The Dual Quaternion Approach to Hand-Eye Calibration

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Abstract

In order to relate measurements made by a sensor mounted on a mechanical link to the robot's coordinate frame we must first estimate the transformation between the sensor and the link frame. In this paper we introduce the use of dual quaternions which are the algebraic counterpart of screws. We algebraically prove that if we consider the camera and motor transformations as screws then only the line coefficients of the screw axes are relevant regarding the hand-eye calibration. This new parametrization enables us to simultaneously solve for the hand-eye rotation and translation using the Singular Value Decomposition.

1 Introduction

Hand-eye calibration is called the computation of the relative position and orientation between the robot gripper and a camera mounted rigidly on the gripper. This problem concerns also all sensors that are rigidly mounted on mechanical links, like a camera mounted on a binocular head with mechanical degrees of freedom as well as a camera mounted on the vehicle. Although the term sensor-actuator calibration is more appropriate we will throughout this paper use the well-known term "hand-eye".

The hand-eye transformation is required in a number of sensing-acting tasks. Using a camera mounted on a gripper or a vehicle we can estimate the position of a target to grasp or to reach in camera coordinates. However, the control commands can be expressed only in the coordinate system of the gripper or the vehicle. Even if the desired control criterion is given in camera coordinates we have to know which is the effect of a robot motion in the camera frame.

The second task group is the placement of sensors at desired positions. We can perform stereo by placing a camera mounted on a gripper at multiple poses sharing the same field of view. However, in order to reconstruct the 3D positions we must know the relative orientation from the camera coordinate systems. But the only transformations we know are in the robot coordinates. The same applies for mounting cameras on binocular heads. As the cameras are manually mounted a hand-eye calibration is necessary in order to align the camera coordinate system with the tilt-vergence link.

The usual way to describe the hand-eye calibration is by means of homogeneous transformation matrices. We denote[1] by $\boldsymbol{X}$ the transformation from camera to gripper, by $\boldsymbol{A}_i$ the transformation matrix from the camera to the world coordinate system and by $\boldsymbol{B}_i$ the transformation matrix from the robot base to the gripper at the i-th pose (Fig. 1). The camera-world transformation $\boldsymbol{A}_i$ is obtained with the extrinsic calibration techniques. The robot base to gripper transformation $\boldsymbol{B}_i$ is given by the direct kinematic chain from the joint angle readings. We see that for one pose we have two transformations as unknowns: robot base to world and the camera to gripper $\boldsymbol{X}$. In order to eliminate the

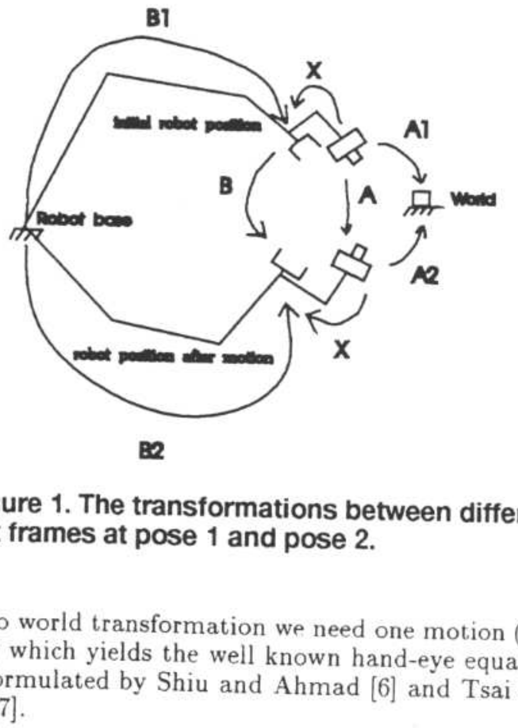

Figure 1. The transformations between different frames at pose 1 and pose 2.

base to world transformation we need one motion (two poses) which yields the well known hand-eye equation first formulated by Shiu and Ahmad [6] and Tsai and Lenz [7].

