14. A Unified Description of Multiple View Geometry

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14.1 Introduction

Multiple view tensors play a central role in many areas of Computer Vision. The *Fundamental Matrix*, *Trifocal Tensor* and *Quadfocal Tensor* have been investigated by many researchers using a number of different formalisms. For example, standard matrix analysis has been used in [106] and [215]. An analysis of multiple view tensors in terms of Grassmann-Cayley (GC) algebra can be found in [82], [184], [80]. Geometric Algebra (GA) has also been applied to the problem [189], [190], [147], [146].

In this article we will show how Geometric Algebra can be used to give a unified *geometric* picture of multiple view tensors. It will be seen that with the GA approach multiple view tensors can be derived from simple geometric considerations. In particular, constraints on the internal structure of multiple view tensors will all be derived from the trivial fact that the intersection points of a line with three planes, all lie along a line. Our analysis will also show how closely linked the numerous different expressions for multiple view tensors are.

The structure of this article will be as follows. First we give a short introduction to projective geometry, mainly to introduce our notation. We then describe the Fundamental Matrix, the Trifocal Tensor and the Quadfocal Tensor in detail, investigating their derivations, inter-relations and other properties. Following on from these analytical investigations, we show how the self-consistency of a trifocal tensor influences its reconstruction quality. We end this article with some conclusions and a table summarising the main properties of the three multiple view tensors described here.

14.2 Projective Geometry

In this section we will outline the GA framework for projective geometry. We assume that the reader is familiar with the basic ideas of GA and is able to manipulate GA expressions.

We define a set of 4 orthonormal basis vectors $\{e_1, e_2, e_3, e_4\}$ with signature $\{---+\}$. The pseudoscalar of this space is defined as $I = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. A vector in this 4D-space ($\mathbb{P}^3$), which will be called a *homogeneous* vector, can then be regarded as a projective line which describes a point in the corresponding 3D-space ($\mathbb{E}^3$). Also, a line in $\mathbb{E}^3$ is represented in $\mathbb{P}^3$ by the outer product of two homogeneous vectors, and a plane in $\mathbb{E}^3$ is given by the outer product of three homogeneous vectors in $\mathbb{P}^3$. In the following, homogeneous vectors in $\mathbb{P}^3$ will be written as capital letters, and their corresponding 3D-vectors in $\mathbb{E}^3$ as lower case letters in bold face.

Note that the set of points $\{X\}$ that lie on a line $(A \wedge B)$ are those that satisfy $X \wedge (A \wedge B) = 0$. Similarly, a plane is defined through the set of points $\{X\}$ that satisfy $X \wedge (A \wedge B \wedge C) = 0$. Therefore, it is clear that if two lines, or a line and a plane intersect, their outer product is zero.

The projection of a 4D vector A into $\mathbb{E}^3$ is given by,

$$\boldsymbol{a} = \frac{A \wedge e_4}{A \cdot e_4}$$

This is called the *projective split*. Note that a homogeneous vector with no e_4 component will be projected onto the plane at infinity.

A set $\{A_\mu\}$ of four homogeneous vectors forms a basis or *frame* of $\mathbb{P}^3$ if and only if $(A_1 \wedge A_2 \wedge A_3 \wedge A_4) \neq 0$. The *characteristic pseudoscalar* of this frame for 4 such vectors is defined as $I_a = A_1 \wedge A_2 \wedge A_3 \wedge A_4$. Note that $I_a = \rho_a I$, where ρ_a is a scalar. This and results relating the inner products of multivectors with the pseudoscalars of the space are given in [190].

Another concept which is very important in the analysis to be presented is that of the dual of a multivector X. This is written as X^* and is defined as $X^* = X I^{-1}$. It will be extremely useful to introduce the *dual bracket* and the *inverse dual bracket*. They are related to the bracket notation as used in GC algebra and GA, [146]. The bracket of a pseudoscalar P is a scalar, defined as the dual of P in GA. That is, $[P] = P I^{-1}$. The dual and inverse dual brackets are defined as

$$[\![A_{\mu_1} \cdots A_{\mu_n}]\!]_a \equiv (A_{\mu_1} \wedge \ldots \wedge A_{\mu_n}) I_a^{-1} \tag{14.1a}$$

$$[\![A_{\mu_1} \cdots A_{\mu_n}]\!] \equiv (A_{\mu_1} \wedge \ldots \wedge A_{\mu_n}) I^{-1} \tag{14.1b}$$

$$\langle\!\langle A_{\mu_1} \cdots A_{\mu_n} \rangle\!\rangle_a \equiv (A_{\mu_1} \wedge \ldots \wedge A_{\mu_n}) I_a \tag{14.2a}$$
$$\langle\!\langle A_{\mu_1} \cdots A_{\mu_n} \rangle\!\rangle \equiv (A_{\mu_1} \wedge \ldots \wedge A_{\mu_n}) I \tag{14.2b}$$

with $n \in \{0, 1, 2, 3, 4\}$. The range given here for n means that in $\mathbb{P}^3$ none, one, two, three or four homogeneous vectors can be bracketed with a dual or inverse dual bracket. For example, if $P = A_1 \wedge A_2 \wedge A_3 \wedge A_4$, then $[\![A_1 A_2 A_3 A_4]\!] = [\![P]\!] = [P] = \rho_a$.

Using this bracket notation the *normalized reciprocal A-frame*, written $\{A_a^{\mu}\}$, is defined as $A_a^{\mu_1} = [\![A_{\mu_2} A_{\mu_3} A_{\mu_4}]\!]_a$. It is also useful to define a *standard reciprocal A-frame*: $A^{\mu_1} = [\![A_{\mu_2} A_{\mu_3} A_{\mu_4}]\!]$. Then, $A_\mu \cdot A_a^\nu = \delta_\mu^\nu$ and $A_\mu \cdot A^\nu = \rho_a \delta_\mu^\nu$, where δ_μ^ν is the Kronecker delta. That is, *a reciprocal frame vector is nothing else but the dual of a plane*. In the GC algebra these reciprocal vectors would be defined as elements of a *dual space*, which is indeed what is done in [80]. However, because GC algebra does not have an explicit inner product, elements of this dual space cannot operate on elements of the "normal" space. Hence, the concept of reciprocal frames cannot be defined in the GC algebra.

A reciprocal frame can be used to transform a vector from one frame into another. That is, $X = (X \cdot A_a^\mu) A_\mu = (X \cdot A_\nu) A_a^\nu$. Note that in general we will use greek indices to count from 1 to 4 and latin indices to count from 1 to 3. We also adopt the convention that if a subscript index is repeated as a superscript, or vice versa, it is summed over its implicit range, unless stated otherwise. That is, $\sum_{\mu=1}^{4} (X \cdot A_a^\mu) A_\mu \equiv (X \cdot A_a^\mu) A_\mu$.

It will be important later not only to consider vector frames but also line frames. The A-line frame $\{L_a^i\}$ is defined as $L_a^{i_1} = A_{i_2} \wedge A_{i_3}$. The $\{i_1, i_2, i_3\}$ are assumed to be an even permutation of $\{1, 2, 3\}$. The *normalised reciprocal A-line frame* $\{\bar{L}_i^a\}$ and the *standard reciprocal A-line frame* $\{L_i^a\}$ are given by $\bar{L}_i^a = [\![A_i A_4]\!]_a$ and $L_i^a = [\![A_i A_4]\!]$, respectively. Hence, $L_a^i \cdot \bar{L}_j^a = \delta_j^i$ and $L_a^i \cdot L_j^a = \rho_a \delta_j^i$. Again, this shows the universality of the inner product: bivectors can be treated in the same fashion as vectors.

The *meet* and *join* are the two operations needed to calculate intersections between two lines, two planes or a line and a plane – these are discussed in more detail in [190], [146] and [118]; here we will give just the most relevant expression for the meet. If A and B represent two planes or a plane and a line in $\mathbb{P}^3$ their meet may be written as

$$A \vee B = \langle\!\langle [\![A]\!][\![B]\!] \rangle\!\rangle = [\![A]\!] \cdot B \equiv (A I^{-1}) \cdot B \tag{14.3}$$

From this equation it also follows that

$$\langle\!\langle A \rangle\!\rangle \vee \langle\!\langle B \rangle\!\rangle = \langle\!\langle AB \rangle\!\rangle \tag{14.4}$$

Later on we will need the *dual representations* of points and lines. For lines they are given by,

$$L_a^{i_1} = A_{i_2} \wedge A_{i_3} \simeq \langle\!\langle A_a^{i_1} A_a^4 \rangle\!\rangle \quad \text{and} \quad A_{i_1} \wedge A_4 \simeq \langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \tag{14.5}$$

The symbol $\simeq$ denotes equality up to a scalar factor. This shows that a line can either be expressed as the outer product of two vectors or by the intersection of two planes, since $\langle\!\langle A_a^{i_1} A_a^4 \rangle\!\rangle = \langle\!\langle A_a^{i_1} \rangle\!\rangle \vee \langle\!\langle A_a^4 \rangle\!\rangle$. Similarly, for points we have

$$A_{\mu_1} \simeq \langle\!\langle A_a^{\mu_2} A_a^{\mu_3} A_a^4 \rangle\!\rangle \tag{14.6}$$

That is, a point can also be described as the intersection of three planes.

A pinhole camera can be defined by 4 homogeneous vectors in $\mathbb{P}^3$: one vector gives the optical centre and the other three define the image plane [147], [146]. Thus, the vectors needed to define a pinhole camera also define a frame for $\mathbb{P}^3$. Conventionally the fourth vector of a frame, eg. A_4, defines the optical centre, and the outer product of the other three defines the image plane.

Suppose that X is given in some frame $\{Z_\mu\}$ as $X = \zeta^\mu Z_\mu$, it can be shown [190] that the projection of some point X onto image plane A can be written as

$$X_a = (\, X \cdot A^i\,) A_i = (\zeta^\mu \, Z_\mu \cdot A^i\,) A_i = \zeta^\mu K_{\underset{\mu}{i}} A_i \,; \quad K_{\underset{\mu}{i}} \equiv Z_\mu \cdot A^i \tag{14.7}$$

The matrix $K_{\underset{\mu}{i}}$ is the *camera matrix* of camera A, for projecting points given in the Z-frame onto image plane[1] A. In general we will write the projection of some point X onto image plane P as $X \xrightarrow{P} X_p$.

In [80] the derivations begin with the camera matrices by noting that the row vectors *refer* to planes. As was shown here, the row vectors of a camera matrix are the reciprocal frame vectors $\{A^i\}$, whose dual *is* a plane.

With the same method as before, lines can be projected onto an image plane. For example, let L be some line in $\mathbb{P}^3$, then its projection onto image plane A is: $(L \wedge A_4) \vee (A_1 \wedge A_2 \wedge A_3) = (\, L \cdot L_i^a \,) L_a^i$.

An *epipole* is the projection of the optical centre of one camera onto the image plane of another. Therefore epipoles contain important information about the relative placements of cameras.

As an example consider two cameras A and B represented by frames $\{A_i\}$ and $\{B_i\}$, respectively. The projection of the optical centre of camera B onto image plane A will be denoted E_{ab}. That is, $E_{ab} = B_4 \cdot A^i A_i$ or simply $E_{ab} = \varepsilon_{ab}^i A_i$, with $\varepsilon_{ab}^i \equiv B_4 \cdot A^i$. Note, that we adopted the general GA convention that the inner product takes precedence over the geometric product[2].The only other epipole in this two camera set-up is E_{ba} given by

[1] Note that the indices of K are not given as super- and subscripts of K but are raised (or lowered) relative to each other. This notation was adopted since it leaves the superscript position of K free for other usages.

[2] Also, the outer product has precedence over the inner product. That is, $A \cdot B \wedge C = A \cdot (B \wedge C)$.

$E_{ba} = A_4 \cdot B^i B_i$. This may also be written as $E_{ba} = \varepsilon^i_{ba} B_i$, with $\varepsilon^i_{ba} \equiv A_4 \cdot B^i$. If there are three cameras then each image plane contains two epipoles. With four cameras each image plane contains three epipoles. In general the total number of epipoles is $N(N-1)$ where N is number of cameras present.

Let $\{B_\mu\}$ define a camera in $\mathbb{P}^3$ and $\{A_\mu\}$ be some other frame of the same projective space. Also, define A_4 to be the origin of $\mathbb{P}^3$. Then E_{ba} contains some information about the placement of camera B relative to the origin. Therefore, $A_4 \cdot B^j$ may be regarded as a *unifocal* tensor U_b.

$$U^i_b \equiv \varepsilon^i_{ba} = A_4 \cdot B^i = K^b_{i_4} \simeq \langle\!\langle A^1 A^2 A^3 B^i \rangle\!\rangle \tag{14.8}$$

Obviously the unifocal tensor is of rank 1. The definition of a unifocal tensor is only done for completeness and is not strictly necessary since every unifocal tensor is also an epipole vector.

