m$ (see also figure 1). The Hilbert transform of f and with it also the one dimensional phase can be calculated by: $\sqrt{[R_x\{f\}(0,0)]^2 + [R_y\{f\}(0,0)]^2} = h_1 * f_{\theta_m}$ with the partial Hilbert transform $(h_1 * f_\theta)(0) = -\frac{1}{\pi}\int_{\tau\in\mathbb{R}} \frac{f(\tau\cos\theta, \tau\sin\theta)}{\tau} d\tau$. The first order Riesz transform of any i2D signal consisting of a number

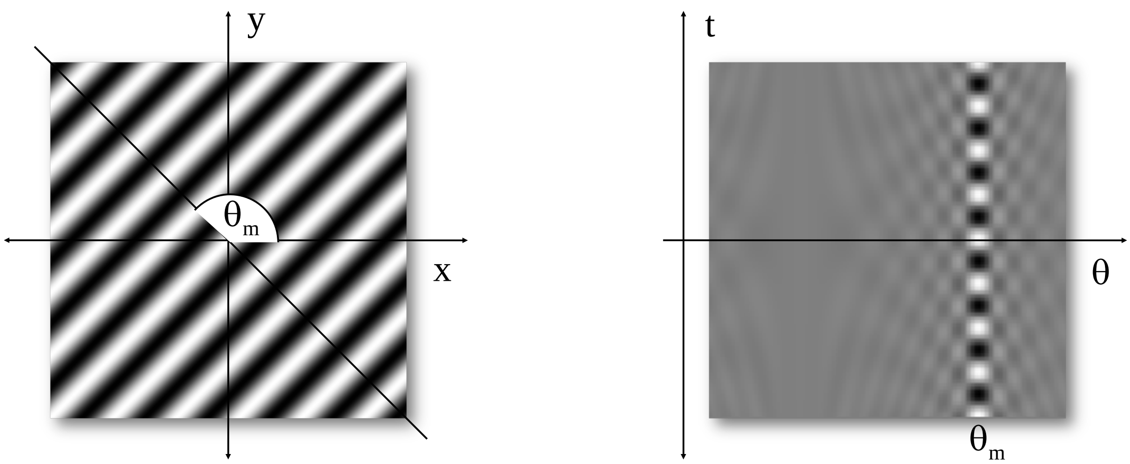

FIGURE 1. From left to right: i1D signal in spatial domain with x- and y-axis and with main orientation θ_m and in Radon space with θ- and t-axis. Even though in the right figure some artifacts can be seen in the Radon space, those artifacts do not exist in case of an infinite support of the image patterns and therefore the artifacts will be neglected in this work: $\frac{\partial}{\partial t} r(t,\theta) = 0\ \forall t\ \forall\theta \neq \theta_m$ and in the left figure $r(t_1,\theta) = r(t_2,\theta)\ \forall t_1,t_2\ \forall\theta \neq \theta_m$

$|I|$ of i1D signals reads:

$$\begin{bmatrix} R_x\{f\}(0,0) \\ R_y\{f\}(0,0) \end{bmatrix} = \sum_{i\in I}\left[-\frac{1}{2\pi^2}\int_{t\in\mathbb{R}} \frac{1}{t} h_1(t) * \frac{\partial}{\partial t} r(t,\theta_i) dt\right] n_{\theta_i} = \sum_{i\in I} s(\theta_i) n_{\theta_i} \tag{7}$$

THE MONOGENIC CURVATURE TENSOR

So far the monogenic curvature tensor [Zang] has only been known in Fourier domain. This drawback makes interpretation impossible when applying the monogenic curvature tensor to a certain signal model. This problem can be solved in spatial domain of the Riesz transform. Now any i2D signal will be regarded as the superposition of a finite number of i1D signals $f = \sum_{i\in I} f_i$. With the properties $\mathscr{R}\{\mathscr{R}^{-1}\{r\}\} = r$ and $\mathscr{R}\{\sum_{i\in I} f_i\} = \sum_{i\in I} \mathscr{R}\{f_i\}$ [Toft] and because of $h_1 * h_1 * f = -f$ the monogenic curvature tensor $T = T_e + T_o \in M(2, \mathbb{R}_3)$ can be also written in terms of the concatenation of two and three Riesz transforms respectively and therefore also in terms of the Radon transform and its inverse [Stein]:

$$T_e(f) = \mathscr{F}^{-1}\left\{\mathscr{F}\{f\}(\alpha,\rho)\begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha\mathbf{e}_{12} \\ \sin\alpha\cos\alpha\mathbf{e}_{12} & \sin^2\alpha \end{bmatrix}\right\} = \begin{bmatrix} R_x\{R_x\{f\}\} & -R_x\{R_y\{f\}\}\mathbf{e}_{12} \\ R_x\{R_y\{f\}\}\mathbf{e}_{12} & R_y\{R_y\{f\}\} \end{bmatrix} \quad (8)$$