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{B} \tag{1}$$

[1] We use boldface capitals for matrices $\boldsymbol{X}$, arrowed boldface for 3D-vectors $\vec{\boldsymbol{x}}$, boldface for real quaternions $\boldsymbol{x}$, checked normal fonts $\check{x}$ for dual scalars, checked arrowed boldface for dual vectors $\check{\vec{\boldsymbol{x}}}$, and checked boldface for dual quaternions $\check{\boldsymbol{x}}$. The natural inner product of two vectors or quaternions is denoted by $\boldsymbol{x}^T\boldsymbol{y}$ and the cross product between 3D-vectors by $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$.

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where $\boldsymbol{A} = \boldsymbol{A}_1\boldsymbol{A}_2^{-1}$ and $\boldsymbol{B} = \boldsymbol{B}_1\boldsymbol{B}_2^{-1}$. As every homogeneous transformation matrix has the form

$$\begin{pmatrix} \boldsymbol{R} & \vec{t} \\ 0^T & 1 \end{pmatrix}$$

from (1) follows one matrix and one vector equation

$$\begin{aligned} \boldsymbol{R}_A\boldsymbol{R}_X &= \boldsymbol{R}_X\boldsymbol{R}_B & (2) \\ (\boldsymbol{R}_A - \boldsymbol{I})\vec{t}_X &= \boldsymbol{R}_X\vec{t}_B - \vec{t}_A & (3) \end{aligned}$$

The majority of the approaches regards the rotation estimation in (2) decoupled from translation estimation, the latter following the former. At least two rotations containing motions with not parallel rotation axes are required to solve the problem [7]. Several approaches [7, 6, 2] have been proposed for the estimation of $\boldsymbol{R}_X$ from (2) - a survey can be found in [9].

Horaud and Dornaika [5] are the first who applied a simultaneous non-linear minimization with respect to the rotation quaternion and the translation vector. However, the first simultaneous consideration of rotation and translation in a geometric way was presented by Chen [1] who first introduced the screw theory in the hand-eye calibration. A general displacement can be represented by a rotation about an axis not through the origin - the screw axis- and a translation parallel to this axis - the screw pitch.

In this paper we introduce the algebraic entity for a screw: the unit dual quaternion. Dual quaternions are an extension of the real quaternions by means of the dual numbers and were first introduced by Clifford [3]. Dual numbers and dual quaternions have been earlier used in robotics [4] and in computer vision [8]. Based on the dual quaternions we prove that

1. the hand-eye transformation is independent of the angle and the pitch of the camera and hand motions and depends only on the line parameters of their screw axes (geometrically proved in [1])
2. the unknown screw parameters including both rotation and translation can be simultaneously recovered using the Singular Value Decomposition.

We next briefly introduce the dual quaternions. Quaternions are an extension of the complex numbers to $\mathbb{R}^4$. Among other formalisms one definition of quaternions is as pairs $(s, \vec{q})$ where $s \in \mathbb{R}$ and $\vec{q} \in \mathbb{R}^3$. The quaternions are a vector space over the reals - we will call $\mathbb{H}$ - with the zero element $(0, \mathbf{0})$. The multiplication between quaternions defined as

$$q_1 q_2 = (s_1 s_2 - \vec{q}_1^T\vec{q}_2, s_1\vec{q}_2 + s_2\vec{q}_1 + \vec{q}_1 \times \vec{q}_2) \qquad (4)$$

has a unit element $(1, \mathbf{0})$ and is associative but not commutative. Therefore the quaternions are an associative algebra and since they do not contain zero-divisors they are a division algebra. The norm of a quaternion is defined as $||q|| = q\bar{q}$ where $\bar{q}$ is the conjugate quaternion $(s, -\vec{q})$. A subgroup of $\mathbb{H}$ regarding only the multiplication operation are the unit quaternions with norm equal one. For every rotation (element of SO(3)) about an axis $\vec{n}$ ($||\vec{n}|| = 1$) with an angle θ exists a corresponding unit quaternion $q = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{n})$ that maps a vector $\vec{x} \in \mathbb{R}^3$ to the vector $q(0, \vec{x})\bar{q}$.

