The Relation of Inverse Problems and Isotropic 2D Signal Analysis*

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Abstract. This paper shows the fundamental relation of exact rotationally invariant local phase based 2D signal analysis in scale space and the task of the solution of the corresponding inverse problem. The generalized 2D Hilbert transform in spatial domain will be derived from order one to three to result in the appropriate system of equations to solve for an assumed local signal model.

1 Introduction

We assume that a 2D signal $f \in L_2(\mathbb{R}^2, \mathbb{R})$ can be locally modeled by a superposition of arbitrarily orientated one-dimensional cosine waves [6]

$$\mathcal{P}\{f\}(x, y; s) = (p * f)(x, y; s) = \sum_{\nu=1}^{n} a_\nu \cos(x \cos\theta_\nu + y \sin\theta_\nu + \phi_\nu)$$

in scale space for each scale space parameter $s > 0$ and $p(x, y; s)$ (see figure 1) as the Poisson filter [3] which acts as a low pass filter to the original 2D signal. Without loss of generality this signal model degrades locally at the origin $(x, y; \cdot) := (0, 0; \cdot)$ of a local coordinate system to

$$f_p = \sum_{\nu=1}^{n} a_\nu \cos\phi_\nu \; .$$

In case of image analysis lines, edges and junctions can be modeled exactly in this way. The task is now to determine the local amplitude a_ν, the local orientation θ_ν and the local phase ϕ_ν. This problem has been already solved for one-dimensional signals by the *analytic signal* [4] by means of the Hilbert transform [5] and for intrinsically one-dimensional [7] signals ($n = 1$) by the *2D monogenic signal* [2] by means of the generalized first order Hilbert transform. This paper shows that 2D signal processing can be regarded as an *inverse problem* [1] where generalized 2D Hilbert transforms up to order three are applied to the original signal f whose model will be restricted to $n = 2$.

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2 Generalized Hilbert Transform Kernels

The resulting convolution kernels in Poisson scale space and in spatial domain of the first order generalized 2D Hilbert transform read

$$q^{(1)}(x,y;s) = \frac{1}{2\pi\left(s^2+x^2+y^2\right)^{\frac{3}{2}}}\begin{bmatrix} x \\ y \end{bmatrix} \, ,$$

the second order generalized 2D Hilbert transform kernels read

$$q^{(2)}(x,y;s) = \frac{s(2s^2+3(x^2+y^2)) - 2(s^2+x^2+y^2)^{\frac{3}{2}}}{2\pi(x^2+y^2)^2(s^2+x^2+y^2)^{\frac{3}{2}}}\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

and the third order generalized 2D Hilbert transform kernels read

$$q^{(3)}(x,y;s) = \frac{4s^2(2s^2+3(x^2+y^2)) + 3(x^2+y^2)^2 - 8s(s^2+x^2+y^2)^{\frac{3}{2}}}{(x^2+y^2)^3(s^2+x^2+y^2)^{\frac{3}{2}}}\begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix}$$

(see figure 1). These generalized Hilbert transform kernels can be now used for convolution in spatial domain with the original 2D signal which results in $[f_x, f_y]^T = f * q^{(1)}$, $[f_{xx}, f_{xy}, f_{yy}]^T = f * q^{(2)}$ and $[f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}]^T = f * q^{(3)}$.

3 Relation of the Radon Transform and the Generalized Hilbert Transform

By the relation of the 2D Radon transform and the generalized 2D Hilbert transform [6] the appropriate system of equations can be determined as

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \mathcal{R}^{-1}\left\{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}(h(t) * f_r(t;\theta;s))\right\}(x,y;s)$$

for the first order generalized Hilbert transform with $f_r(t,\theta;s) := \mathcal{R}\left\{\mathcal{P}\{f\}\right\}(t,\theta;s)$ as the 2D Radon transformed signal, its inverse $\mathcal{R}^{-1}\{\cdot\}(x,y;s)$ and the one-dimensional Hilbert transform kernel $h(t) = \frac{1}{\pi t}$. The second order generalized Hilbert transformed signal reads

$$\begin{bmatrix} f_{xx} \\ f_{xy} \\ f_{yy} \end{bmatrix} = -\mathcal{R}^{-1}\left\{\begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix} f_r(t,\theta;s)\right\}(x,y;s)$$

and the third order generalized Hilbert transform results in

$$\begin{bmatrix} f_{xxx} \\ f_{xxy} \\ f_{xyy} \\ f_{yyy} \end{bmatrix} = -\mathcal{R}^{-1}\left\{\begin{bmatrix} \cos^3\theta \\ \cos^2\theta\sin\theta \\ \cos\theta\sin^2\theta \\ \sin^3\theta \end{bmatrix}(h(t) * f_r(t;\theta;s))\right\}(x,y;s)$$