Later on we will have to deal with determinants of various 3×3 matrices. Such a determinant can be written in terms of the ϵ_{ijk} operator, which is defined as

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if the } \{ijk\} \text{ form an even permutation of } \{123\} \\ 0 & \text{if any two indices of } \{ijk\} \text{ are equal} \\ -1 & \text{if the } \{ijk\} \text{ form an odd permutation of } \{123\} \end{cases} \tag{14.9}$$

Let $\alpha^{i_a}_1$, $\alpha^{i_b}_2$ and $\alpha^{i_c}_3$ give the three rows of a 3×3 matrix M. Then the determinant of M is $\det(M) = \epsilon_{i_a i_b i_c} \alpha^{i_a}_1 \alpha^{i_b}_2 \alpha^{i_c}_3$. Note that there is an implicit summation over all indices. It will simplify the notation later on if we define

$$\det(\alpha^{i_a}_1, \alpha^{i_b}_2, \alpha^{i_c}_3)_{i_a i_b i_c} = \det(\alpha^i_j)_{ij} \equiv \epsilon_{i_a i_b i_c} \alpha^{i_a}_1 \alpha^{i_b}_2 \alpha^{i_c}_3 = \det(M) \tag{14.10}$$

Furthermore, if the rows of the matrix M are written as vectors $\boldsymbol{a}_j = \alpha^i_j e_i$, then we can also adopt the notation

$$\det(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = |\boldsymbol{a}_1 \boldsymbol{a}_2 \boldsymbol{a}_3| \equiv \det(M) \tag{14.11}$$

As an example, let the $\{A_\mu\}$ form a frame of $\mathbb{P}^3$, with reciprocal frame $\{A^\mu\}$. Then from the definition of the square and angle brackets, it follows that

$$\epsilon_{i_a i_b i_c} = [\![A_{i_a} A_{i_b} A_{i_c} A_4]\!]_a \quad \text{and} \quad \epsilon^{i_a i_b i_c} = \langle\!\langle A^{i_a} A^{i_b} A^{i_c} A^4 \rangle\!\rangle_a \tag{14.12}$$

Therefore, we may, for example, express a determinant as $\det(\alpha^i_j)_{ij} = \alpha^{i_a}_1 \alpha^{i_b}_2 \alpha^{i_c}_3 [\![A_{i_a} A_{i_b} A_{i_c} A_4]\!]_a$.

14.3 The Fundamental Matrix

14.3.1 Derivation

Let $\{A_\mu\}$ and $\{B_\mu\}$ define two cameras in $\mathbb{P}^3$. A point X in $\mathbb{P}^3$ may be transformed into the A and B frames via

$$X = X \cdot A^{\mu}_{a} A_{\mu} = X \cdot B^{\nu}_{b} B_{\nu} \tag{14.13}$$

Recall that there is an implicit summation over μ and ν. From that follows that the line $A_4 \wedge X$ can also be written as

$$\begin{aligned} A_4 \wedge X &= X \cdot A^{i}_{a}\, A_4 \wedge A_i \\ &= \rho_a^{-1}\, A_4 \wedge X_a \end{aligned} \tag{14.14}$$

where $X_a = X \cdot A^i\, A_i$. Let X_a and X_b be the images of some point $X \in \mathbb{P}^3$ taken by cameras A and B, respectively. Then, since the lines from A and B to X intersect at X

$$\begin{aligned} 0 &= (\, A_4 \wedge X \wedge B_4 \wedge X\,) I^{-1} \\ &\simeq (\, A_4 \wedge X_a \wedge B_4 \wedge X_b\,) I^{-1} \\ &= \alpha^i \beta^j [\![A_4 A_i B_4 B_j]\!] \end{aligned} \tag{14.15}$$

where $\alpha^i \equiv X \cdot A^i$ and $\beta^j \equiv X \cdot B^j$ are the image point coordinates of X_a and X_b, respectively. Therefore, for a *Fundamental Matrix* defined as

$$F_{ij} \equiv [\![A_4 A_i B_4 B_j]\!] \tag{14.16}$$

we have

$$\alpha^i \beta^j F_{ij} = 0 \tag{14.17}$$

if the image points given by $\{\alpha^i\}$ and $\{\beta^j\}$ are images of the same point in space. Note, however, that equation (14.17) holds as long as X_a is the image of any point along $A_4 \wedge X_a$ and X_b is the image of any point along $B_4 \wedge X_b$. In other words, the condition in equation (14.17) only ensures that lines $A_4 \wedge X_a$ and $B_4 \wedge X_b$ are co-planar.

In the following let any set of indices of the type $\{i_1, i_2, i_3\}$ be an even permutation of $\{1, 2, 3\}$. It may be shown that

$$[\![B_4 B_{j_1}]\!] \simeq B^{j_2} \wedge B^{j_3} \tag{14.18}$$

Thus, equation (14.16) can also be written as

$$F_{ij_1} \simeq (A_i \wedge A_4) \cdot (B^{j_2} \wedge B^{j_3}) \tag{14.19}$$

This may be expanded to

$$\begin{aligned} F_{ij_1} &= (A_4 \cdot B^{j_2})(A_i \cdot B^{j_3}) - (A_4 \cdot B^{j_3})(A_i \cdot B^{j_2}) \\ &= U^{j_2}_{b} K^{b}_{j_3 i} - U^{j_3}_{b} K^{b}_{j_2 i} \end{aligned} \tag{14.20}$$

That is, the Fundamental Matrix is just the standard cross product between the epipole[3] $U^{\bullet}_{b}$ and the column vectors $K^{b}_{\bullet i}$.

$$F_{i\bullet} \simeq U^{\bullet}_{b} \times K^{b}_{\bullet i} \tag{14.21}$$

In order to have a unified naming convention the Fundamental Matrix will be refered to as the *bifocal tensor*.

[3] Recall that $U_b \equiv E_{ba}$.

14.3.2 Rank of F

Note that we use the term "rank" in relation to tensors in order to generalise the notion of rank as used for matrices. That is, we would describe a rank 2 matrix as a rank 2, 2-valence tensor.

In general a tensor may be decomposed into a linear combination of rank 1 tensors. The minimum number of terms necessary for such a decomposition gives the *rank* of the tensor. For example, a rank 1, 2-valence tensor M is created by combining the components $\{\alpha^i\}$, $\{\beta^i\}$ of two vectors as $M^{ij} = \alpha^i \beta^j$.

The rank of F can be found quite easily from geometric considerations. Equation (14.16) can also be written as

$$F_{ij} \simeq A_i \cdot [\![A_4 B_4 B_j]\!] \tag{14.22}$$

The expression $[\![A_4 B_4 B_j]\!]$ gives the normal to the plane $(A_4 \wedge B_4 \wedge B_j)$. This defines three planes, one for each value of j, all of which contain the line $A_4 \wedge B_4$. Hence, all three normals lie in a plane. Furthermore, no two normals are identical since the $\{B_j\}$ are linearly independent by definition. It follows directly that at most two columns of F_{ij} can be linearly independent. Therefore, F is of rank 2.

The rank of the bifocal tensor F can also be arrived at through a *minimal* decomposition of F into rank 1 tensors. To achieve this we first define a new A-image plane frame $\{A'_i\}$ as

$$A'_i \equiv s(A_i + t_i A_4) \tag{14.23}$$

where s and the $\{t_i\}$ are some scalar components. Thus we have

$$\begin{aligned} A_4 \wedge A'_i &= s A_4 \wedge (A_i + t_i A_4) \\ &= s A_4 \wedge A_i \end{aligned} \tag{14.24}$$

Hence, F is left unchanged up to an overall scale factor under the transformation $A_i \longrightarrow A'_i$. In other words, the image plane bases $\{A_i\}$ and $\{B_j\}$ can be changed along the projective rays $\{A_4 \wedge A_i\}$ and $\{B_4 \wedge B_j\}$, respectively, without changing the bifocal tensor relating the two cameras. This fact limits the use of the bifocal tensor, since it cannot give any information about the actual placement of the image planes.

Define two bifocal tensors F and F' as

$$F_{ij} = [\![A_4 A_i B_4 B_j]\!] \tag{14.25a}$$
$$F'_{ij} = [\![A_4 A'_i B_4 B_j]\!] \tag{14.25b}$$

From equation (14.24) it follows directly that $F_{ij} \simeq F'_{ij}$. Since the $\{A'_i\}$ can be chosen arbitrarily along the line $A_4 \wedge A_i$ we may write

$$A'_i = (A_4 \wedge A_i) \vee P \tag{14.26}$$

where P is some plane in $\mathbb{P}^3$. $P = (B_4 \wedge B_1 \wedge B_2)$ seems a good choice, since then the $\{A'_i\}$ all lie in a plane together with B_4. The effect of this is that the projections of the $\{A'_i\}$ on image plane B will all lie along a line. The matrix $A'_i \cdot B^j$ therefore only has two linearly independent columns because the column vectors *are* the projections of the $\{A'_i\}$ onto image plane B. That is, $A'_i \cdot B^j$, which is the 3×3 minor of K^b, is of rank 2.

This matrix could only be of rank 1, if the $\{A'_i\}$ were to project to a single point on image plane B, which is only possible if they lie along a line in $\mathbb{P}^3$. However, then they could not form a basis for image plane A which they were defined to be.

Thus $A'_i \cdot B^j$ can minimally be of rank 2. Such a minimal form is what we need to find a minimal decomposition of F into rank 1 tensors using equation (14.20). Substituting $P = (B_4 \wedge B_1 \wedge B_2)$ into equation (14.26) gives

$$\begin{aligned} A'_i &= (A_4 \wedge A_i) \vee (B_4 \wedge B_1 \wedge B_2) \\ &= [\![A_4 A_i]\!] \cdot (B_4 \wedge B_1 \wedge B_2) \\ &= [\![A_4 A_i B_4 B_1]\!] B_2 - [\![A_4 A_i B_4 B_2]\!] B_1 + [\![A_4 A_i B_1 B_2]\!] B_4 \\ &= F_{i1} B_2 - F_{i2} B_1 + [\![A_4 A_i B_1 B_2]\!] B_4 \end{aligned} \tag{14.27}$$

Expanding F' in the same way as F in equation (14.20) and substituting the above expressions for the $\{A'_i\}$ gives

$$\begin{aligned} F'_{ij_1} &= (A_4 \cdot B^{j_2})(A'_i \cdot B^{j_3}) - (A_4 \cdot B^{j_3})(A'_i \cdot B^{j_2}) \\ &= \quad (A_4 \cdot B^{j_2}) \Big[- F_{i2}(B_1 \cdot B^{j_3}) + F_{i1}(B_2 \cdot B^{j_3}) \Big] \\ &\quad - (A_4 \cdot B^{j_3}) \Big[- F_{i2}(B_1 \cdot B^{j_2}) + F_{i1}(B_2 \cdot B^{j_2}) \Big] \\ &= \quad \varepsilon^{j_2}_{ba} \Big[- F_{i2} \delta^{j_3}_1 + F_{i1} \delta^{j_3}_2 \Big] \\ &\quad - \varepsilon^{j_3}_{ba} \Big[- F_{i2} \delta^{j_2}_1 + F_{i1} \delta^{j_2}_2 \Big] \\ &= \quad F_{i1} \Big[\varepsilon^{j_2}_{ba} \delta^{j_3}_2 - \varepsilon^{j_3}_{ba} \delta^{j_2}_2 \Big] \\ &\quad - F_{i2} \Big[\varepsilon^{j_2}_{ba} \delta^{j_3}_1 - \varepsilon^{j_3}_{ba} \delta^{j_2}_1 \Big] \end{aligned} \tag{14.28}$$

where we used the fact that $B_4 \cdot B^j = 0$. Clearly, F_{i1}, F_{i2} and the expressions in the square brackets all represent vectors. Therefore, equation (14.28) expresses F' as a linear combination of two rank 1 tensors (matrices). This shows again that the bifocal tensor is of rank 2.

But why should we do all this work of finding a minimal decomposition of F if its rank can be found so much more easily from geometric considerations? There are two good reasons:

1. for the trifocal and quadfocal tensor, a minimal decomposition will be the easiest way to find the rank, and

2. such a decomposition is useful for evaluating F with a non-linear algorithm, since the self-consistency constraints on F are automatically satisfied.

14.3.3 Degrees of Freedom of F

Equation (14.28) is in fact a minimal parameterisation of the bifocal tensor. This can be seen by writing out the columns of F'.

$$F'_{i1} = -\varepsilon^3_{ba} F_{i1}\,; \quad F'_{i2} = -\varepsilon^3_{ba} F_{i2}\,; \quad F'_{i3} = \varepsilon^1_{ba} F_{i1} + \varepsilon^2_{ba} F_{i2} \tag{14.29}$$

As expected, the third column (F_{i3}) is a linear combination of the first two. Since an overall scale is not important we can also write

$$F'_{i1} = F_{i1}\,; \quad F'_{i2} = F_{i2}\,; \quad F'_{i3} = -\bar{\varepsilon}^1_{ba} F_{i1} - \bar{\varepsilon}^2_{ba} F_{i2} \tag{14.30}$$

where $\bar{\varepsilon}^i_{ba} \equiv \varepsilon^i_{ba}/\varepsilon^3_{ba}$. This is the most general form of a rank 2, 3×3 matrix. Furthermore, since there are no more constraints on F_{i1} and F_{i2} this is also a minimal parameterisation of the bifocal tensor. That is, eight parameters are minimally necessary to form the bifocal tensor. It follows that since an overall scale is not important the bifocal tensor has *seven* degrees of freedom (DOF).