With $\mathscr{F}\{f\}(\alpha,\rho)$ as the Fourier transformed of the signal f in polar coordinates with α as the angular component and ρ as the radial component and $\mathscr{F}^{-1}$ as the inverse Fourier transform. Now the even tensor in Radon space reads:

$$T_e(f) = -\begin{bmatrix} \mathscr{R}^{-1}\{\cos^2\theta\mathscr{R}\{f\}\} & -\mathscr{R}^{-1}\{\sin\theta\cos\theta\mathscr{R}\{f\}\}\mathbf{e}_{12} \\ \mathscr{R}^{-1}\{\sin\theta\cos\theta\mathscr{R}\{f\}\}\mathbf{e}_{12} & \mathscr{R}^{-1}\{\sin^2\theta\mathscr{R}\{f\}\} \end{bmatrix} \quad (9)$$

The odd tensor is defined as the Riesz transform of the even part: $T_o(f) = T_e(R_x\{f\} + R_y\{f\}\mathbf{e}_{12})$. Please note that the monogenic curvature tensor and even the monogenic signal are not restricted to any intrinsic dimension.

Interpretation of the Monogenic Curvature Tensor for Two Superimposed i1D Signals

In the following two superimposed i1D signals with orientations θ_1, θ_2 and arbitrary but same phase φ for both i1D signals will be analyzed in Radon space (see figure 2). With the abbreviations: $a := \cos\theta_1,\ b := \cos\theta_2,\ c := \sin\theta_1,\ d := \sin\theta_2$ the even tensor for two i1D signals reads:

$$T_e = f(0,0)\begin{bmatrix} a^2+b^2 & -(ca+db)\mathbf{e}_{12} \\ (ca+db)\mathbf{e}_{12} & c^2+d^2 \end{bmatrix} \quad (10)$$

and the odd tensor for two i1D signals reads:

$$T_o = -s\begin{bmatrix} (a^3+b^3)+(ca^2+db^2)\mathbf{e}_{12} & -(ca^2+db^2)\mathbf{e}_{12}+(c^2a+d^2b) \\ (ca^2+db^2)\mathbf{e}_{12}-(c^2a+d^2b) & (c^2a+d^2b)+(c^3+d^3)\mathbf{e}_{12} \end{bmatrix} \quad (11)$$

with assumption $s := s(\theta_1) = s(\theta_2)$. The determinant of the odd tensor reads:

$$\mathbf{e}_1\det T_o = \mathbf{e}_1\underbrace{s^2(a^3d^2b+b^3c^2a-ca^2d^3-db^2c^3-2ca^2db^2+2c^2ad^2b)}_{=:B}+\mathbf{e}_2\underbrace{s^2(a^3d^3+b^3c^3-db^2c^2a-ca^2d^2b)}_{=:C} \quad (12)$$

Using that notation, the main orientation can be derived by: $\theta_m = \frac{\theta_1+\theta_2}{2} = \frac{1}{2}\arctan\frac{C}{B}$. The apex angle of the two i1D signals can be derived by the Gaussian and main curvature known from differential geometry and with the basic ideas of Euler's and Meusnier's theorems [Baer]: $H := \frac{1}{2}\mathrm{trace}(T_e) = \frac{1}{2}(a^2+b^2+c^2+d^2)f(0,0) = f(0,0)$ and $K := \det T_e = [f(0,0)]^2(a^2d^2+b^2c^2-2abcd)$. This yields the apex angle:

$$\left|\frac{\theta_1-\theta_2}{2}\right| = \arctan\sqrt{\left|\frac{H-\sqrt{H^2-K}}{H+\sqrt{H^2-K}}\right|} = \arctan\sqrt{\left|\frac{1-\sqrt{1-a^2d^2-b^2c^2+2abcd}}{1+\sqrt{1-a^2d^2-b^2c^2+2abcd}}\right|} \quad (13)$$

The phase φ of both signals can be derived in many different ways. One possibility would be by the determinants of the even and odd tensors: $\varphi = \arctan\frac{B^2+C^2}{\det^2 T_e}$.