A dual number is defined as $\check{z} = a + \epsilon b$ with $\epsilon^2 = 0$. The operations addition and multiplication make them an abelian ring called Δ but not a field because only dual numbers with real part not zero possess an inverse element. An important property is associated with the derivatives of functions with dual arguments. Since all powers greater equal two of ϵ vanish a Taylor expansion yields always $f(a + \epsilon b) = f(a) + \epsilon b f'(a)$.

Dual quaternions are defined in a similar way like real quaternions as $(\check{s}, \check{\vec{q}})$ where $\check{s}$ a dual number and $\check{\vec{q}}$ a dual vector. The product has the same definition as in real quaternions (4) making the dual quaternions a non-abelian ring with unit element $(1, \mathbf{0})$. The product with a dual number makes the dual quaternions a Δ-module. The norm of a dual quaternion is defined as $||\check{q}|| = \check{q}\bar{\check{q}}$ and is a dual number with positive real part. If the norm has a non vanishing real part than the dual quaternion has an inverse $\check{q}^{-1} = ||\check{q}||^{-1}\bar{\check{q}}$. If the norm is equal one then an inverse element exists and is equal to the conjugate quaternion.

2 Line transformations with unit dual quaternions

As already known the rotation of a point $\vec{p}_b$ to a point $\vec{p}_a$ can be written by means of a unit quaternion q as the product $\vec{p}_a = q\vec{p}_b\bar{q}$. This form allows the concatenation of rotations be represented by a simple quaternion product. Unfortunately, no such quaternion representation exists for a general rigid transformation including translation. We will describe in this section that the introduction of dual quaternions allows a rigid transformation rule as simple as the one for pure rotations, however, not for a point but for a line.

A line in space with direction $\vec{l}$ through a point $\vec{p}$ can be represented with the 6-tuple $(\vec{l}, \vec{m})$ where $\vec{m}$ is called the line moment and is equal to $\vec{p} \times \vec{l}$. The line moment is normal to the plane through the line and the origin with magnitude equal to the distance from the line to the origin. The constraints $||\vec{l}|| = 1$ and $\vec{l}^T\vec{m} = 0$ guarantee that the degrees of freedom of an arbitrary line in space are four.

We next give an answer to the following problem:

> *A line given by its dual quaternion $\check{l}_a = l_a + \epsilon m_a$ is transformed with $(\boldsymbol{R}, \vec{t})$ into a line $\check{l}_b$. Show that a unit dual quaternion exists such that $\check{l}_a = \check{q}\check{l}_b\bar{\check{q}}$.*

Applying a rotation $\boldsymbol{R}$ and a translation $\vec{t}$ to a line

$(\vec{l}_b, \vec{m}_b)$ we obtain the transformed line $(\vec{l}_a, \vec{m}_a)$

$$\vec{l}_a = R\vec{l}_b \tag{5}$$

$$\begin{aligned}\vec{m}_a &= \vec{p}_a \times \vec{l}_a = (R\vec{p}_b + \vec{t}) \times R\vec{l}_b \\ &= R\vec{m}_b + \vec{t} \times R\vec{l}_b.\end{aligned} \tag{6}$$

We change from vector to quaternion notation which means that the vector $\vec{l}$ is represented by a quaternion with zero scalar part $l = (0, \vec{l})$. The terms containing rotation can be easily written with quaternions. The difficulty with the cross-product is tackled with the identity $(0, \vec{t} \times \vec{q}) = \frac{1}{2}(q\bar{t} + tq)$ where t is the translation quaternion $(0, \vec{t})$ and q the rotation quaternion $(0, \vec{q})$. We then obtain

$$\begin{aligned}l_a &= q l_b \bar{q} \\ m_a &= q m_b \bar{q} + \frac{1}{2}(q l_b \bar{q} \bar{t} + t q l_b \bar{q}).\end{aligned} \tag{7}$$