This DOF count can also be arrived at from more general considerations: each camera matrix has 12 components. However, since an overall scale is not important, each camera matrix adds only 11 DOF. Furthermore, the bifocal tensor is independent of the choice of basis. Therefore, it is invariant under a projective transformation, which has 16 components. But again, an overall scale is not important. Thus only 15 DOF can be subtracted from the DOF count due to the camera matrices. For two cameras we therefore have $2 \times 11 - 15 = 7$ DOF.

14.3.4 Transferring Points with F

The bifocal tensor can also be used to transfer a point in one image to a line in the other. Starting again from equation (14.16) the bifocal tensor can be written as

$$\begin{aligned} F_{ij} &= [\![A_i A_4 B_j B_4]\!] \\ &= (A_i \wedge A_4) \cdot [\![B_j B_4]\!] \\ &= (A_i \wedge A_4) \cdot L^b_j \end{aligned} \tag{14.31}$$

This shows that F_{ij} gives the components of the projection of line $(A_i \wedge A_4)$ onto image plane B. Therefore,

$$(A_i \wedge A_4) \xrightarrow{B} F_{ij} L^j_b. \tag{14.32}$$

Since $A_4 \xrightarrow{B} E_{ba}$ (the epipole on image plane B), $F_{ij}L_b^j$ defines an epipolar line.

Thus, contracting F with the coordinates of a point on image plane A, results in the homogeneous line coordinates of a line passing through the corresponding point on image plane B and the epipole E_{ba}.

$$\alpha^i F_{ij} = \lambda_j^b \tag{14.33}$$

where the $\{\alpha^i\}$ are some point coordinates and the $\{\lambda_j^b\}$ are the homogeneous line coordinates of an epipolar line.

14.3.5 Epipoles of *F*

Recall that if there are two cameras then two epipoles are defined;

$$E_{ab} \equiv B_4 \cdot A^i A_i = \varepsilon_{ab}^i A_i \tag{14.34a}$$

$$E_{ba} \equiv A_4 \cdot B^i B_i = \varepsilon_{ba}^i B_i \tag{14.34b}$$

Contracting F_{ij} with ε_{ab}^i gives

$$\begin{aligned} \varepsilon_{ab}^i F_{ij} &= \varepsilon_{ab}^i [\![A_4 A_i B_4 B_j]\!] \\ &= \rho_a [\![A_4 (B_4 \cdot A_a^i \, A_i) B_4 B_j]\!] \\ &= \rho_a [\![A_4 B_4 B_4 B_j]\!] \ ; \quad \text{from equation (14.14)} \\ &= 0 \end{aligned} \tag{14.35}$$

Similarly,

$$\varepsilon_{ba}^j F_{ij} = 0 \tag{14.36}$$

Therefore, vectors $\{\varepsilon_{ab}^i\}$ and $\{\varepsilon_{ba}^j\}$ can be regarded respectively as the left and right null spaces of matrix F. Given a bifocal tensor F, its epipoles can therefore easily be found using, for example, a singular value decomposition (SVD).

14.4 The Trifocal Tensor

14.4.1 Derivation

Let the frames $\{A_\mu\}$, $\{B_\mu\}$ and $\{C_\mu\}$ define three distinct cameras. Also, let $L = X \wedge Y$ be some line in P^3. The plane $L \wedge B_4$ is then the same as the plane $\lambda_i^b L_b^i \wedge B_4$, up to a scalar factor, where $\lambda_i^b = L \cdot L_i^b$. But,

$$L_b^{i_1} \wedge B_4 = B_{i_2} \wedge B_{i_3} \wedge B_4 = \langle\!\langle B^{i_1} \rangle\!\rangle$$

Intersecting planes $L \wedge B_4$ and $L \wedge C_4$ has to give L. Therefore, $(\lambda_i^b \langle\!\langle B^i \rangle\!\rangle) \vee (\lambda_j^c \langle\!\langle C^j \rangle\!\rangle)$ has to give L up to a scalar factor. Now, if two lines intersect, their outer product is zero. Thus, the outer product of lines $X \wedge A_4$ (or $Y \wedge A_4$) and L has to be zero. Note that $X \wedge A_4$ defines the same line as $(\alpha^i A_i) \wedge A_4$, up to a scalar factor, where $\alpha^i = X \cdot A^i$. Figure 14.1 shows this construction.

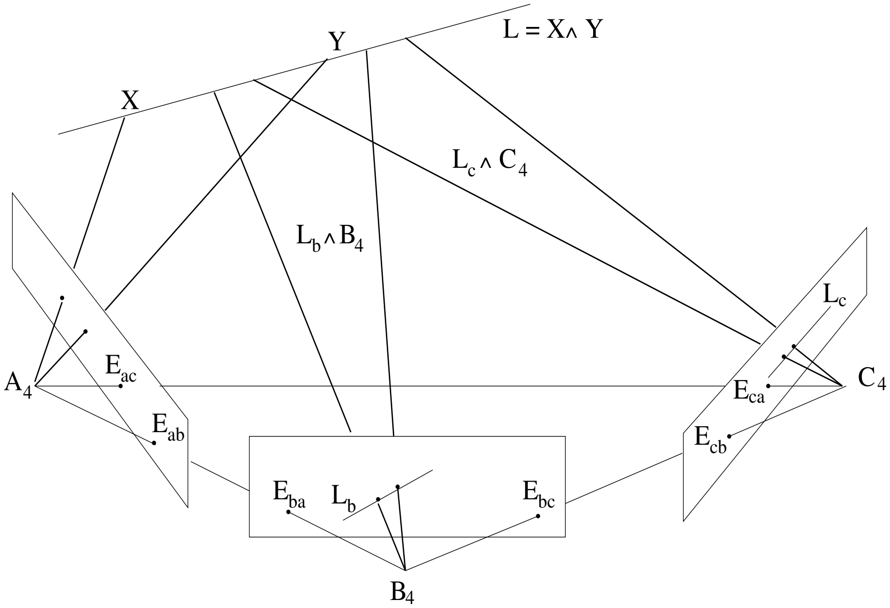

Fig. 14.1. Line projected onto three image planes. Note that although the figure is drawn in $\mathbb{E}^3$, lines and points are denoted by their corresponding vectors in $\mathbb{P}^3$

Combining all these expressions gives

$$\begin{aligned} 0 &= (X \wedge A_4 \wedge L) I^{-1} \\ &= \alpha^i \lambda^b_j \lambda^c_k \left\llbracket (A_i \wedge A_4)(\langle\!\langle B^j \rangle\!\rangle \vee \langle\!\langle C^k \rangle\!\rangle) \right\rrbracket \\ &= \alpha^i \lambda^b_j \lambda^c_k \left\llbracket (A_i \wedge A_4)\langle\!\langle B^j C^k \rangle\!\rangle \right\rrbracket \end{aligned} \tag{14.37}$$

where the identity from equation (14.4) was used. If the trifocal tensor $T_{i}{}^{jk}$ is defined as

$$T_{i}{}^{jk} = \left\llbracket (A_i \wedge A_4)\langle\!\langle B^j C^k \rangle\!\rangle \right\rrbracket \tag{14.38}$$

then, from equation (14.37) it follows that it has to satisfy $\alpha^i \lambda^b_j \lambda^c_k T_{i}{}^{jk} = 0$. This expression for the trifocal tensor can be expanded in a number of different ways. One of them is,

$$\begin{aligned} T_{i}{}^{jk} &= (A_i \wedge A_4) \cdot \llbracket \langle\!\langle B^j C^k \rangle\!\rangle \rrbracket \\ &= (A_i \wedge A_4) \cdot (B^j \wedge C^k) \\ &= (A_4 \cdot B^j)(A_i \cdot C^k) - (A_4 \cdot C^k)(A_i \cdot B^j) \\ &= U^j_b K^c_{k_i} - U^k_c K^b_{j_i} \end{aligned} \tag{14.39}$$

where $K^b_{j_i} \equiv A_i \cdot B^j$ and $K^c_{k_i} \equiv A_i \cdot C^k$ are the camera matrix minors for cameras B and C, respectively, relative to camera A. This is the expression for the trifocal tensor given by Hartley in [106]. Note that the camera matrix for camera A would be written as $K^a_{j_\mu} \equiv A_\mu \cdot A^j \simeq \delta^j_i$. That is, $K^a = [I|0]$ in standard matrix notation. In many other derivations of the trifocal tensor (eg. [106]) this form of the camera matrices is assumed at the beginning. Here, however, the trifocal tensor is defined first geometrically and we then find that it implies this particular form for the camera matrices.

14.4.2 Transferring Lines

The trifocal tensor can be used to transfer lines from two images to the third. That is, if the image of a line in $\mathbb{P}^3$ is known on two image planes, then its image on the third image plane can be found. This can be seen by expanding equation (14.38) in the following way,

$$\begin{aligned} T_i{}^{jk} &= [\![A_i A_4]\!] \cdot \langle\!\langle B^j C^k \rangle\!\rangle \\ &= L^a_i \cdot \langle\!\langle B^j C^k \rangle\!\rangle \end{aligned} \tag{14.40}$$

This shows that the trifocal tensor gives the homogeneous line components of the projection of line $\langle\!\langle B^j C^k \rangle\!\rangle$ onto image plane A. That is,

$$\langle\!\langle B^j C^k \rangle\!\rangle \xrightarrow{A} T_i{}^{jk} L^i_a \tag{14.41}$$

It will be helpful later on to define the following two lines.

$$T^{jk} \equiv \langle\!\langle B^j C^k \rangle\!\rangle \tag{14.42a}$$

$$T^{jk}_a \equiv T_i{}^{jk} L^i_a \tag{14.42b}$$

such that $T^{jk} \xrightarrow{A} T^{jk}_a$. Let the $\{\lambda^b_j\}$ and $\{\lambda^c_k\}$ be the homogeneous line coordinates of the projection of some line $L \in \mathbb{P}^3$ onto image planes B and C, respectively. Then recall that $\lambda^b_j \lambda^c_k \langle\!\langle B^j C^k \rangle\!\rangle$ gives L up to an overall scalar factor, i.e.

$$L \simeq \lambda^b_j \lambda^c_k \langle\!\langle B^j C^k \rangle\!\rangle \; ; \qquad \lambda^b_j \equiv L \cdot L^b_j \;\text{ and }\; \lambda^c_k \equiv L \cdot L^c_k \tag{14.43}$$

The image of L on image plane A, L_a, can therefore be found via

$$\begin{aligned} L_a &= L \cdot L^a_i L^i_a \\ &\simeq \lambda^b_j \lambda^c_k \langle\!\langle B^j C^k \rangle\!\rangle \cdot L^a_i L^i_a \\ &= \lambda^b_j \lambda^c_k T_i{}^{jk} L^i_a \end{aligned} \tag{14.44}$$

Thus, we have

$$\lambda^a_i \simeq \lambda^b_j \lambda^c_k T_i{}^{jk} \tag{14.45}$$

14.4.3 Transferring Points

It is also possible to find the image of a point on one image plane if its image is known on the other two. To see this, the expression for the trifocal tensor needs to be expanded in yet another way. Substituting the dual representation of line $A_{i_1} \wedge A_4$, i.e. $\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle$ into equation (14.38) gives

$$\begin{aligned} T_{i_1}{}^{jk} &= \left[\!\!\left[(A_{i_1} \wedge A_4) \langle\!\langle B^j C^k \rangle\!\rangle \right]\!\!\right] \\ &= \left[\!\!\left[\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \langle\!\langle B^j C^k \rangle\!\rangle \right]\!\!\right] \\ &= \langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \cdot (B^j \wedge C^k) \\ &= \langle\!\langle A_a^{i_2} A_a^{i_3} B^j C^k \rangle\!\rangle \end{aligned} \tag{14.46}$$

It can be shown that this form of the trifocal tensor is equivalent to the determinant form given by Heyden in [120]. Now only one more step is needed to see how the trifocal tensor may be used to transfer points.

$$\begin{aligned} T_{i_1}{}^{jk} &= \langle\!\langle A_a^{i_2} A_a^{i_3} B^j C^k \rangle\!\rangle \\ &= \langle\!\langle A_a^{i_2} A_a^{i_3} B^j \rangle\!\rangle \cdot C^k \\ &= X^T_{i_1}{}^{j} \cdot C^k \;; \quad X^T_{i_1}{}^{j} \equiv \langle\!\langle A_a^{i_2} A_a^{i_3} B^j \rangle\!\rangle \end{aligned} \tag{14.47}$$

