Wavelet Filter Design via Linear Independent Basic Filters

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Abstract. A new point of view for wavelet filters is presented. This leads to a description of wavelet filters in terms of certain linear independent *basic filters* which can be designed to construct wavelets with special properties. Furthermore, it is shown, that this approach makes explicit closed form descriptions for higher order DAUBECHIES wavelet filters (at least for D_8 and D_{10}) possible, which were unaccessible before. Additionally, some biorthogonal examples are discussed and finally, a conceptual generalization to the twodimensional case is given.

1 Introduction

Since its introduction in the early 1980s, the evolution of wavelet analysis caused a deep impact in nearly all tasks of signal processing as well as computer vision applications and related questions (e.g. image compression, feature detection, optic flow estimation, treatment of PDEs). Though, the onedimensional theory has grown rapidly in the last two decades, there are several open questions concerning the general, multidimensional wavelet theory, for example the lack of factorization theorems like the FEJER-RIESZ-Lemma, which makes the design of scaling (and wavelet) filters with *desirable* properties in more than one dimension quite tricky. The aim of this paper is the presentation of a framework for onedimensional scaling filter design, which can be easily generalized to higher dimensions and may therefore help to overcome some of the existing problems. The reason for this is the fact, that a direct design method is used, which is independent of factorization questions. Finally, we shall mention that similar results were presented in the article [AHC93], which the author was unaware of during the first writing of this text. However, in [AHC93] the concept of linear independence was not used and no multidimensional generalization was intended; additionally, the closed form descriptions for higher order maximally flat orthogonal wavelet filters are a new contribution (although, they are mostly of theoretical interest).

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2 Basic Material

All of the following investigations are restricted to wavelets that come from a dyadic multiresolution analysis (however, the generalization to arbitrary integer dilations is straightforward). Consider a scaling function $\varphi \in L_2(\mathbb{R})$ and suppose, that the related scaling filter symbol $m_0(\omega)$ is given by

$$m_0(\omega) \;=\; \sum_{k=0}^{n} \gamma_k \cdot e^{i\omega k} \qquad , \; a_k \in \mathbb{R}.$$

To yield an orthonormal basis for $L_2(\mathbb{R})$, the symbol has to satisfy the *orthogonality criterion*

$$\begin{aligned} 1 \;&\equiv\; |m_0(\omega)|^2 \;+\; |m_0(\omega+\pi)|^2 \\ &=\; 2 \cdot \sum_{k=0}^{n} \gamma_k^2 \;+\; 4 \cdot \sum_{k=1}^{\frac{n-1}{2}} \sum_{j=0}^{n-2k} \gamma_j \cdot \gamma_{j+2k} \cdot \cos(2k\omega). \end{aligned}$$

From this, we can directly derive the following $(n+1)/2$ constraint equations of order two:

$$\sum_{j=0}^{n} \gamma_j^2 \;=\; 1 \qquad \text{and} \qquad \sum_{j=0}^{n-2k} \gamma_j \cdot \gamma_{j+2k} = 0, \quad k = 1, \ldots, (n-1)/2. \tag{1}$$

For most applications, wavelets with a sufficient high regularity and a number of vanishing moments (this gives polynomial reproducibility) are desirable. Moreover, it is a well known fact, that both of the mentioned properties are in some sense connected to the zero order, say m, of the scaling filter symbol $m_0(\omega)$ at the aliasing frequency $\omega = \pi$. These zeros can be characterized by the STRANG-FIX *conditions* or *sum rules* of order $m-1$, that is

$$\sum_{k=0}^{n} (-1)^k \, k^l \, h_k \;=\; 0 \qquad \text{for} \;\; l = 0, 1, \ldots, m-1.$$

From the constraints in (1) one easily shows that an orthogonal scaling filter symbol of length $n+1$ can have at most a zero of order $(n+1)/2$ at π.

3 The Framework

The main idea of our framework is the following: consider a linear combination of linear independent (in vectorial sense) *basic filters* and solve the equation system (1) for the coefficients of the linear combination; from this point of view the linear independence means that no redundancy can occur and all solutions (if they exist) are accessible. In this section we will now successively build such families of linear independent basic filters. These will additionally be chosen such that they satisfy the STRANG-FIX conditions up to a certain order.