We define a new quaternion $q' = \frac{1}{2}tq$ and a dual quaternion $\check{q} = q + \epsilon q'$. It can be easily shown that (7) is equivalent to

$$l_a + \epsilon m_a = (q + \epsilon q')(l_b + \epsilon m_b)(\bar{q} + \epsilon \bar{q'}). \tag{8}$$

Denoting also the lines by dual quaternions $\check{l}_a$ and $\check{l}_b$ we obtain

$$\check{l}_a = \check{q} \check{l}_b \bar{\check{q}}.$$

This formula resembles to the rotation of points with quaternions. Lines can thus be rigidly transformed using a single operation (multiplying left and right with the conjugate) in the non-abelian ring of dual quaternions. The norm

$$|\check{q}|^2 = \check{q}\bar{\check{q}} = q\bar{q} + \epsilon(q\bar{q'} + q'\bar{q}) = q\bar{q} + \epsilon/2(q\bar{q}\bar{t} + tq\bar{q}) = 1$$

hence $\check{q}$ is a unit quaternion. The above relations give also explicitly the transformation from $(R, \vec{t})$ to $q + \epsilon q'$. The dual part $q' = \frac{1}{2}tq$ and the quaternion q can be obtained from the rotation matrix by finding the axis and the angle of rotation. If $\check{q}$ is a solution then $-\check{q}$ is also a solution. It is sufficient to enforce like in non-dual quaternions that the scalar non-dual part is positive in order to eliminate this ambiguity. Reversely, the translation t can be recovered from the dual quaternion as $t = 2q'\bar{q}$. The unit dual quaternion $\check{q}$ can be written as the concatenation of a pure translational unit dual quaternion and a pure rotational quaternion with dual part equal zero i.e. $\check{q} = (1, \epsilon\frac{\vec{t}}{2})q$.

3 Unit dual quaternions and screws

This section shows that the scalar and the vector part of the dual quaternion have a specific meaning which relates them to the kinematic notion of a screw.

According to Chasles' theorem [1] a rigid transformation can be modeled as a rotation about an axis not through the origin and a translation along this axis. As the screw axis is a line in space it depends on four parameters which together with the rotation angle θ and the translation along the axis d (pitch) constitute the six degrees of freedom of a rigid transformation.

In the following we will solve the problem

Compute d as well as the screw axis given by its direction and moment pair $(\vec{l}, \vec{m})$ from R and $\vec{t}$.

The direction $\vec{l}$ is parallel to the rotation axis. The pitch d is the projection of translation on the rotation axis, therefore equal $\vec{t}^T\vec{l}$. In order to recover the moment $\vec{m}$ we introduce a point $\vec{c}$ on the screw axis being the projection of the origin on the axis (Fig. 2).

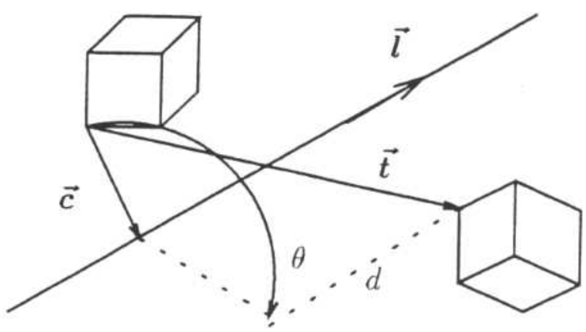

Figure 2. The geometry of a screw: Every motion can be modeled as a rotation with angle θ about an axis at $\vec{c}$ with direction $\vec{l}$ and a subsequent translation d along the axis.

The coordinate system is shifted to this point and then transformed. The resulting translation is then $d\vec{l} + (I - R)\vec{c}$. The so called pitch $d = \vec{l}^T\vec{t}$. Using the Rodrigues formula

$$R\vec{c} = \vec{c} + \sin(\theta)\vec{l} \times \vec{c} + (1 - \cos\theta)\vec{l} \times (\vec{l} \times \vec{c})$$

and $\vec{c}^T\vec{l} = 0$ it follows that [1]

$$\vec{c} = \frac{1}{2}(\vec{t} - (\vec{t}^T\vec{l})\vec{l} + \cot\frac{\theta}{2}\vec{l} \times \vec{t}). \tag{9}$$

This point and hence the screw axis is not defined if the angle θ is either 0 or 180. Otherwise the moment vector reads then

$$\vec{m} = \vec{c} \times \vec{l} = \frac{1}{2}(\vec{t} \times \vec{l} + \vec{l} \times (\vec{t} \times \vec{l})\cot\frac{\theta}{2}). \tag{10}$$

We proceed then with the computation of the corresponding quaternion:

Given the screw parameters $(\theta, d, \vec{l}, \vec{m})$ compute the corresponding dual quaternion $\check{q}$.