Note that the points $\{X^T_{i_1}{}^{j}\}$ are defined through their dual representation as the set of intersection points of lines $\{A_{i_1} \wedge A_4\}$ ($\simeq \{\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle\}$) and planes $\{\langle\!\langle B^j \rangle\!\rangle\}$ ($\simeq \{L_b^j \wedge B_4\}$). Let $L = X \wedge Y$ be a line in $\mathbb{P}^3$. Then

$$X \xrightarrow{A} X_a = \alpha^i A_i \tag{14.48a}$$

$$L \xrightarrow{B} L_B = \lambda_j^b L_b^j \tag{14.48b}$$

Hence

$$\begin{aligned} X &\simeq (\alpha^{i_1} \underbrace{A_{i_1} \wedge A_4}_{\langle\!\langle A^{i_2} A^{i_3} \rangle\!\rangle}) \vee (\lambda_j^b \underbrace{L_b^j \wedge B_4}_{\langle\!\langle B^j \rangle\!\rangle}) \\ &= \textstyle\sum_{i_1} \alpha^{i_1} \lambda_j^b \langle\!\langle A^{i_2} A^{i_3} B^j \rangle\!\rangle \\ &= \alpha^{i_1} \lambda_j^b X^T_{i_1}{}^{j} \end{aligned} \tag{14.49}$$

Now, the projection of X onto image plane C is simply

$$\begin{aligned} X_c &= X \cdot C^k C_k \\ &\simeq \alpha^i \lambda_j^b X^T_{i_1}{}^{j} \cdot C^k C_k \\ &= \alpha^i \lambda_j^b T_i{}^{jk} C_k \end{aligned} \tag{14.50}$$

That is,

$$\eta^k \simeq \alpha^i \lambda_j^b T_{i}{}^{jk} \tag{14.51}$$

with $\eta^k \equiv X \cdot C^k$. Similarly we also have,

$$\beta^k \simeq \alpha^i \lambda_k^c T_{i}{}^{jk} \tag{14.52}$$

Therefore, if the image of a point and a line through that point are known on two image planes, respectively, then the image of the point on the third image plane can be calculated. Note that the line defined by the $\{\lambda_j^b\}$ can be any line that passes through the image of X on image plane B. That is, we may choose the point $(0, 0, 1)$ as the other point the line passes through. Then we have

$$\lambda_1^b = \beta^2 \,; \quad \lambda_2^b = -\beta^1 \,; \quad \lambda_3^b = 0 \tag{14.53}$$

Hence, equation (14.51) becomes

$$\eta^k \simeq \alpha^i (\beta^2 T_{i}{}^{1k} - \beta^1 T_{i}{}^{2k}) \tag{14.54}$$

and equation (14.52) becomes

$$\beta^k \simeq \alpha^i (\eta^2 T_{i}{}^{j1} - \eta^1 T_{i}{}^{j2}) \tag{14.55}$$

14.4.4 Rank of *T*

Finding the rank of T is somewhat harder than for the bifocal tensor, mainly because there is no simple geometric construction which yields its rank. As was mentioned before the rank of a tensor is given by the minimum number of terms necessary for a linear decomposition of it in terms of rank 1 tensors[4]. As for the bifocal tensor, the transformation $A_i \to A'_i = s(A_i + t_i A_4)$ leaves the trifocal tensor unchanged up to an overall scale. A good choice for the $\{A'_i\}$ seems to be

$$A'_i = (A_i \wedge A_4) \vee (B_3 \wedge B_4 \wedge C_4) \tag{14.56}$$

since then all the $\{A'_i\}$ lie in a plane together with B_4 and C_4. Therefore, the camera matrix minors $K^b_{j_i} = A'_i \cdot B^j$ and $K^c_{k_i} = A'_i \cdot C^k$ are of rank 2. As was shown before, this is the minimal rank camera matrix minors can have. To see how this may help to find a minimal decomposition of T recall equation (14.39);

$$T_{i}{}^{jk} = U_b^j K^c_{k_i} - U_c^k K^b_{j_i}$$

[4] For example, a rank 1 3-valence tensor is created by combining the components $\{\alpha^i\}$, $\{\beta^i\}$, $\{\eta^i\}$ of three vectors as $T^{ijk} = \alpha^i \beta^j \eta^k$.

This decomposition of T shows that its rank is at most 6, since U_b and U_c are vectors, and K^c and K^b cannot be of rank higher than 3. Using the above choice for K^b and K^c however shows that the rank of T is 4, since then the rank of the camera matrices is minimal, and we thus have a minimal linear decomposition of T.

14.4.5 Degrees of Freedom of T

As for the bifocal tensor we can also write down an explicit parameterisation for the trifocal tensor. Starting with equation (14.56) we get

$$\begin{aligned}
A'_i &= (A_i \wedge A_4) \vee (B_3 \wedge B_4 \wedge C_4) \\
&= [\![A_i A_4]\!] \cdot (B_3 \wedge B_4 \wedge C_4) \\
&= [\![A_i A_4 B_4 C_4]\!] B_3 - [\![A_i A_4 B_3 C_4]\!] B_4 + [\![A_i A_4 B_3 B_4]\!] C_4 \\
&= \alpha_i^1 B_3 + \alpha_i^2 B_4 + \alpha_i^3 C_4
\end{aligned} \tag{14.57}$$

where α_i^1, α_i^2 and α_i^3 are defined appropriately. The trifocal tensor may be expressed in terms of the $\{A'_i\}$ as follows (see equation (14.39)).

$$\begin{aligned}
T_{i}{}^{jk} &= (A_4 \cdot B^j)(A'_i \cdot C^k) - (A_4 \cdot C^k)(A'_i \cdot B^j) \\
&= \quad (A_4 \cdot B^j)\Big[\alpha_i^1 B_3 \cdot C^k + \alpha_i^2 B_4 \cdot C^k\Big] \\
&\quad - (A_4 \cdot C^k)\Big[\alpha_i^1 B_3 \cdot B^j + \alpha_i^3 C_4 \cdot B^j\Big] \\
&= \quad \varepsilon_{ba}^{j}\Big[\alpha_i^1 B_3 \cdot C^k + \alpha_i^2 \varepsilon_{cb}^{k}\Big] \\
&\quad - \varepsilon_{ca}^{k}\Big[\alpha_i^1 \delta_3^j + \alpha_i^3 \varepsilon_{bc}^{j}\Big]
\end{aligned} \tag{14.58}$$

This decomposition of T has $5 \times 3 + 3 \times 3 - 1 = 23$ DOF. The general formula for finding the DOF of T gives $3 \times 11 - 15 = 18$ DOF. Therefore, equation (14.58) is an overdetermined parameterisation of T. However, it will still satisfy the self-consistency constraints of T.

14.4.6 Constraints on T

To understand the structure of T further, we will derive self-consistency constraints for T. Heyden derives the constraints on T using the "quadratic p-relations" [120]. In GA these relations can easily be established from geometric considerations.

The simplest constraint on T may be found as follows. Recall equation (14.47), where the trifocal tensor was expressed in terms of the projection of points $X^T_{i_1}{}^{j} = \langle\!\langle A_a^{i_2} A_a^{i_3} B^j \rangle\!\rangle$ onto image plane C, i.e.

$$T_{i_1}{}^{jk} = X^T_{i_1}{}^{j} \cdot C^k$$

Now consider the following trivector.

$$
\begin{aligned}
& X^T_{i_1 j_a} \wedge X^T_{i_1 j_b} \wedge X^T_{i_1 j_c} \\
&= \Big(\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \vee \langle\!\langle B^{j_a} \rangle\!\rangle \Big) \\
&\quad \wedge \Big(\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \vee \langle\!\langle B^{j_b} \rangle\!\rangle \Big) \wedge \Big(\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle \vee \langle\!\langle B^{j_c} \rangle\!\rangle \Big) \\
&= 0
\end{aligned}
\tag{14.59}
$$

The first step follows from equation (14.4). It is clear that this expression is zero because we take the outer product of the intersection points of line $\langle\!\langle A_a^{i_2} A_a^{i_3} \rangle\!\rangle$ with the planes $\langle\!\langle B^{j_1} \rangle\!\rangle$, $\langle\!\langle B^{j_2} \rangle\!\rangle$ and $\langle\!\langle B^{j_3} \rangle\!\rangle$. In other words, this equation says that the intersection points of a line with three planes all lie along a line (see figure 14.2).

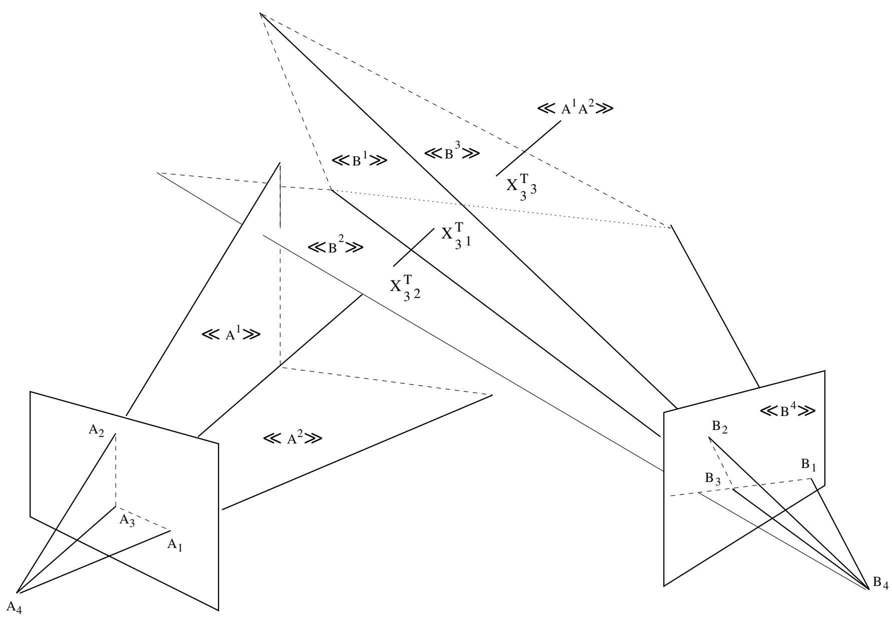

Fig. 14.2. This demonstrates the constraint from equation (14.59) for $i_2 = 1$, $i_3 = 2$ and $j_a = 1$, $j_b = 2$, $j_c = 3$. The figure also visualises the use of the inverse dual bracket to describe planes and lines

When projecting the three intersection points onto image plane C they still have to lie along a line. That is,

$$\begin{aligned}
&0 = (X^T_{i^{j_a}} \cdot C^{k_a})(X^T_{i^{j_b}} \cdot C^{k_b})(X^T_{i^{j_c}} \cdot C^{k_c}) C_{k_a} \wedge C_{k_b} \wedge C_{k_c} \\
\iff &0 = T_{i^{j_a k_a}} T_{i^{j_b k_b}} T_{i^{j_c k_c}} [\![C_{k_a} C_{k_b} C_{k_c} C_4]\!]_c \\
&\quad = \epsilon_{k_a k_b k_c} T_{i^{j_a k_a}} T_{i^{j_b k_b}} T_{i^{j_c k_c}} \\
&\quad = \det(T_{i^{jk}})_{jk}
\end{aligned} \tag{14.60}$$

14.4.7 Relation between $\boldsymbol{T}$ and $\boldsymbol{F}$

We mentioned before that the quadratic p-relations can be used to find constraints on T [120]. The equivalent expressions in GA are of the form

$$\langle\!\langle B^1 B^2 \rangle\!\rangle \wedge \langle\!\langle A^1 A^2 A^3 \rangle\!\rangle \wedge \langle\!\langle B^1 B^2 C^1 \rangle\!\rangle = 0 \tag{14.61}$$

This expression is zero because $\langle\!\langle B^1 B^2 \rangle\!\rangle \wedge \langle\!\langle B^1 B^2 C^1 \rangle\!\rangle = 0$. This becomes obvious immediately from a geometric point of view: the intersection point of line $\langle\!\langle B^1 B^2 \rangle\!\rangle$ with plane $\langle\!\langle C^1 \rangle\!\rangle$ clearly lies on line $\langle\!\langle B^1 B^2 \rangle\!\rangle$.