Lemma 1. *Suppose, the filter* $[\ \gamma_0\ \gamma_1\ \ldots\ \gamma_n\ ]$ *satisfies the sum rules exactly up to order* $n-1$*. Then the filter*

$$\left[\ \tilde{\gamma}_0\ \ \tilde{\gamma}_1\ \ldots\ \tilde{\gamma}_{n+1}\ \right] := \left[\ 1\ \ 1\ \right] * \left[\ \gamma_0\ \ \gamma_0\ \ \gamma_1\ \ \ldots\ \ \gamma_n\ \right]$$

satisfies the sum rules exactly up to order n*.*

Proof. The proof is straightforward. Just evaluate

$$\sum_{k=0}^{n+1} (-1)^k \cdot k^p \cdot \tilde{\gamma}_k = -\sum_{j=0}^{p-1} \binom{p}{j} \cdot \sum_{k=0}^{n} (-1)^k \cdot k^j \cdot \gamma_k .$$

The inner sum on the right side vanishes for $p = 0, 1 \ldots n$ and is different from zero for $p = n+1$ by the assumptions that were made. □

Corollary 1. *For every* $n \in \mathbb{N}_*$ *the filter*

$$h^n := \left[\ 1\ \ 1\ \right]^{*n}$$

satisfies the sum rules up to order $n-1$*, where* γ^{*n} *denotes the* n *times subsequently repeated discrete convolution of* γ*.*

Lemma 2. *Define*

$$g^{l,m} := \left[\ 1\ \ -1\ \right]^{*l} * \left[\ 1\ \ 1\ \right]^{*m} .$$

Then, for all $m, l \in \mathbb{N}_*$ *the filter* $g^{m,l}$ *satisfies exactly the sum rules of order* $m-1$*.*

The filters h^n and the convolution filters $g^{l,m}$ with $l + m = n$ form a linear independent family of basic filters and will be very useful in the design of several scaling filters, as we shall see in the following.

Proposition 1. *Let* $\mathbf{A}_n$ *be the* $(n+1) \times (n+1)$ *matrix*

$$\mathbf{A}_n = \begin{bmatrix} h_0^n & g_0^{1,n-1} & g_0^{2,n-2} & \cdots & g_0^{n,0} \\ h_1^n & g_1^{1,n-1} & g_1^{2,n-2} & \cdots & g_1^{n,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n^n & g_n^{1,n-1} & g_n^{2,n-2} & \cdots & g_n^{n,0} \end{bmatrix},$$

then

$$\det \mathbf{A}_n = (-2)^{\frac{n(n+1)}{2}} .$$

Especially, $\det \mathbf{A}_n \neq 0$ for all $n \in \mathbb{N}_*$ and from this, we directly deduce that the $n+1$ filters

$$\left\{ h^n, g^{1,n-1}, g^{2,n-2}, \ldots g^{n,0} \right\}$$

form a linear independent family of basic filters and moreover, every subfamily

$$\{h^n, g^{1,n-1}, g^{2,n-2}, \dots g^{n-k,k}\}$$

additionally satisfies all sum rules up to order $k-1$ by Corollary 1 and Lemma 2. Furthermore, this family of basic filters yields a natural decomposition of every scaling filter that satisfies the sum rules up to order $k-1$ into an even (symmetric) and an odd (antisymmetric) part since h^n is always even and $g^{j,n-j}$ is even for j even and odd for j odd. Note further that the sum rule order decreases by one with each symmetry switch.

Proof of Proposition 1. We will make use of the induction principle. For $n = 1$ we obtain $\mathbf{A}_1 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and $\det \mathbf{A}_1 = -2$. In the second step, we will evaluate $\det \mathbf{A}_{n+1}$ from $\det \mathbf{A}_n$ by elementary matrix operations. In particular, we acquire

$$\det \mathbf{A}_{n+1} = (-2)^{n+1} \cdot \det \mathbf{A}_n.$$

By induction, we obtain the desired relation. □

4 Examples

Maximally flat filters. We will start with an example, that leads to the classical D_8 filter, i.e. an orthogonal filter with a zero of fourth order at the aliasing frequency $\omega = \pi$. Therefore, consider a combination of linear independent basic filters of length eight, that satisfy the STRANG-FIX conditions up to order three. By Proposition 1 such a filter is given by

$$\gamma \;=\; \lambda_0 \cdot h^7 \;+\; \lambda_1 \cdot g^{1,6} \;+\; \lambda_2 \cdot g^{2,5} \;+\; \lambda_3 \cdot g^{3,4}.$$