The quaternion derived from the rotation matrix $\boldsymbol{R}$ reads

$$(q_0, \vec{q}) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{l}) \tag{11}$$

hence the moment equation (10) can be written

$$\sin\frac{\theta}{2}\vec{m} = \frac{1}{2}(\vec{t} \times \vec{q} + q_0\vec{t} - \cos\frac{\theta}{2}(\vec{l}^T\vec{t})\vec{l}).$$

Using $(\vec{l}^T\vec{t}) = d$ and rewriting

$$\sin\frac{\theta}{2}\vec{m} + \frac{d}{2}\cos\frac{\theta}{2}\vec{l} = \frac{1}{2}(\vec{t} \times \vec{q} + q_0\vec{t})$$

which is the vector part of the dual part q' of the dual quaternion $\check{q}$. Applying (11) and $q' = \frac{1}{2}\boldsymbol{tq}$ we obtain

$$\check{q} = \begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix} + \epsilon \begin{pmatrix} -\frac{1}{2}\vec{q}^T\vec{t} \\ \frac{1}{2}(q_0\vec{t} + \vec{t} \times \vec{q}) \end{pmatrix}. \tag{12}$$

Every function f of dual numbers obeys the rule

$$f(a + \epsilon b) = f(a) + \epsilon b f'(a)$$

hence $\cos(\frac{\theta + \epsilon d}{2}) = \cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}$ and $\sin(\frac{\theta+\epsilon d}{2}) = \sin\frac{\theta}{2} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}$.

It is now straightforward to see that a dual quaternion can also be written as

$$\check{q} = \begin{pmatrix} \cos(\frac{\theta+\epsilon d}{2}) \\ \sin(\frac{\theta+\epsilon d}{2})(\vec{l} + \epsilon\vec{m}) \end{pmatrix}. \tag{13}$$

This representation is very powerful since first it algebraically separates the angle and pitch information from the line information characterizing the pose of the screw axis. Second writing the dual angle $\check{\theta} = \theta + \epsilon d$ and the dual vector $\check{\vec{l}} = \vec{l} + \epsilon\vec{m}$ (13) becomes equivalent to the pure rotation non-dual equation (11). We can easily verify that

$$\check{q} = (\cos\check{\theta}/2, \sin\check{\theta}/2\check{\vec{l}})$$

is a unit quaternion $\check{q}\bar{\check{q}} = 1$.

4 Hand-eye transformation with unit dual quaternions

The concatenation of two rigid displacements or screws can be written as the product of two dual quaternions. Let $\check{a}$ denote the screw of a camera motion and $\check{b}$ denote the screw of the motor motion. Motor (hand) and camera (eye) are rigidly attached to each other. The rigid transformation between them is unknown and it will be denoted by the unit dual quaternion $\check{q}$. The screw concatenation yields then

$$\check{a} = \check{q}\check{b}\bar{\check{q}} \tag{14}$$

which is the most compact equation since the dual quaternion components are eight and not twelve like in the homogeneous matrices of (1). The scalar part of a dual quaternion $\check{a}$ is $(\check{a} + \bar{\check{a}})/2$, hence

$$\begin{aligned} Sc(\check{a}) &= \frac{1}{2}(\check{a} + \bar{\check{a}}) = \frac{1}{2}(\check{q}\check{b}\bar{\check{q}} + \check{q}\bar{\check{b}}\bar{\check{q}}) = \frac{1}{2}\check{q}(\check{b} + \bar{\check{b}})\bar{\check{q}} \\ &= \check{q}Sc(\check{b})\bar{\check{q}} = Sc(\check{b})\,\check{q}\bar{\check{q}} = Sc(\check{b}). \end{aligned} \tag{15}$$

According to (13) the scalar parts are equal to the cosine of the respective dual angles, hence the angle and the pitch of the motor screw are equal to the angle and the pitch of the camera screw, or the angle and the pitch remain invariant under coordinate transformations. This is also known as the Screw Congruence Theorem [1], its proof without dual unit quaternions is, however, considerably longer than the one line proof in (15).