In the following we will write $T^{XYZ}_{i_1{}^{jk}}$ to denote the trifocal tensor

$$T^{XYZ}_{i_1{}^{jk}} = \langle\!\langle X^{i_2} X^{i_3} Y^j Z^k \rangle\!\rangle$$

We will similarly write $F^{XY}_{i_1 j_1}$ to denote the bifocal tensor

$$F^{XY}_{i_1 j_1} = \langle\!\langle X^{i_2} X^{i_3} Y^{j_2} Y^{j_3} \rangle\!\rangle$$

If no superscripts are given then $T_{i^{jk}}$ and F_{ij} take on the same meaning as before. That is,

$$T_{i^{jk}} \equiv T^{ABC}_{i^{jk}} \tag{14.62a}$$

$$F_{ij} \equiv F^{AB}_{ij} \tag{14.62b}$$

We can obtain a constraint on T by expanding equation (14.61).

$$\begin{aligned}
0 &= \langle\!\langle B^1 B^2 \rangle\!\rangle \wedge \langle\!\langle A^1 A^2 A^3 \rangle\!\rangle \wedge \langle\!\langle B^1 B^2 C^1 \rangle\!\rangle \\
&= \quad \langle\!\langle A^1 A^2 B^1 B^2 \rangle\!\rangle \langle\!\langle B^1 B^2 A^3 C^1 \rangle\!\rangle \\
&\quad + \langle\!\langle A^3 A^1 B^1 B^2 \rangle\!\rangle \langle\!\langle B^1 B^2 A^2 C^1 \rangle\!\rangle \\
&\quad + \langle\!\langle A^2 A^3 B^1 B^2 \rangle\!\rangle \langle\!\langle B^1 B^2 A^1 C^1 \rangle\!\rangle \\
&= F_{33} T^{BAC}_{3^{31}} + F_{23} T^{BAC}_{3^{21}} + F_{13} T^{BAC}_{3^{11}} \\
&= F_{i3} T^{BAC}_{3^{i1}}
\end{aligned} \tag{14.63}$$

Note that there is an implicit summation over i, because it is repeated as a (relative) superscript. Of course, we could have chosen different indices for the reciprocal B vectors and the reciprocal C vector. Therefore, we can obtain the following relation between the trifocal tensor and the bifocal tensor.

$$F_{ij}\, T^{BAC}_{\underset{j}{i}k} = 0 \tag{14.64}$$

Again there is an implicit summation over the i index but not over the j index. From this equation it follows that the three column vectors of the bifocal tensor give the three "left" null vectors of the three matrices $T_{\underset{i}{}\bullet\bullet}$, respectively. Equation (14.64) has two main uses: it can be used to find some epipoles of the trifocal tensor via equations (14.35) and (14.36), but it also serves to give more constraints on T since $\det F = 0$.

The columns of F may be found from equation (14.64) using, for example, an SVD. However, since the columns are found separately they will not in general be scaled consistently. Therefore, F found from equation (14.64) has only a limited use. Nonetheless, we can still find the correct left null vector of F, i.e. ε^{i}_{ab}, because each column is consistent in itself. Note also that, the determinant of F is still zero, since the rank F cannot be changed by scaling its columns separately. We cannot use this F, though, to find the right null vector, i.e. ε^{i}_{ba}, or to check whether image points on planes A and B are images of the same world point. Finding a consistent F is not necessary to find the right null vector of F, as will be shown later on. Therefore, unless we need to find a bifocal tensor from T which we can use to check image point pair matches, a fully consistent F is not necessary. A consistent F can, however, be found as shown in the following.

We can find the bifocal tensor row-wise in the following way.

$$\begin{aligned} 0 &= \langle\!\langle A^{i_2} A^{i_3} \rangle\!\rangle \wedge \langle\!\langle B^1 B^2 B^3 \rangle\!\rangle \wedge \langle\!\langle A^{i_2} A^{i_3} C^k \rangle\!\rangle \\ &= F_{i_1 j}\, T_{\underset{i_1}{}jk} \end{aligned} \tag{14.65}$$

Knowing F row-wise and column-wise we can find a consistently scaled bifocal tensor. What remains is to find T^{BAC} from T. To do so we define the following intersection points in terms of the lines $T^{i_a j_a} \equiv \langle\!\langle B^{i_a} C^{j_a} \rangle\!\rangle$ (see equation (14.42a)).

$$\begin{aligned} p(i_a j_a, i_b j_b) &\equiv (A_4 \wedge T^{i_a j_a}) \vee T^{i_b j_b} \\ &= \left\langle\!\!\left\langle \left[\!\!\left[A_4 \langle\!\langle B^{i_a} C^{j_a} \rangle\!\rangle \right]\!\!\right] \left[\!\!\left[\langle\!\langle B^{i_b} C^{j_b} \rangle\!\rangle \right]\!\!\right] \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \Big(A_4 \cdot \left[\!\!\left[\langle\!\langle B^{i_a} C^{j_a} \rangle\!\rangle \right]\!\!\right] \Big) B^{i_b} C^{j_b} \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \Big(A_4 \cdot (B^{i_a} \wedge C^{j_a}) \Big) B^{i_b} C^{j_b} \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \begin{aligned} &(A_4 \cdot B^{i_a}) C^{j_a} B^{i_b} C^{j_b} \\ &- (A_4 \cdot C^{j_a}) B^{i_a} B^{i_b} C^{j_b} \end{aligned} \right\rangle\!\!\right\rangle \\ &= \varepsilon^{i_a}_{ba} \langle\!\langle C^{j_a} B^{i_b} C^{j_b} \rangle\!\rangle + \varepsilon^{j_a}_{ca} \langle\!\langle B^{i_a} C^{j_b} B^{i_b} \rangle\!\rangle \end{aligned} \tag{14.66}$$

Two useful special cases are

$$p(i_1 j, i_2 j) = \varepsilon^j_{ca} \langle\!\langle B^{i_1} C^j B^{i_2} \rangle\!\rangle \tag{14.67a}$$

$$p(ij_1, ij_2) = \varepsilon^i_{ba} \langle\!\langle C^{j_1} B^i C^{j_2} \rangle\!\rangle \tag{14.67b}$$

The projection of $p(i_1 j, i_2 j)$ onto image plane A, denoted by $p_a(i_1 j, i_2 j)$ gives

$$\begin{aligned} p_a(i_2 k, i_3 k) &= \varepsilon^k_{ca} \Big(A^j \cdot \langle\!\langle B^{i_2} C^k B^{i_3} \rangle\!\rangle \Big) A_j \\ &= \varepsilon^k_{ca} \langle\!\langle A^j B^{i_2} C^k B^{i_3} \rangle\!\rangle A_j \\ &= -\varepsilon^k_{ca} \langle\!\langle B^{i_2} B^{i_3} A^j C^k \rangle\!\rangle A_j \\ &= -\varepsilon^k_{ca} T^{BAC}_{\substack{\\ i_1}jk} A_j \end{aligned} \tag{14.68}$$

We can also calculate $p_a(j_a k_a, j_b k_b)$ by immediately using the projections of the T^{jk} onto image plane A (see equation (14.42b)). That is,

$$\begin{aligned} p_a(j_a k_a, j_b k_b) &= (A_4 \wedge T^{i_a j_a}_a) \vee T^{i_b j_b}_a \\ &= T_{i_a}{}^{j_a k_a} T_{i_b}{}^{j_b k_b} (A_4 \wedge L^{i_a}_a) \vee L^{i_b}_a \\ &= T_{i_a}{}^{j_a k_a} T_{i_b}{}^{j_b k_b} (A_4 \wedge \langle\!\langle A^{i_a}_a A^4_a \rangle\!\rangle) \vee \langle\!\langle A^{i_b}_a A^4_a \rangle\!\rangle \\ &= T_{i_a}{}^{j_a k_a} T_{i_b}{}^{j_b k_b} \left\langle\!\!\left\langle \Big(A_4 \cdot (A^{i_a}_a \wedge A^4_a) \Big) A^{i_b}_a A^4_a \right\rangle\!\!\right\rangle \\ &\simeq T_{i_a}{}^{j_a k_a} T_{i_b}{}^{j_b k_b} \langle\!\langle A^{i_a}_a A^{i_b}_a A^4_a \rangle\!\rangle \end{aligned} \tag{14.69}$$

From the definition of the inverse dual bracket we have

$$A_{i_3} = \langle\!\langle A^{i_1}_a A^{i_2}_a A^4_a \rangle\!\rangle_a$$

Therefore, from equation (14.69) we find

$$p_a(j_1 k, j_2 k) \simeq (T_{i_1}{}^{j_1 k} T_{i_2}{}^{j_2 k} - T_{i_2}{}^{j_1 k} T_{i_1}{}^{j_2 k}) A_{i_3} \tag{14.70}$$

Equating this with equation (14.68) gives

$$T^{BAC}_{j_3}{}^{i_3 k} \simeq (\varepsilon^k_{ca})^{-1} (T_{i_1}{}^{j_1 k} T_{i_2}{}^{j_2 k} - T_{i_2}{}^{j_1 k} T_{i_1}{}^{j_2 k}) \tag{14.71}$$

Since ε^k_{ca} can be found from T (as will be shown later) we can find T^{BAC} from T up to an overall scale. Equation (14.71) may also be written in terms of the standard cross product.

$$T^{BAC}_{j_3}{}^{\bullet k} \simeq (\varepsilon^k_{ca})^{-1} (T_{\bullet}{}^{j_1 k} \times T_{\bullet}{}^{j_2 k}) \tag{14.72}$$

Had we used equation (14.67b) instead of equation (14.67a) in the previous calculation, we would have obtained the following relation.

$$T^{CBA}_{k_3{}^{ji_3}} \simeq (\varepsilon^{j}_{ba})^{-1}(T_{i_1}{}^{jk_1}T_{i_2}{}^{jk_2} - T_{i_2}{}^{jk_1}T_{i_1}{}^{jk_2}) \tag{14.73}$$

Or, in terms of the standard cross product

$$T^{CBA}_{k_3{}^{j\bullet}} \simeq (\varepsilon^{j}_{ba})^{-1}(T_{\bullet}{}^{jk_1} \times T_{\bullet}{}^{jk_2}) \tag{14.74}$$

Hence, we can also obtain T^{CBA} from T up to an overall scale. Note that since

$$\begin{aligned} T^{ABC}_{i_1{}^{jk}} &= \langle\!\langle A^{i_2}A^{i_3}B^{j}C^{k}\rangle\!\rangle \\ &= -\langle\!\langle A^{i_2}A^{i_3}C^{k}B^{j}\rangle\!\rangle \\ &= -T^{ACB}_{i_1{}^{kj}} \end{aligned} \tag{14.75}$$

we have found *all* possible trifocal tensors for a particular camera setup from T.

Equations (14.72) and (14.74) simply express that the projections of the intersection points between some lines onto image plane A are the same as the intersection points between the projections of the same lines onto image plane A. This implies that independent of the intersection points, i.e. the components of $T_{i}{}^{jk}$, equations (14.72) and (14.74) will always give a self-consistent tensor, albeit not necessarily one that expresses the correct camera geometry.

14.4.8 Second Order Constraints

There are more constraints on T which we will call “second order” because they are products of determinants of components of T. Their derivation is more involved and can be found in [189] and [190]. Here we will only state the results. These constraints may be used to check the self-consistency of T when it is calculated via a non-linear method.

$$\begin{aligned} 0 = \ & |T^{j_a k_a}_{a} T^{j_b k_a}_{a} T^{j_a k_b}_{a}|\ |T^{j_b k_b}_{a} T^{j_a k_c}_{a} T^{j_b k_c}_{a}| \\ - & |T^{j_a k_a}_{a} T^{j_b k_a}_{a} T^{j_b k_b}_{a}|\ |T^{j_a k_b}_{a} T^{j_a k_c}_{a} T^{j_b k_c}_{a}| \end{aligned} \tag{14.76}$$

$$\begin{aligned} 0 = \ & |T^{j_a k_a}_{a} T^{j_a k_b}_{a} T^{j_b k_a}_{a}|\ |T^{j_b k_b}_{a} T^{j_c k_a}_{a} T^{j_c k_b}_{a}| \\ - & |T^{j_a k_a}_{a} T^{j_a k_b}_{a} T^{j_b k_b}_{a}|\ |T^{j_b k_a}_{a} T^{j_c k_a}_{a} T^{j_c k_b}_{a}| \end{aligned} \tag{14.77}$$

$$\begin{aligned} 0 = \ & |T^{i_a j_a}_{a} T^{i_b j_a}_{a} T^{i_a j_b}_{a}|\ |T^{i_b j_b}_{a} T^{i_a j_b}_{a} T^{i_b j_c}_{a}| \\ - & |T^{i_a j_a}_{a} T^{i_b j_a}_{a} T^{i_b j_b}_{a}|\ |T^{i_a j_b}_{a} T^{i_a j_c}_{a} T^{i_b j_b}_{a}| \end{aligned} \tag{14.78}$$

Where the determinants are to be be interpreted as

$$|T^{j_a k_a}_{a} T^{j_b k_a}_{a} T^{j_a k_b}_{a}| = \det(T_{i_a}{}^{j_a k_a}, T_{i_b}{}^{j_b k_a}, T_{i_c}{}^{j_a k_b})_{i_a i_b i_c}$$

14.4.9 Epipoles

The epipoles of T can be found indirectly via the relation of bifocal tensors to T (e.g. equation (14.64)). Also recall that the right null vector of some F^{XY}_{ij} is ε^{j}_{yx}, whereas the left null vector is ε^{i}_{xy} (equations (14.35) and (14.36)). From equation (14.65) we know that

$$F_{ij}T_{i}{}^{jk} = 0$$

When calculating F from this equation, we cannot guarantee that the rows are scaled consistently. Nevertheless, this does not affect the right null space of F. Hence, we can find ε^{j}_{ba} from this F. In the following we will list the necessary relations to find all epipoles of T.