CONCLUSION AND FUTURE WORK

The odd-order Riesz transform of any i1D or i2D signal can be analyzed in Radon space in which a one dimensional partial Hilbert transform will be applied in direction of each i1D signal with its individual orientation $\theta_i,\ i \in I$.

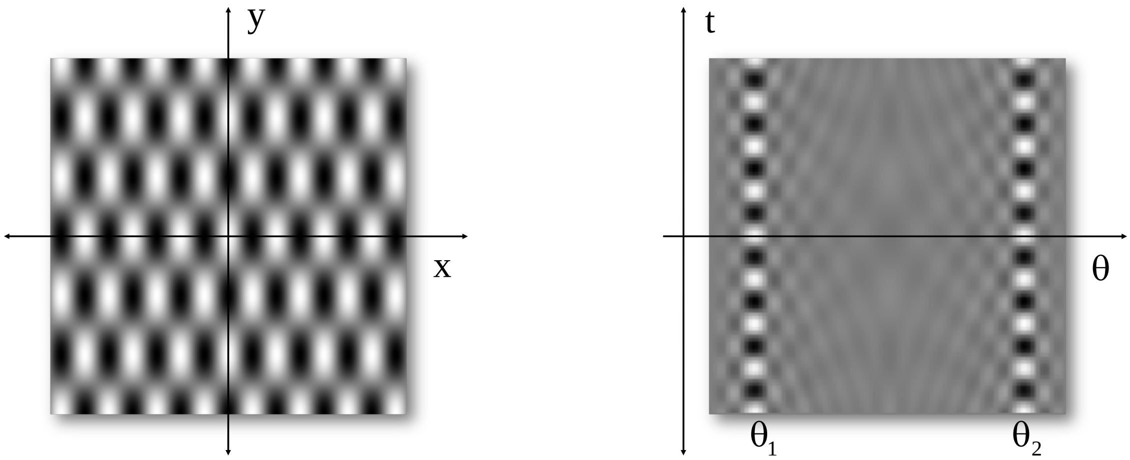

FIGURE 2. From left to right: Two superimposed i1D signals with orientation θ_1 and θ_2 in spatial domain and in Radon space. Assuming that both signals have same phase φ the signal information can be separated by the Riesz transform.

Assuming that two superimposing i1D signals have arbitrary but same phase φ, the orientations θ_1, θ_2 and the phase can be calculated. The advantage of analyzing the Riesz transform in Radon space is that the signal properties consisting of orientation and phase are explicitly given after applying the general operator (e.g. the monogenic curvature operator) to the specific i1D or i2D signal function. Future work will include analyzing the superposition of i1D signals with individual phases φ_i. Assume the orientations θ_1, θ_2 of the two signals to be known. Then the Hilbert transforms of each signal can be derived by the following linear system of equations:

$$\begin{bmatrix} h_1 * f_{\theta_1} \\ h_1 * f_{\theta_2} \end{bmatrix} = \frac{1}{ad - cb} \begin{bmatrix} R_x\{f\}d - R_y\{f\}b \\ R_y\{f\}a - R_x\{f\}c \end{bmatrix} \tag{14}$$

Note that arbitrary but same phase of both signals has been assumed in this paper for deriving the orientations, so that only one common phase can be calculated by this assumption. The analysis of i2D signals in Radon space presented in this work realizes to interpret the monogenic curvature tensor for the first time in an exact way.

ACKNOWLEDGMENTS

We acknowledge funding by the German Research Foundation (DFG) under the projects *SO 320/4-2* and *We 2602/5-1*.

REFERENCES

Beyerer. Jürgen Beyerer and Fernando Puente Leon, The Radon Transform in Digital Image Processing, Publisher: Oldenbourg, 2002, Vol 50, Part 10, pages: 472-480, Country of publication: Germany, Journal title: Automatisierungstechnik, ISSN: 0178-2312

Toft. Peter Toft, The Radon Transform - Theory and Implementation, PhD Thesis, 1996, Department of Mathematical Modelling, Technical University of Denmark, June 1996

Baer. Christian Baer, Elementare Differentialgeometrie, de Gruyter, 2001, Vol 1, ISBN-10: 3110155206, ISBN-13: 978-3110155204

Felsberg. Michael Felsberg, Low-Level Image Processing with the Structure Multivector, PhD Thesis, Kiel University, 2002

Zang. Di Zang, Signal Modeling for Two-Dimensional Image Structures and Scale-Space Based Image Analysis, PhD Thesis, Kiel University, 2007

Stein. Elias M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, Princeton University Press, Princeton, New Jersey, 1970