The fundamental equation $\check{a} = \check{q}\check{b}\bar{\check{q}}$ consists of four dual equations. Since the scalar parts are equal only the vector components contribute to the computation of the unknown $\check{q}$:

$$\sin\frac{\check{\theta}_a}{2}(0, \check{\vec{a}}) = \check{q}(0, \sin\frac{\check{\theta}_b}{2}\check{\vec{b}})\bar{\check{q}} = \sin\frac{\check{\theta}_b}{2}\check{q}(0, \check{\vec{b}})\bar{\check{q}}.$$

If the angles $\theta_{a,b}$ are not 0 or 360 degrees the sines can be simplified yielding

$$(0, \check{\vec{a}}) = \check{q}(0, \check{\vec{b}})\bar{\check{q}} \tag{16}$$

which is nothing else then the motion of the lines of the screw axes.

Thus,

1. The hand-eye estimation is independent of the angle and the pitch of the camera and the motor motions.
2. The hand-eye calibration is equivalent to the 3D motion estimation problem from 3D-line correspondences where the lines are the screw axes of the motors and the cameras.

We should note here that all other hand-eye calibration methods make use of the rotation angle and the pitch at least at the translation estimation step (3) which turns out in (16) to be unnecessary. Having shown that the problem is equivalent to the 3D-motion problem we already know from computer vision that the minimum requirement are two non parallel lines.

5 Estimation of the hand-eye screw with SVD

Although we showed in the last section that only the vector part of the dual quaternions is relevant for the estimation of the unknown hand eye unit dual quaternion $\check{q}$ let us keep the same notation $\check{a}$ and $\check{b}$ for $(0, \check{\vec{b}})$ and $(0, \check{\vec{a}})$, respectively.

We split the fundamental equation (14) into the non-dual and dual parts and we obtain

$$\begin{aligned} a &= qb\bar{q} \\ a' &= qb\bar{q}' + qb'\bar{q} + q'b\bar{q}. \end{aligned}$$

Multiplying on the left with q and applying the identity

$$\bar{q}q' + \bar{q}'q = 0$$

in the first term of the right hand side of the first equation yields

$$\begin{aligned} aq - qb &= 0 \\ (a'q - qb') + (aq' - q'b) &= 0. \end{aligned}$$

We keep in mind that from every of the two equations above the scalar part is redundant because they are equivalent to (16). Hence, we have in total six equations with eight unknowns which can be written in matrix form as follows. Let $a = (0, \vec{a})$ and $a' = (0, \vec{a}')$ as well as $b = (0, \vec{b})$ and $b' = (0, \vec{b}')$. The quaternion equations above can then be written as a matrix vector equation

$$\begin{pmatrix} \vec{a} - \vec{b} & [\vec{a} + \vec{b}]_\times & 0_{3\times 1} & 0_{3\times 3} \\ \vec{a}' - \vec{b}' & [\vec{a}' + \vec{b}']_\times & \vec{a} - \vec{b} & [\vec{a} + \vec{b}]_\times \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} = 0 \tag{17}$$

where the matrix - we will call S - is a 6×8 matrix and the vector of unknowns (q^T, q'^T) is 8-dimensional. The operator $[]_\times$ denotes the antisymmetric matrix equivalent to a cross product.