$$\begin{aligned} 0 &= \langle\!\langle A^{i_2}A^{i_3}\rangle\!\rangle \wedge \langle\!\langle B^1B^2B^3\rangle\!\rangle \wedge \langle\!\langle A^{i_2}A^{i_3}C^k\rangle\!\rangle \\ &= F_{i_1 j}T_{i_1}{}^{jk} && \to \varepsilon^{j}_{ba} \end{aligned} \tag{14.79a}$$

$$\begin{aligned} 0 &= \langle\!\langle A^{i_2}A^{i_3}\rangle\!\rangle \wedge \langle\!\langle C^1C^2C^3\rangle\!\rangle \wedge \langle\!\langle A^{i_2}A^{i_3}B^j\rangle\!\rangle \\ &= F^{AC}_{i_1 k}T_{i_1}{}^{jk} && \to \varepsilon^{k}_{ca} \end{aligned} \tag{14.79b}$$

$$\begin{aligned} 0 &= \langle\!\langle B^{i_2}B^{i_3}\rangle\!\rangle \wedge \langle\!\langle A^1A^2A^3\rangle\!\rangle \wedge \langle\!\langle B^{i_2}B^{i_3}C^k\rangle\!\rangle \\ &= F^{BA}_{i_1 j}T^{BAC}_{i_1}{}^{jk} && \to \varepsilon^{j}_{ab} \end{aligned} \tag{14.80a}$$

$$\begin{aligned} 0 &= \langle\!\langle B^{i_2}B^{i_3}\rangle\!\rangle \wedge \langle\!\langle C^1C^2C^3\rangle\!\rangle \wedge \langle\!\langle B^{i_2}B^{i_3}A^j\rangle\!\rangle \\ &= F^{BC}_{i_1 k}T^{BAC}_{i_1}{}^{jk} && \to \varepsilon^{k}_{cb} \end{aligned} \tag{14.80b}$$

$$\begin{aligned} 0 &= \langle\!\langle C^{i_2}C^{i_3}\rangle\!\rangle \wedge \langle\!\langle A^1A^2A^3\rangle\!\rangle \wedge \langle\!\langle C^{i_2}C^{i_3}B^j\rangle\!\rangle \\ &= F^{CA}_{i_1 k}T^{CBA}_{i_1}{}^{jk} && \to \varepsilon^{k}_{ac} \end{aligned} \tag{14.81a}$$

$$\begin{aligned} 0 &= \langle\!\langle C^{i_2}C^{i_3}\rangle\!\rangle \wedge \langle\!\langle B^1B^2B^3\rangle\!\rangle \wedge \langle\!\langle C^{i_2}C^{i_3}A^k\rangle\!\rangle \\ &= F^{CB}_{i_1 j}T^{CBA}_{i_1}{}^{jk} && \to \varepsilon^{j}_{bc} \end{aligned} \tag{14.81b}$$

By $\to \varepsilon^{j}_{xy}$ we denote the epipole that can be found from the respective relation[5] . Note that since

$$\begin{aligned} F^{XY}_{i_1 j_1} &= \langle\!\langle X^{i_2}X^{i_3}Y^{j_2}Y^{j_3}\rangle\!\rangle \\ &= \langle\!\langle Y^{j_2}Y^{j_3}X^{i_2}X^{i_3}\rangle\!\rangle \\ &= F^{YX}_{j_1 i_1} \end{aligned} \tag{14.82}$$

we have also found all fundamental matrices.

[5] Initial computations evaluating the quality of the epipoles found via this method indicate that this may not be the best way to calculate the epipoles. It seems that better results can be obtained when the epipoles are found directly from T^{ABC}.

14.5 The Quadfocal Tensor

14.5.1 Derivation

Let L be a line in $\mathbb{P}^3$ and let $\{A_\mu\}$, $\{B_\mu\}$, $\{C_\mu\}$ and $\{D_\mu\}$ define four cameras A, B, C and D, respectively. The projection of L onto the image planes of these four cameras is

$$L \xrightarrow{A} L_A = L \cdot L_i^a L_a^i = \lambda_i^a L_a^i \tag{14.83a}$$

$$L \xrightarrow{B} L_B = L \cdot L_i^b L_b^i = \lambda_i^b L_b^i \tag{14.83b}$$

$$L \xrightarrow{C} L_C = L \cdot L_i^c L_c^i = \lambda_i^c L_c^i \tag{14.83c}$$

$$L \xrightarrow{D} L_D = L \cdot L_i^d L_d^i = \lambda_i^d L_d^i \tag{14.83d}$$

The intial line L can be recovered from these projections by intersecting any two of the planes $(L_A \wedge A_4)$, $(L_B \wedge B_4)$, $(L_C \wedge C_4)$ and $(L_D \wedge D_4)$. For example,

$$L \simeq (L_A \wedge A_4) \vee (L_B \wedge B_4) \simeq (L_C \wedge C_4) \vee (L_D \wedge D_4) \tag{14.84}$$

Therefore,

$$\begin{aligned}
0 &= \Big[\!\!\Big[\Big((L_A \wedge A_4) \vee (L_B \wedge B_4) \Big) \wedge \Big((L_C \wedge C_4) \vee (L_D \wedge D_4) \Big) \Big]\!\!\Big] \\
&= \lambda_i^a \lambda_j^b \lambda_k^c \lambda_l^d \Big[\!\!\Big[\Big((L_a^i \wedge A_4) \vee (L_b^j \wedge B_4) \Big) \\
&\qquad \Big((L_c^k \wedge C_4) \vee (L_d^l \wedge D_4) \Big) \Big]\!\!\Big] \\
&= \lambda_i^a \lambda_j^b \lambda_k^c \lambda_l^d \Big[\!\!\Big[\Big(\langle\!\langle A^i \rangle\!\rangle \vee \langle\!\langle B^j \rangle\!\rangle \Big) \Big(\langle\!\langle C^k \rangle\!\rangle \vee \langle\!\langle D^l \rangle\!\rangle \Big) \Big]\!\!\Big] \\
&= \lambda_i^a \lambda_j^b \lambda_k^c \lambda_l^d \langle\!\langle A^i B^j C^k D^l \rangle\!\rangle
\end{aligned} \tag{14.85}$$

Therefore, a *quadfocal tensor* may be defined as

$$Q^{ijkl} = \langle\!\langle A^i B^j C^k D^l \rangle\!\rangle \tag{14.86}$$

If the quadfocal tensor is contracted with the homogeneous line coordinates of the projections of one line onto the four camera image planes, the result is zero. In this way the quadfocal tensor encodes the relative orientation of the four camera image planes. However, note that contracting the quadfocal tensor with the line coordinates of the projection of one line onto only three image planes gives a zero vector. This follows directly from geometric considerations. For example,

$$\begin{aligned}
\lambda_i^a \lambda_j^b \lambda_k^c Q^{ijkl} &= \lambda_i^a \lambda_j^b \lambda_k^c \langle\!\langle A^i B^j C^k \rangle\!\rangle \cdot D^l \\
&\simeq \Big(L \vee (\lambda_k^c C^k) \Big) \cdot D^l
\end{aligned} \tag{14.87}$$

where L is the line whose images on image planes A, B and C have coordinates $\{\lambda_i^a\}$, $\{\lambda_j^b\}$ and $\{\lambda_k^c\}$, respectively. Hence, L lies on plane $\lambda_k^c C^k$, and thus their meet is zero. This also shows that the quadfocal tensor does not add any new information to what can be known from the trifocal tensor, since the quadfocal tensor simply relates any three image planes out of a group of four.

The form for Q given in equation (14.86) can be shown to be equivalent to the form given by Heyden in [120]. In this form it is also immediately clear that changing the order of the reciprocal vectors in equation (14.86) at most changes the overall sign of Q.

14.5.2 Transferring Lines

If the image of a line is known on two image planes, then the quadfocal tensor can be used to find its image on the other two image planes. This can be achieved through a somewhat indirect route. Let L be a line projected onto image planes A and B with coordinates $\{\lambda_i^a\}$ and $\{\lambda_j^b\}$, respectively. Then we know that

$$L \simeq \lambda_i^a \lambda_j^b \langle\!\langle A^i B^j \rangle\!\rangle \tag{14.88}$$

Therefore, we can define three points $\{X_L^k\}$ that lie on L as

$$\begin{aligned} X_L^k &\equiv \lambda_i^a \lambda_j^b (\langle\!\langle A^i B^j \rangle\!\rangle \vee \langle\!\langle C^k \rangle\!\rangle) \\ &= \lambda_i^a \lambda_j^b \langle\!\langle A^i B^j C^k \rangle\!\rangle \end{aligned} \tag{14.89}$$

The projections of the $\{X_L^k\}$ onto image plane D, denoted by $\{X_{L_d}^k\}$ are given by

$$\begin{aligned} X_{L_d}^k &\equiv X_L^k \cdot D^l \\ &= \lambda_i^a \lambda_j^b \langle\!\langle A^i B^j C^k \rangle\!\rangle \cdot D^l \\ &= \lambda_i^a \lambda_j^b \langle\!\langle A^i B^j C^k D^l \rangle\!\rangle \\ &= \lambda_i^a \lambda_j^b Q^{ijkl} \end{aligned} \tag{14.90}$$

From the points $\{X_{L_d}^k\}$ the projection of line L onto image plane D can be recovered.

14.5.3 Rank of Q

The form for the quadfocal tensor as given in equation (14.86) may be expanded in a number of ways. For example,

$$\begin{aligned} Q^{i_1 jkl} &= (A_{i_2} \wedge A_{i_3} \wedge A_4) \cdot (B^j \wedge C^k \wedge D^l) \\ &= \quad U_b^j \Big[{K_k^c}_{i_3} {K_l^d}_{i_2} - {K_l^d}_{i_3} {K_k^c}_{i_2} \Big] \\ &\quad - U_c^k \Big[{K_j^b}_{i_3} {K_l^d}_{i_2} - {K_l^d}_{i_3} {K_j^b}_{i_2} \Big] \\ &\quad + U_d^l \Big[{K_j^b}_{i_3} {K_k^c}_{i_2} - {K_k^c}_{i_3} {K_j^b}_{i_2} \Big] \end{aligned} \tag{14.91}$$

In terms of the standard cross product this may be written as

$$Q^{\bullet jkl} = U^j_b (K^c_{k_\bullet} \times K^d_{l_\bullet}) - U^k_c (K^b_{j_\bullet} \times K^d_{l_\bullet}) + U^l_d (K^b_{j_\bullet} \times K^c_{k_\bullet}) \tag{14.92}$$

This decomposition of Q shows that the quadfocal tensor can be at most of rank 9. From equation (14.91) it becomes clear that, as for the trifocal tensor, the transformation $A_i \mapsto s(A_i + t_i A_4)$ leaves Q unchanged up to an overall scale.

Let $P = B_4 \wedge C_4 \wedge D_4$. As for the trifocal tensor case, define a basis $\{A'_i\}$ for image plane A by

$$A'_i = (A_i \wedge A_4) \vee P \tag{14.93}$$

All the $\{A'_i\}$ lie on plane P, that is they lie on the plane formed by B_4, C_4 and D_4. Therefore, $K^{b\prime}_{j_i} = A'_i \cdot B^j$, $K^{c\prime} = A'_i \cdot C^k$ and $K^{d\prime} = A'_i \cdot D^l$ are of rank 2. As was shown previously, this is the minimum rank the camera matrices can have. Hence, forming Q with the $\{A'_i\}$ should yield its rank. However, it is not immediately obvious from equation (14.91) what the rank of Q is when substituting the $\{A'_i\}$ for the $\{A_i\}$. A more yielding decomposition of Q is achieved by expanding equation (14.93).

$$\begin{aligned} A'_i &= (A_i \wedge A_4) \vee P \\ &\simeq [\![A_i A_4]\!] \cdot (B_4 \wedge C_4 \wedge D_4) \\ &= [\![A_i A_4 B_4 C_4]\!] D_4 - [\![A_i A_4 B_4 D_4]\!] C_4 + [\![A_i A_4 C_4 D_4]\!] B_4 \\ &= \alpha^1_i B_4 + \alpha^2_i C_4 + \alpha^3_i D_4 \end{aligned} \tag{14.94}$$

where the $\{\alpha^j_i\}$ are defined accordingly. Furthermore,

$$A'_{i_1} \wedge A'_{i_2} = \lambda^1_{i_3}\, C_4 \wedge D_4 + \lambda^2_{i_3}\, D_4 \wedge B_4 + \lambda^3_{i_3}\, B_4 \wedge C_4 \tag{14.95}$$

with $\lambda^{j_3}_{i_3} = \alpha^{j_1}_{i_1} \alpha^{j_2}_{i_2} - \alpha^{j_1}_{i_2} \alpha^{j_2}_{i_1}$. Equation (14.91) may also be written as

$$\begin{aligned} Q^{i_1 jkl} &= (A_{i_2} \wedge A_{i_3} \wedge A_4) \cdot (B^j \wedge C^k \wedge D^l) \\ &= \quad U^j_b \Big[(A'_{i_2} \wedge A'_{i_3}) \cdot (C^k \wedge D^l) \Big] \\ &\quad - U^k_c \Big[(A'_{i_2} \wedge A'_{i_3}) \cdot (B^j \wedge D^l) \Big] \\ &\quad + U^l_d \Big[(A'_{i_2} \wedge A'_{i_3}) \cdot (B^j \wedge C^k) \Big] \end{aligned} \tag{14.96}$$