Solving (1) for this filter yields the solution

$$\begin{aligned}
\lambda_0 &= \frac{1}{128} \\
\lambda_1 &= \frac{1}{384} \cdot \left(\sqrt{21 + 3\mu - 42\mu^{-1}} + \sqrt{42 - 3\mu + 42\mu^{-1} + 18\sqrt{105} \cdot (7 + \mu - 14\mu^{-1})^{-1/2}} \right) \\
\lambda_2 &= 64 \cdot \lambda_1^2 - \frac{7}{256} \\
\lambda_3 &= \frac{\sqrt{35}}{128},
\end{aligned}$$

where we used the abbreviation $\mu = \sqrt[3]{154 + 42\sqrt{15}}$. Thus we found a closed form description for a DAUBECHIES filter of length eight, which was impossible using other filter design methods. In the same manner we can also find an explicit analytical form for D_{10}. For bigger filters, the complexity increases too much and permits explicit forms. However, if one considers a filter

$$\gamma \;=\; \lambda_0 \cdot h^n \;+\; \lambda_1 \cdot g^{1,n-1} \;+\; \lambda_2 \cdot g^{2,n-2} \;\dots\; +\; \lambda_{(n-1)/2} \cdot g^{(n-1)/2,(n+1)/2}$$

of arbitrary even length and solves the system (1), one can at least verify that

$$\begin{aligned} \lambda_0 &= 2^{-n} \\ \lambda_2 &= 2^{n-1} \cdot \lambda_1^2 - n \cdot 2^{-n-1} \\ \lambda_{(n-1)/2} &= 2^{-n} \cdot \sqrt{\binom{n}{(n+1)/2}}. \end{aligned}$$

We additionally remark, that the solutions for $\lambda_3, \ldots, \lambda_{(n-1)/2-1}$ can all be written as rational functions in λ_1, while λ_1 itself is a root of a polynomial of degree $2^{(n-3)/2}$ for $n \geq 7$.

Biorthogonal wavelets. Our framework can also be used to design biorthogonal filters, which have some advantages over orthogonal filters in special applications (e.g. symmetry for image compression). We differ between two cases of biorthogonal filters: if a primal filter is given, dual filters can always be found by solving a system of linear equations; this is the *easy* case and not considered here (since this solutions can be obtained by several other design methods). On the other hand, one can take two linear combinations of even basic filters and solve their coefficients for the *biorthogonality constraints*

$$\tilde{m}_0(\omega) \cdot \overline{m_0(\omega)} + \tilde{m}_0(\omega + \pi) \cdot \overline{m_0(\omega + \pi)} \equiv 1, \qquad \tilde{m}_0(0) = m_0(0) = 1,$$

which again leads to a quadratic equation system. For example, considering a symmetric pair of length 9 and 7 and imposing the maximal number of sum rules on these filters, in particular, we take

$$\gamma = \lambda_0 \cdot h^9 + \lambda_1 \cdot g^{2,7} + \lambda_2 \cdot g^{4,5} \qquad \text{and} \qquad \tilde{\gamma} = \mu_0 \cdot h^7 + \mu_1 \cdot g^{2,5},$$

we obtain the *classical* and till today widely used 9/7 image compression filter, that was first presented in [ABMD92]. Estimating the joint spectral radius of the associated linear operators $(\mathbf{T}_0)_{jk} = \gamma_{2j-k-1}$ and $(\mathbf{T}_1)_{jk} = \gamma_{2j-k}$ reduced to a certain invariant subspace E, we obtain the smoothness values $\alpha \approx 1.068$ and $\tilde{\alpha} \approx 1.701$ in terms of the Hölder exponent (these techniques are discussed in detail in[DL92] and [Gri96]). To obtain better smoothness results, one could give up one zero order of $\tilde{\gamma}$ (by adding $\mu_2 \cdot g^{4,3}$), and use this degree of freedom to find *better* filters. Another wish could be the property, that the coefficients are rationals (as in the easy case), because this can reduce the computational amount of the wavelet transform. In order to achieve these requirements, we use a numerical heuristic that approximates