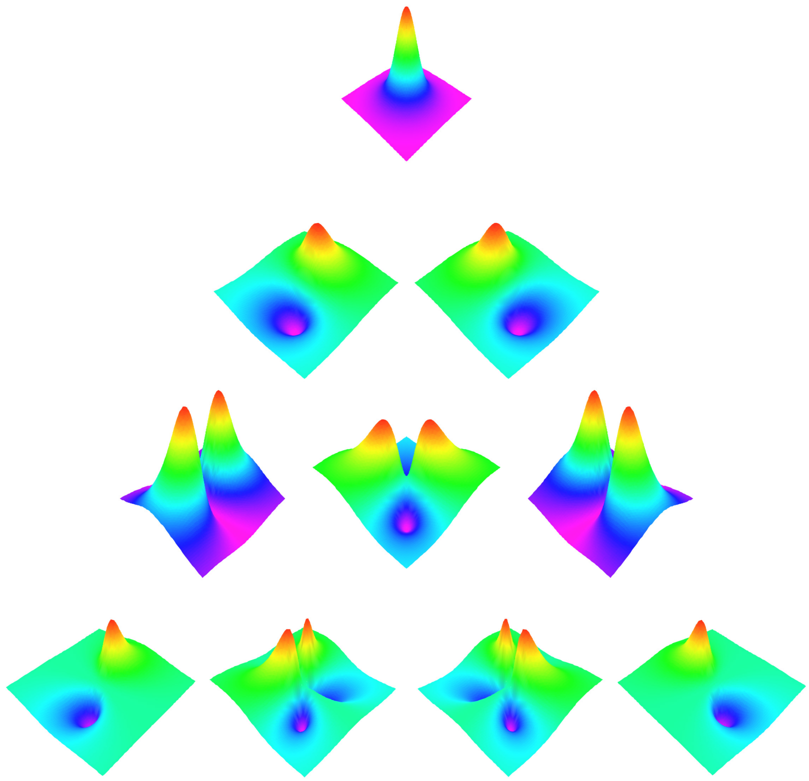

Fig. 1. Illustration from left to right of the 2D convolution kernels in spatial domain for a certain scale space parameter $s > 0$. Top row: Poisson kernel p. Second row: $q_x^{(1)}$ and $q_y^{(1)}$. Third row: $q_{xx}^{(2)}$, $q_{xy}^{(2)}$ and $q_{yy}^{(2)}$. Bottom row: $q_{xxx}^{(3)}$, $q_{xxy}^{(3)}$, $q_{xyy}^{(3)}$ and $q_{yyy}^{(3)}$.

4 Corresponding Inverse Problem

By means of the results of the Radon space interpretation of the generalized Hilbert transformed signal the system of equations can be identified at the origin $(x, y; \cdot) := (0, 0; \cdot)$ of the local coordinate system as

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \sum_{\nu=1}^{n} a_\nu \sin\phi_\nu \begin{bmatrix} \cos\theta_\nu \\ \sin\theta_\nu \end{bmatrix}$$

for the first order,

$$\begin{bmatrix} f_{xx} \\ f_{xy} \\ f_{yy} \end{bmatrix} = \sum_{\nu=1}^{n} a_\nu \cos\phi_\nu \begin{bmatrix} \cos^2\theta_\nu \\ \frac{1}{2}\sin(2\theta_\nu) \\ \sin^2\theta_\nu \end{bmatrix}$$

for the second order and

$$\begin{bmatrix} f_{xxx} \\ f_{xxy} \\ f_{xyy} \\ f_{yyy} \end{bmatrix} = \sum_{\nu=1}^{n} a_\nu \sin\phi_\nu \begin{bmatrix} \cos^3\theta_\nu \\ \cos^2\theta_\nu \sin\theta_\nu \\ \cos\theta_\nu \sin^2\theta_\nu \\ \sin^3\theta_\nu \end{bmatrix}$$

for the third order [6]. Signal analysis in general now corresponds to the inverse problem to solve those system of equations for the unknown parameters a_ν, ϕ_ν and θ_ν for a certain n and using the results of the generalized Hilbert transformed signal. For $n = 2$ the results of the generalized Hilbert transformed signal $[f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}]^T$ can be used.

5 Conclusion

It has been shown that isotropic phase based signal analysis is fixed to a certain signal model. By means of the generalized 2D Hilbert transform and the relation to the 2D Radon transform the appropriate system of equations can be determined which corresponds to the known results of the convolution in spatial domain and the local structural and geometrical features of the assumed signal model. Therefore, the problem of signal analysis can be written as an *inverse problem*. The solution and the geometrical interpretation of this problem for certain signal models will be part of our future work.

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