From equation (14.95) it then follows

$$
\begin{aligned}
(A'_{i_2} \wedge A'_{i_3}) \cdot (C^k \wedge D^l) = {} & \lambda^1_{i_1}\, D_4 \cdot C^k \; C_4 \cdot D^l \\
& - \lambda^2_{i_1}\, D_4 \cdot C^k \; B_4 \cdot D^l \\
& - \lambda^3_{i_1}\, B_4 \cdot C^k \; C_4 \cdot D^l
\end{aligned} \tag{14.97a}
$$

$$
\begin{aligned}
(A'_{i_2} \wedge A'_{i_3}) \cdot (B^j \wedge D^l) = {} & \lambda^1_{i_1}\, D_4 \cdot B^j \; C_4 \cdot D^l \\
& - \lambda^2_{i_1}\, D_4 \cdot B^j \; B_4 \cdot D^l \\
& + \lambda^3_{i_1}\, C_4 \cdot B^j \; B_4 \cdot D^l
\end{aligned} \tag{14.97b}
$$

$$
\begin{aligned}
(A'_{i_2} \wedge A'_{i_3}) \cdot (B^j \wedge C^k) = {} & - \lambda^1_{i_1}\, C_4 \cdot B^j \; D_4 \cdot C^k \\
& - \lambda^2_{i_1}\, D_4 \cdot B^j \; B_4 \cdot C^k \\
& + \lambda^3_{i_1}\, C_4 \cdot B^j \; B_4 \cdot C^k
\end{aligned} \tag{14.97c}
$$

Each of these three equations has a linear combination of three rank 1, 3-valence tensors on its right hand side. Furthermore, none of the rank 1, 3-valence tensors from one equation is repeated in any of the others. Therefore, substituting equations (14.97) into equation (14.96) gives a decomposition of Q in terms of 9 rank 1 tensors. Since this is a minimal decomposition, Q is of rank 9.

14.5.4 Degrees of Freedom of Q

Substituting equations (14.97) back into equation (14.96) gives

$$
\begin{aligned}
Q^{ijkl} = {} & \varepsilon^{j}_{ba}\Big[\lambda^1_i \varepsilon^{k}_{cd}\varepsilon^{l}_{dc} - \lambda^2_i \varepsilon^{k}_{cd}\varepsilon^{l}_{db} + \lambda^3_i \varepsilon^{k}_{cb}\varepsilon^{l}_{dc}\Big] \\
& - \varepsilon^{k}_{ca}\Big[\lambda^1_i \varepsilon^{j}_{bd}\varepsilon^{l}_{dc} - \lambda^2_i \varepsilon^{j}_{bd}\varepsilon^{l}_{db} + \lambda^3_i \varepsilon^{j}_{bc}\varepsilon^{l}_{db}\Big] \\
& + \varepsilon^{l}_{da}\Big[\lambda^1_i \varepsilon^{j}_{bc}\varepsilon^{k}_{cd} - \lambda^2_i \varepsilon^{j}_{bd}\varepsilon^{k}_{cb} + \lambda^3_i \varepsilon^{j}_{bc}\varepsilon^{k}_{cb}\Big]
\end{aligned} \tag{14.98}
$$

This decomposition of Q has $9 \times 3 + 3 \times 3 - 1 = 35$ DOF. The general formula for the DOF of Q gives $4 \times 11 - 15 = 29$ DOF. Therefore the parameterisation of Q in equation (14.98) is overdetermined. However, it will still give a self-consistent Q.

14.5.5 Constraints on Q

The constraints on Q can again be found very easily through geometric considerations. Let the points $\{X^{ijk}_Q\}$ be defined as

$$
X^{ijk}_Q \equiv \langle\!\langle A^i B^j C^k \rangle\!\rangle \tag{14.99}
$$

A point X^{ijk}_Q can be interpreted as the intersection of line $\langle\!\langle A^i B^j \rangle\!\rangle$ with plane $\langle\!\langle C^k \rangle\!\rangle$. Therefore,

$$
X^{ijk_a}_Q \wedge X^{ijk_b}_Q \wedge X^{ijk_c}_Q = 0 \tag{14.100}
$$

because the three intersection points $X_Q^{ijk_a}$, $X_Q^{ijk_b}$ and $X_Q^{ijk_c}$ lie along line $\langle\!\langle A^i B^j \rangle\!\rangle$. Hence, also their projections onto an image plane have to lie along a line. Thus, projecting the intersection points onto an image plane D we have

$$\begin{aligned} & 0 = (X_Q^{ijk_a} \cdot D^{l_a})\,(X_Q^{ijk_b} \cdot D^{l_b})\,(X_Q^{ijk_c} \cdot D^{l_c}) \\ & \quad\;\; (D_{l_a} \wedge D_{l_b} \wedge D_{l_c}) \\ \Longleftrightarrow\; & 0 = Q^{ijk_a l_a}\, Q^{ijk_b l_b}\, Q^{ijk_c l_c} [\![D_{l_a} D_{l_b} D_{l_c} D_4]\!]_d \\ & \;\; = \epsilon_{l_a l_b l_c}\, Q^{ijk_a l_a}\, Q^{ijk_b l_b}\, Q^{ijk_c l_c} \\ & \;\; = \det(Q^{ijkl})_{kl} \end{aligned} \tag{14.101}$$

Similarly, this type of constraint may be shown for every pair of indices. We therefore get the following constraints on Q.

$$\begin{aligned} & \det(Q^{ijkl})_{ij} = 0;\; \det(Q^{ijkl})_{ik} = 0;\; \det(Q^{ijkl})_{il} = 0 \\ & \det(Q^{ijkl})_{jk} = 0;\; \det(Q^{ijkl})_{jl} = 0;\; \det(Q^{ijkl})_{kl} = 0 \end{aligned} \tag{14.102}$$

14.5.6 Relation between Q and T

We can find the relation between Q and T via the method employed to find the relation between T and F. For example,

$$\begin{aligned} 0 &= \langle\!\langle A^1 A^2 A^3 \rangle\!\rangle \wedge \langle\!\langle B^j C^k D^l \rangle\!\rangle \wedge \langle\!\langle B^j C^k \rangle\!\rangle \\ &= \textstyle\sum_{i_1} \Big(\langle\!\langle A^{i_1} B^j C^k D^l \rangle\!\rangle \langle\!\langle A^{i_2} A^{i_3} B^j C^k \rangle\!\rangle \Big) \\ &= Q^{ijkl} T_i^{\;jk} \end{aligned} \tag{14.103}$$

Similarly, equations for the other possible trifocal tensors can be found. Because of the trifocal tensor symmetry detailed in equation (14.75) all trifocal tensors may be evaluated from the following set of equations.

$$\begin{aligned} & Q^{ijkl}\, T_{i}^{ABC\,jk} = 0;\; Q^{ijkl}\, T_{i}^{ABD\,jl} = 0;\; Q^{ijkl}\, T_{i}^{ACD\,kl} = 0 \\ & Q^{ijkl}\, T_{j}^{BAC\,ik} = 0;\; Q^{ijkl}\, T_{j}^{BAD\,il} = 0;\; Q^{ijkl}\, T_{j}^{BCD\,kl} = 0 \\ & Q^{ijkl}\, T_{k}^{CAB\,ij} = 0;\; Q^{ijkl}\, T_{k}^{CAD\,il} = 0;\; Q^{ijkl}\, T_{k}^{CBD\,jl} = 0 \\ & Q^{ijkl}\, T_{l}^{DAB\,ij} = 0;\; Q^{ijkl}\, T_{l}^{DAC\,ik} = 0;\; Q^{ijkl}\, T_{l}^{DBC\,jk} = 0 \end{aligned} \tag{14.104}$$

Note that the trifocal tensors found in this way will not be of consistent scale. To fix the scale we start by defining intersection points

$$\begin{aligned} X_{BCD}^{jkl} &\equiv \Big[A_4 \wedge \langle\!\langle B^j C^k \rangle\!\rangle \Big] \vee \langle\!\langle C^k D^l \rangle\!\rangle \\ &\simeq \varepsilon_{ca}^{k} \langle\!\langle B^j C^k D^l \rangle\!\rangle \end{aligned} \tag{14.105}$$

Projecting these points onto image plane A gives

$$\begin{aligned} X^{jkl}_{BCD_a} &\equiv X^{jkl}_{BCD} \cdot A^i A_i \\ &\simeq \varepsilon^k_{ca} \langle\!\langle B^j C^k D^l \rangle\!\rangle \cdot A^i A_i \\ &\simeq \varepsilon^k_{ca} \langle\!\langle A^i B^j C^k D^l \rangle\!\rangle A_i \\ &= \varepsilon^k_{ca} Q^{ijkl} A_i \end{aligned} \tag{14.106}$$

But we could have also arrived at an expression for $X^{jkl}_{BCD_a}$ via

$$\begin{aligned} X^{jkl}_{BCD_a} &\simeq \Big(\langle\!\langle B^j C^k \rangle\!\rangle \cdot L^a_{i_a} \Big) \Big(\langle\!\langle C^k D^l \rangle\!\rangle \cdot L^a_{i_b} \Big) \Big[A_4 \wedge L^{i_a}_a \Big] \vee L^{i_b}_a \\ &\simeq \Big(T^{ABC}_{i_1 jk} T^{ACD}_{i_2 kl} - T^{ABC}_{i_2 jk} T^{ACD}_{i_1 kl} \Big) A_{i_3} \end{aligned} \tag{14.107}$$

Equating this with equation (14.106) gives

$$T^{ABC}_{i_1 jk} T^{ACD}_{i_2 kl} - T^{ABC}_{i_2 jk} T^{ACD}_{i_1 kl} \simeq \varepsilon^k_{ca} Q^{i_3 jkl} \tag{14.108}$$

This equation may be expressed more concisely in terms of the standard cross product.

$$T^{ABC}_{\bullet jk} \times T^{ACD}_{\bullet kl} \simeq \varepsilon^k_{ca} Q^{\bullet jkl} \tag{14.109}$$

Furthermore, from the intersection points

$$X^{kjl}_{CBD} \equiv \Big[A_4 \wedge \langle\!\langle C^k B^j \rangle\!\rangle \Big] \vee \langle\!\langle B^j D^l \rangle\!\rangle$$

and their projections onto image plane A we get

$$T^{ABC}_{\bullet jk} \times T^{ABD}_{\bullet jl} \simeq \varepsilon^j_{ba} Q^{\bullet jkl} \tag{14.110}$$

We can now find the correct scales for T^{ABC} by demanding that

$$\frac{T^{ABC}_{i_1 jk} T^{ACD}_{i_2 kl} - T^{ABC}_{i_2 jk} T^{ACD}_{i_1 kl}}{Q^{i_3 jkl}} = \phi \tag{14.111}$$

for all j while keeping i_1, k and l constant, where ϕ is some scalar. Furthermore, we know that

$$\frac{T^{ABC}_{i_1 jk} T^{ABD}_{i_2 jl} - T^{ABC}_{i_2 jk} T^{ABD}_{i_1 jl}}{Q^{i_3 jkl}} = \phi \tag{14.112}$$

for all k while keeping i_1, k and l constant, where ϕ is some different scalar. Equations (14.111) and (14.112) together fix the scales of T^{ABC} completely. Note that we do not have to know the epipoles ε^k_{ca} and ε^j_{ba}.

Similarly, all the other trifocal tensors can be found. These in turn can be used to find the fundamental matrices and the epipoles.

14.6 Reconstruction and the Trifocal Tensor

In the following we will investigate a computational aspect of the trifocal tensor. In particular we are interested in the effect the determinant constraints have on the "quality" of a trifocal tensor. That is, a trifocal tensor calculated only from point matches has to be compared with a trifocal tensor calculated form point matches while enforcing the determinant constraints.

For the calculation of the former a simple linear algorithm is used that employs the trilinearity relationships, as, for example, given by Hartley in [106]. In the following this algorithm will be called the "7pt algorithm".

To enforce all the determinant constraints, an estimate of the trifocal tensor is first found using the 7pt algorithm. From this tensor the epipoles are estimated. Using these epipoles the image points are transformed into the epipolar frame. With these transformed point matches the trifocal tensor can then be found in the epipolar basis.

It can be shown [147] that the trifocal tensor in the epipolar basis has only 7 non-zero components[6]. Using the image point matches in the epipolar frame these 7 components can be found linearly. The trifocal tensor in the "normal" basis is then recovered by tranforming the trifocal tensor in the epipolar basis back with the initial estimates of the epipoles. The trifocal tensor found in this way has to be fully self-consistent since it was calculated from the minimal number of parameters. That also means that the determinant constraints have to be fully satisfied. This algorithm will be called the "MinFact" algorithm.