$$\min_{\mu_2} \left\{ \max_{|\lambda|} \left\{ \lambda \in \text{Spectrum} \left(\mathbf{T}_{0|E}(\mu_2), \mathbf{T}_{1|E}(\mu_2), \tilde{\mathbf{T}}_{0|\tilde{E}}(\mu_2), \tilde{\mathbf{T}}_{1|\tilde{E}}(\mu_2) \right) \right\} \right\}$$

at dyadic rational values μ_2. Thereby one finds

$$\gamma = \begin{bmatrix} \frac{9}{320} & \frac{-3}{160} & \frac{-3}{40} & \frac{43}{160} & \frac{19}{32} & \frac{43}{160} & \frac{-3}{40} & \frac{-3}{160} & \frac{9}{320} \end{bmatrix}$$

and the dual filter

$$\tilde{\gamma} \quad = \quad \left[\begin{array}{ccccccc} \frac{-3}{64} & \frac{-1}{32} & \frac{19}{64} & \frac{9}{16} & \frac{19}{64} & \frac{-1}{32} & \frac{-3}{64} \end{array} \right].$$

This new pair of filters is indeed very promising in applications such as image compression. Its smoothness values are $\alpha \approx 1.48409$ and $\tilde{\alpha} \approx 1.67807$, respectively. Note that $\tilde{\alpha}$ is only minimally worse than in the classical 9/7 case, while the value for α is significantly better and additionally, all the filter coefficients are rational. Finally, we shall mention that this heuristic method does not guarantee, that there exist no *better* solutions than the given one.

5 The 2D Case

In the same manner one can build twodimensional filters from linear combinations of basic filters. The main ideas of this conceptual generalization will be described in this section. First, we will state a similar result to Lemma 1. In that case, repeated convolutions with the sequences $[\,1\;\;1\,]$ and $[\,1\;\;-1\,]$ were used to successively build longer filters with a higher sum rule order and it turns out that a similar thing can be done in higher dimensions.

Lemma 3. *Suppose, the twodimensional filter* $\left[\, \{\gamma_{jk}\}_{j\in 1\ldots n_x, k\in 1\ldots n_y} \,\right]$ *satisfies the (twodimensional) sum rules up to order* $m-1$*. Then the filter*

$$[\,\gamma_{jk}\,] * \begin{bmatrix} & \alpha & \\ \beta & 2\alpha+2\beta & \beta \\ & \alpha & \end{bmatrix}$$

satisfies the sum rules at least up to order m*, if* $\alpha, \beta \neq 0$.

The *proof* is similar to the onedimensional case and omitted. However, these filters will not be sufficient; we additionally need some antisymmetric filters, which we will get from the following Lemma.

Lemma 4. *Suppose, the* symmetric *filter* $\left[\, \{\gamma_{jk}\}_{j\in 1\ldots n_x, k\in 1\ldots n_y} \,\right]$ *satisfies the sum rules up to order* $m-1$*. Then for* $\alpha \neq 0$ *both of the* antisymmetric *filters*

$$[\,\gamma_{jk}\,] * \begin{bmatrix} & -\alpha & \\ -\alpha & 0 & \alpha \\ & \alpha & \end{bmatrix} \qquad \textit{and} \qquad [\,\gamma_{jk}\,] * \begin{bmatrix} & -\alpha & \\ \alpha & 0 & -\alpha \\ & \alpha & \end{bmatrix}$$

fulfill the sum rules at least up to order m.