We have two constraints on the unknowns so that the result is a unit dual quaternion

$$q^T q = 1 \qquad \text{and} \qquad q^T q' = 0 \tag{18}$$

We could think that six equations plus two constraints would suffice, however, the vectors $\vec{a}$ and $\vec{b}$ are unit vectors and the vectors $\vec{a}'$ and $\vec{b}'$ are perpendicular to $\vec{a}$ and $\vec{b}$ so that two equations are redundant. This is nothing new, since it is well known that at least two lines are necessary so that 3D motion can be estimated from their correspondences. Thus, we need at least two motions of the hand-eye system in order to get two lines from the corresponding screws.

Suppose now that $n \geq 2$ motions are given. We construct the $6n \times 8$ matrix

$$T = \begin{pmatrix} S_1^T & S_2^T & \dots & S_n^T \end{pmatrix}^T \tag{19}$$

which in the noise-free case has rank 6. Since in the noise-free case the equations arise from natural constraints the null-space contains at least the solution (q, q'). It is trivial to see that an additional orthogonal solution is $(0_{4\times 1}, q)$. Hence, the matrix is maximally of rank 6. If all axes $\vec{b}$ are mutually parallel then the rank of the matrix is 5. The proof is quite lengthy and will not be given here, however, it is plausible that in this case a three-parameter family of solutions can not be constrained by the two conditions (18).

We compute the Singular Value Decomposition (SVD) $T = U\Sigma V^T$ where Σ is a diagonal matrix with the singular values, the columns of U are the left singular vectors, and the columns of V are the right singular vectors. If the rank is 6 than the last two right singular vectors $\vec{v}_7$ and $\vec{v}_8$ - corresponding to the two vanishing singular values - span the nullspace of T. We write them as composed of two 4×1 vectors $\vec{v}_7{}^T = (\vec{u}_1^T, \vec{v}_1^T)$ and $\vec{v}_8{}^T = (\vec{u}_2^T, \vec{v}_2^T)$. A vector (q^T, q'^T) satisfying $T(q^T, q'^T)^T = 0$ must be a linear combination of $\vec{v}_7$ and $\vec{v}_8$, hence

$$\begin{pmatrix} q \\ q' \end{pmatrix} = \lambda_1 \begin{pmatrix} \vec{u}_1 \\ \vec{v}_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} \vec{u}_2 \\ \vec{v}_2 \end{pmatrix}.$$

The two degrees of freedom are fixed by the constraints (18) which imply two quadratic equations in λ_1 and λ_2.

Further details, experimental results, and more references can be found in the extended version available under `http://www.informatik.uni-kiel.de/~kd`.

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References

[1] H. Chen. A screw motion approach to uniqueness analysis of head-eye geometry. In *IEEE Conf. Computer Vision and Pattern Recognition*, pages 145–151, Maui, Hawaii, June 3-6, 1991.

[2] J. Chou and M. Kamel. Finding the position and orientation of a sensor on a robot manipulator using quaternions. *Intern. Journal of Robotics Research*, 10(3):240–254, 1991.

[3] W. Clifford. Preliminary sketch of bi-quaternions. *Proc. London Math. Soc.*, 4:381–395, 1873.

[4] J. Funda and R. Paul. A computational analysis of screw transformations in robotics. *IEEE Trans. Robotics and Automation*, 6:348–356, 1990.

[5] R. Horaud and F. Dornaika. Hand-eye calibration. *Intern. Journal of Robotics Research*, 14:195–210, 1995.

[6] Y. Shiu and S. Ahmad. Calibration of wrist-mounted robotic sensors by solving homogeneous transform equations of the form $ax = xb$. *IEEE Trans. Robotics and Automation*, 5:16–27, 1989.

[7] R. Tsai and R. Lenz. A new technique for fully autonomous and efficient 3d robotics hand/eye calibration. *IEEE Trans. Robotics and Automation*, 5:345–358, 1989.

[8] M. Walker, L. Shao, and R. Volz. Estimating 3d-location using dual number quaternions. *CVGIP: Image Understanding*, 54:358–367, 1991.

[9] C. Wang. Extrinsic calibration of a vision sensor mounted on a robot. *IEEE Trans. Robotics and Automation*, 8:161–175, 1992.