The main problem with the MinFact algorithm is that it depends crucially on the quality of the initial epipole estimates. If these are bad, the trifocal tensor will still be perfectly self-consistent but will not represent the true camera structure particularly well. This is reflected in the fact that typically a trifocal tensor calculated with the MinFact algorithm does not satisfy the trilinearity relationships as well as a trifocal tensor calculated with the 7pt algorithm, which is of course calculated to satisfy these relationships as well as possible.

Unfortunately, there does not seem to be a way to find the epipoles and the trifocal tensor in the epipolar basis simultaneously with a linear method. In fact, the trifocal tensor in a "normal" basis is a *non-linear* combination of the epipoles and the 7 non-zero components of the trifocal tensor in the epipolar basis.

Nevertheless, since the MinFact algorithm produces a fully self-consistent tensor, the camera matrices extracted from it also have to form a self-consistent set. Reconstruction using such a set of camera matrices may be expected to be better than reconstruction using an inconsistent set of camera

[6] From this it follows directly that the trifocal tensor has 18 DOF: 12 epipolar components plus 7 non-zero components of the trifocal tensor in the epipolar basis minus 1 for an overall scale.

matrices, as typically found from an inconsistent trifocal tensor. The fact that the trifocal tensor found with the MinFact algorithm may not resemble the true camera structure very closely, might not matter too much, since reconstruction is only exact up to a projective transformation. The question is,

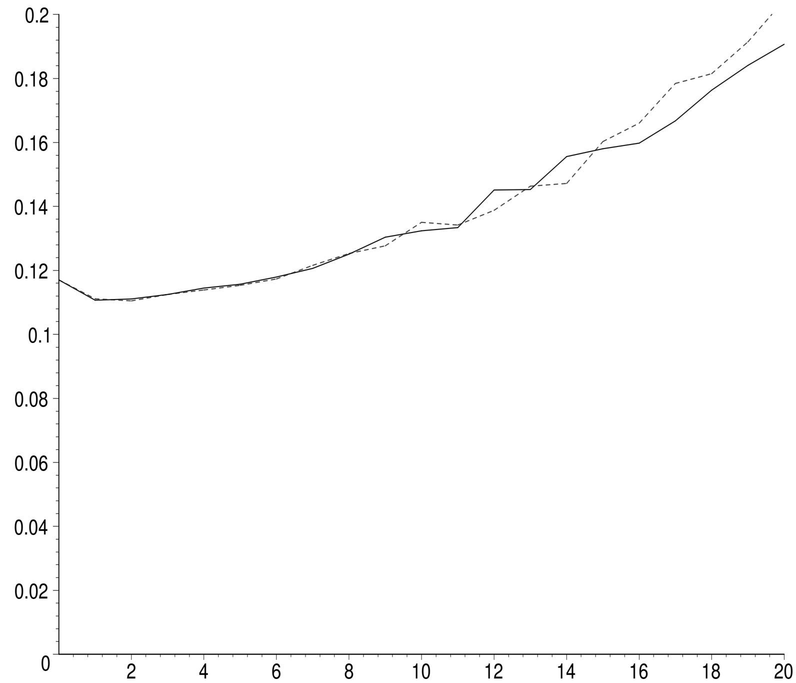

Fig. 14.3. Mean distance between original points and reconstructed points in arbitrary units as a function of mean Gaussian error in pixels introduced by the cameras. The solid line shows the values using the MinFact algorithm, and the dashed line the values for the 7pt algorithm

of course, *how* to measure the quality of the trifocal tensor. Here the quality is measured by how good a reconstruction can be achieved with the trifocal tensor in a geometric sense. This is done as follows:

1. A 3D-object is projected onto the image planes of the three cameras, which subsequently introduce some Gaussian noise into the projected point coordinates. These coordinates are then quantised according to the simulated camera resolution. The magnitude of the applied noise is measured in terms of the mean Gaussian deviation in pixels.

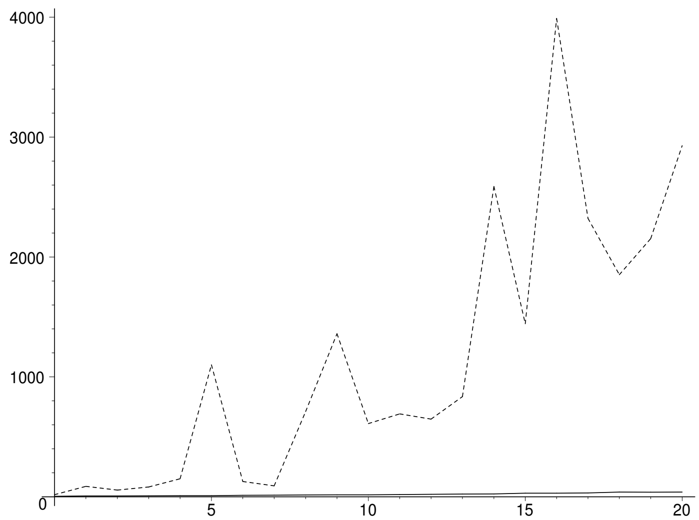

Fig. 14.4. Mean difference between elements of calculated and true tensors in percent. Solid line shows values for trifocal tensor calculated with 7pt algorithm, and dashed line shows values for trifocal tensor calculated with MinFact algorithm

2. The trifocal tensor is calculated in one of two ways from the available point matches:
   a) using the 7pt algorithm, or
   b) using the MinFact algorithm.
3. The epipoles and the camera matrices are extracted from the trifocal tensor. The camera matrices are evaluated using Hartley´s recomputation method [106].
4. The points are reconstructed using a version of what is called "Method 3" in [199] and [200] adapted for three views. This uses a SVD to solve for the homogeneous reconstructed point algebraically using a set of camera matrices. In [199] and [200] this algorithm was found to perform best of a number of reconstruction algorithms.
5. This reconstruction still contains an unknown projective transformation. Therefore it cannot be compared directly with the original object. However, since only synthetic data is used here, the 3D-points of the original object are known exactly. Therefore, a projective transformation matrix that best transforms the reconstructed points into the true points can be calculated. Then the reconstruction can be compared with the original 3D-object geometrically.
6. The final measure of "quality" is arrived at by calculating the mean distance in 3D-space between the reconstructed and the true points.

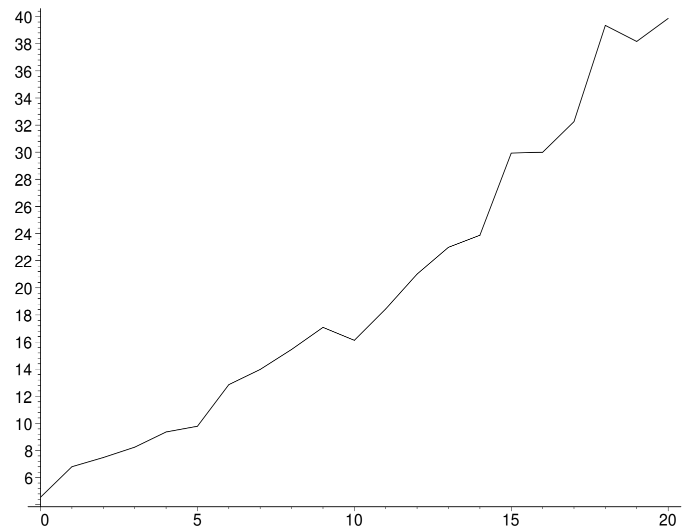

Fig. 14.5. Mean difference between elements of true trifocal tensor and trifocal tensor calculated with 7pt algorithm in percent

These quality values are evaluated for a number of different noise magnitudes. For each particular noise magnitude the above procedure is performed 100 times. The final quality value for a particular noise magnitude is then taken as the average of the 100 trials.

Figure 14.3 shows the mean distance between the original points and the reconstructed points in 3D-space in some arbitrary units[7], as a function of the noise magnitude. The camera resolution was 600 by 600 pixels.

This figure shows that for a noise magnitude of up to approximately 10 pixels both trifocal tensors seem to produce equally good reconstructions. Note that for zero added noise the reconstruction quality is not perfect. This is due to the quantisation noise of the cameras. The small increase in quality for low added noise compared to zero added noise is probably due to the cancellation of the quantisation and the added noise.

Apart from looking at the reconstruction quality it is also interesting to see how close the components of the calculated trifocal tensors are to those of the true trifocal tensor. Figures 14.4 and 14.5 both show the mean of the percentage differences between the components of the true and the calculated trifocal tensors as a function of added noise in pixels. Figure 14.4 compares the trifocal tensors found with the 7pt and the MinFact algorithms. This shows that the trifocal tensor calculated with the MinFact algorithm is indeed very

[7] The particular object used was 2 units wide, 1 unit deep and 1.5 units high in 3D-space. The Y-axis measures in the same units.

different to the true trifocal tensor, much more so than the trifocal tensor calculated with the 7pt algorithm (shown enlarged in figure 14.5).

Table 14.1. Comparison of Multiple View Tensors

Fundamental Matrix	Trifocal Tensor
$F_{i_1 j_1} = \langle\!\langle A^{i_2} A^{i_3} B^{j_2} B^{j_3} \rangle\!\rangle$	$T_{i_1}{}^{jk} = \langle\!\langle A^{i_2} A^{i_3} B^{j} C^{k} \rangle\!\rangle$
$F_{ij_1} = \varepsilon^{j_3}_{ba} K^{b}_{{j_2}_i} - \varepsilon^{j}_{ba} K^{b}_{{j_3}_i}$	$T_{i}{}^{jk} = \varepsilon^{j}_{ba} K^{c}_{k_i} - \varepsilon^{k}_{ca} K^{b}_{j_i}$
$F_{ij} = L^a_i \cdot \underbrace{(B_j \wedge B_4)}_{\text{line}}$	$T_{i}{}^{jk} = L^a_i \cdot \underbrace{\langle\!\langle B^j C^k \rangle\!\rangle}_{\text{line}}$
$\det F = 0$	$\det(T_{i}{}^{jk})_{jk} = 0$ for each i
7 DOF	18 DOF
rank 2	rank 4

Quadfocal Tensor
$Q^{ijkl} = \langle\!\langle A^i B^j C^k D^l \rangle\!\rangle$
$Q^{i_1 jkl} = \varepsilon^{j}_{ba} \Big[K^{c}_{k_{i_3}} K^{d}_{l_{i_2}} - K^{d}_{l_{i_3}} K^{c}_{k_{i_2}} \Big] - \varepsilon^{k}_{ca} \Big[K^{b}_{j_{i_3}} K^{d}_{l_{i_2}} - K^{d}_{l_{i_3}} K^{b}_{j_{i_2}} \Big] + \varepsilon^{l}_{da} \Big[K^{b}_{j_{i_3}} K^{c}_{k_{i_2}} - K^{c}_{k_{i_3}} K^{b}_{j_{i_2}} \Big]$
$Q^{ijkl} = A^i \cdot \underbrace{\langle\!\langle B^j C^k D^l \rangle\!\rangle}_{\text{point}}$
$\det(Q^{ijkl})_{xy} = 0$ where x and y are any pair of $\{ijkl\}$
29 DOF
rank 9

The data presented here seems to indicate that a tensor that obeys the determinant constraints, i.e. is self-consistent, but does not satisfies the trilinearity relationships particularly well is equally as good, in terms of reconstruction ability, as an inconsistent trifocal tensor that satisfies the trilinearity relationships quite well. In particular the fact that the trifocal tensor calculated with the MinFact algorithm is so very much different to the true trifocal tensor (see figure 14.4) does not seem to have a big impact on the final recomputation quality.

14.7 Conclusion

Table 14.1 summarises the expressions for the different tensors, their degrees of freedom, their rank and their main constraints. In particular note the similarities between the expressions for the tensors.

We have demonstrated in this paper how Geometric Algebra can be used to give a unified formalism for multiple view tensors. Almost all properties of the tensors could be arrived at from geometric considerations alone. In this way the Geometric Algebra approach is much more intuitive than traditional tensor methods. We have gained this additional insight into the workings of multiple view tensors because Projective Geometry in terms of Geometric Algebra allows us to describe the geometry on which multiple view tensors are based, directly. Therefore, we can understand their "inner workings" and inter-relations. The best examples of this are probably the derivations of the constraints on T and Q which followed from the fact that the intersection points of a line with three planes all have to lie along a line. It is hard to imagine a more trivial fact.

A similar analysis of multiple view tensors was presented by Heyden in [120]. However, we believe our treatment of the subject is more intuitive due to its geometric nature. In particular the "quadratic p-relations" used by Heyden were here replaced by the geometric fact that the intersection point of a line with a plane lies on that line.

We hope that our unified treatment of multiple view tensors has not just demonstrated the power of Geometric Algebra, but will also give a useful new tool to researchers in the field of Computer Vision.