Local Analysis of Multi-dimensional Signals

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The aim of image analysis is the interpretation of certain subsets of a given multi-dimensional signal by meaningful concepts within the framework of the task at hand. This means e.g. finding a certain object, object classification, texture classification or motion analysis. There is a natural abstraction hierarchy of structure concepts which corresponds to a certain hierarchy of scale. At each scale the decomposition of an image into the set of available concepts may be different. Image analysis is to a main portion local analysis. Here the term local means applying a scaled point operator concept. This local approach of image analysis differs considerably from global analysis which is more well-known to the mathematics community because of its relation to the Fourier transform. In local image analysis the multi-dimensional generalizations of the Hilbert transform take on the key role of getting access to meaningful spectral and geometric feature extraction.

We will report on the design of quadrature filters as linear and shift invariant operators which deliver local signal descriptions as: intrinsic dimension (that is the differential geometric type of structure), local spectral representations as local amplitude and local phase, and the orientation as a geometric feature. While local amplitude indicates the amount of local signal dynamics, local phase is a measure of parity symmetry. By applying the Riesz transform as generalized Hilbert transform, all these features result from the monogenic extension of a real valued n-dimensional signal $f_n(\mathbf{x}) \in \mathbb{R}^n$ to a vector valued signal $\mathbf{f}_{n+1}(\mathbf{x}) \in \mathbb{R}_{n+1}$. In addition, the convolution of the signal with a Poisson kernel of scale s in the upper half space directed to x_{n+1} delivers a signal representation in a Poisson scale-space. Combined with the Riesz transform, the Poisson kernel finally establishes a monogenic scale-space in which all local features exist as both independent and mutually dependent scale-space concepts. Hence, the (intrinsic) scale becomes an additional feature of structure description.

The application of the Riesz transform to scalar valued multi-dimensional signals was the matter of our studies in the past. A survey on that topic was given in [2]. However, this approach has the drawback of establishing only partially an extension of the Hilbert transform from 1D to nD for the purpose of image analysis. This fact is not well recognized in Clifford analysis because the distinction between the intrinsic dimension and the global embedding dimension of a signal (or a function) has not been considered yet. The intrinsic dimension (idD) corresponds to the codimension of a subset of an n-dimensional function. The Riesz transform only represents the i1D case, while in 2D signals also i2D structures are of importance. We developed a possible extension of a monogenic signal representation which delivers for all intrinsic dimensions of a 2D signal meaningful features of local signal analysis. Interestingly, in that approach, i2D signals are transformed by a second order spherical harmonic as a new generalization of the Hilbert transform, see [3]. This was possible by starting with a tensor valued real 2D signal and by embedding that signal tensor into M $(2, \mathbb{R}_3)$.

We will enlighten more in detail the derivation of our tensor valued monogenic signal representation and its interpretation from the viewpoint of image analysis. Finally, we will present a new scale-space concept, derived from the used tensor representation. Our approach is a differential geometric one. In the moment we are interested in evaluating a monogenic representation of the curvature tensor instead of a single scalar valued signal.

Let be $f(\mathbf{x}) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a scalar valued 2D signal. We will consider instead a vector valued signal representation, embedded in $\mathbb{R}_3$, $\mathbf{f}(\mathbf{x}) : \mathbb{R}^2 \longrightarrow \mathbb{R}\mathbf{e}_3$ with $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x\mathbf{e}_1 + y\mathbf{e}_2) = f(x, y)\mathbf{e}_3$. By convolution with a monogenic Hessian operator $h_M \in$ M $(2, \mathbb{R}_3)$,

$$h_M = h_e + h_o = h_e + h_R * h_e,$$

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with

$$h_e = \begin{pmatrix} \partial^2_{xx} & -\partial^2_{xy}\mathbf{e}_{12} \\ \partial^2_{xe}\mathbf{e}_{12} & \partial^2_{yy} \end{pmatrix},$$

h_R being the Riesz transform and h_e, h_o being the even and odd parts of the monogenic Hessian operator, the signal will be transformed to a local monogenic matrix representation, $T(\mathbf{x})\colon \mathbb{R}^2 \longrightarrow \mathrm{M}\,(2, \mathbb{R}_3)$. We call $T(\mathbf{x})$ the monogenic curvature tensor because the Hessian is tightly related to the curvature tensor. It is well known that the trace of the curvature tensor delivers the mean curvature and the determinant of the curvature tensor delivers the Gaussian curvature. From both curvature measures the types of surfaces (plane: i0D, parabolic: i1D, elliptic and hyperbolic: i2D) can be classified. By doing that in the monogenic framework, we will get the monogenic mean curvature,

$$\mathbf{f}_{i1D}(\mathbf{x}) = \mathbf{t}_e(\mathbf{x}) + \mathbf{t}_o(\mathbf{x}),$$

and the monogenic Gaussian curvature,

$$\mathbf{f}_{i2D}(\mathbf{x}) = \mathbf{d}_e(\mathbf{x}) + \mathbf{d}_o(\mathbf{x}),$$

respectively. Interestingly, $\mathbf{f}_{i1D}$ is identical with the monogenic signal,

$$\mathbf{f}_{i1D}(\mathbf{x}) \equiv \mathbf{f}_M(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + (h_R * \mathbf{f})\,(\mathbf{x}),$$

and f_{i2D}, called the monogenic curvature signal, $\mathbf{f}_{MC}$,

$$f_{i2D}(\mathbf{x}) \equiv \mathbf{f}_{MC}(\mathbf{x}) = \mathbf{d}_e(\mathbf{x}) + ((\mathbf{e}_1 h_2 \mathbf{e}_3) * \mathbf{d}_e(\mathbf{x})$$

establishes the second order spherical harmonic, h_2, as a new generalized Hilbert transform, which transforms the even representation of the i2D signal to the odd one. Note that $h_R \equiv h_1$ with h_1 being the first order spherical harmonic. We will show this by representing the monogenic Hessian operator in terms of spherical harmonics, and by evaluating the symmetries of h_e and h_o.

In fact, contemporary we are analyzing only the quadrature phase relation between the even and odd parts of the monogenic curvature tensor, $T = T_e + T_o$. This corresponds to a certain signal model, where two i1D structures of even or odd symmetry are crossing one another with a flexible opening angle. Compared with a former model of i2D signals, called the structure multivector [1], this approach has the advantage of being completely rotation invariant and being not restricted to perpendicularly crossing i1D structures.

The evaluation of $\mathbf{f}_{MC}$ will be performed analogously to that of the monogenic signal, $\mathbf{f}_M$. We will demonstrate some examples of the phase analysis and some first results about the monogenic curvature scale-space. The last one is in fact a scale-space associated to the monogenic Gaussian curvature.

References

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