Taking a little care about some possible redundancies (since specific choices of α and β may lead to linear dependent filters) while using these Lemmata, the linear independence of the basic filters directly carries over to the twodimensional case. The thing that makes everything more difficult is the orthogonality constraint, which now becomes

$$|m_0(\omega_x, \omega_y)|^2 \;+\; |m_0(\omega_x+\pi, \omega_y+\pi)|^2 \;\equiv\; 1.$$

Assuming that the filter is rhombic shaped (this is in some sense the most convenient and applicable case) and consists of ν double diagonals each of length μ, this leads us to $2\nu\mu - \nu - \mu + 1$ equations of the form

$$\sum_{k=1}^{\nu}\sum_{j=1}^{\mu} \gamma_{j-k,j+k-1}^2 \;+\; \gamma_{j-k+1,j+k-1}^2 \;=\; 1,$$

$$\sum_{k=1}^{\nu}\sum_{j=1}^{\mu-1} \gamma_{j-k,j+k-1} \cdot \gamma_{j-k+\tau,j+k+\sigma-1} \;+\; \gamma_{j-k+1,j+k-1} \cdot \gamma_{j-k+\tau+1,j+k+\sigma-1} \;=\; 0$$

with $(\tau, \sigma) \in \{[-\nu+1, -\nu+2, \ldots, \nu-1] \times [0, 1, \ldots, \mu-1]\} \setminus \{(0,0)\}$.

Example. We will now give an example of a twodimensional orthogonal scaling filter, that satisfies the Strang-Fix conditions up to order one. Therefore, we take the simple Haar scaling filters

$$[\,1\;\;1\,] \qquad \text{and} \qquad [\,1\;\;-1\,]$$

and apply the Lemmata 3 and 4 to them. This gives us a linear combination

$$\lambda_0 \cdot \begin{bmatrix} & 1 & 1 & \\ 1 & 5 & 5 & 1 \\ & 1 & 1 & \end{bmatrix} + \lambda_1 \cdot \begin{bmatrix} & -1 & -1 & \\ 1 & 1 & 1 & 1 \\ & -1 & -1 & \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} & 1 & -1 & \\ 1 & 3 & -3 & -1 \\ & 1 & -1 & \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} & -1 & -1 & \\ -1 & -1 & 1 & 1 \\ & 1 & 1 & \end{bmatrix}$$

of four basic filters. Solving the orthogonality criterion for the coefficients λ_i, we obtain

$$\lambda_0 \;=\; \frac{1}{16}, \qquad \lambda_1 \;=\; \frac{-1}{8}, \qquad \lambda_2 \;=\; \frac{\pm\sqrt{3}}{16} \qquad \text{and} \qquad \lambda_3 \;=\; \frac{\pm\sqrt{3}}{8},$$

which reproduces the Kovačević-Vetterli scaling filter (see [KV92]), the first known orthogonal 2D-filter that leads to a continouus wavelet for the important quincunx sampling grid. Note, that we again found a decomposition of the filter into an even and an odd part, where the even part satisfies the sum rules up to order two and the odd part up to order one — everything is very similar to the 1D case. We only need more basic filters because more constraints are to be considered. We should remark, that since $\mu = \nu = 2$ in the previous example, we would have to satisfy five constraint equations and thus we should use five basic filters instead of four, but it turns out that one of the coefficients always gets zero. For filters that satisfy the sum rules up to a higher order (e.g. for order two, one has to choose at least $\nu \geq 3$ and $\mu \geq 4$ or vice versa), the orthogonality constraints seem to be solvable only numerically, because of the rapidly increasing complexity of the related nonlinear equation system.

Finally, we shall remark, that the presented framework could also be used to design twodimensional biorthogonal filters. But due to the symmetry properties of these, the McClellan transform can be used to derive 2D-filters directly from their 1D-*prototypes*, which is much faster to implement. Thus, the direct usage of basic filters seems to make less sense if one is interested in twodimensional biorthogonal filters.

6 Discussion and Conclusion

A framework for the design of wavelet filters was presented, which can be generalized to higher dimensions. There are very few different approaches to direct multidimensional orthogonal filter design. The most important among these is the paraunitary polyphase decomposition due to VAIDYANATHAN ([VH88]). But since his building matrices do not commute in general, the a priori ordering of these matrices is not clear and thus there is no unique representation of all possible orthogonal filters of a given shape, which can be obtained by the proposed method. However, numerical experiments lead to the conjecture, that both methods yield the same filter families. It is intended to apply the multidimensional wavelets, that stem from these approaches to optic flow estimations and to image feature detection within the scope of the authors further research. The presented variations for 1D biorthogonal wavelets and their 2D counterparts (built via MCCLELLAN transform) seem to have nice properties for image compression and some cooperation with researchers from this area is planned.

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