17. Coordinate-Free Projective Geometry for Computer Vision*

Hongbo Li and Gerald Sommer

Institute of Computer Science and Applied Mathematics,
Christian-Albrechts-University of Kiel

17.1 Introduction

How to represent an image point algebraically? Given a Cartesian coordinate system of the retina plane, an image point can be represented by its coordinates (u, v). If the image is taken by a pinhole camera, then since a pinhole camera can be taken as a system that performs the perspective projection from three-dimensional projective space to two-dimensional one with respect to the optical center [77], it is convenient to describe a space point by its homogeneous coordinates $(x, y, z, 1)$ and to describe an image point by its homogeneous coordinates $(u, v, 1)$. In other words, the space of image points can be represented by the space of 3×1 matrices. This is the coordinate representation of image points.

There are other representations which are coordinate-free. The use of algebras of geometric invariants in the coordinate-free representations can lead to remarkable simplifications in geometric computing. Kanatani [128] uses the three-dimensional affine space for space points, and the space of displacements of the affine space for image points. In other words, he uses vectors fixed at the origin of $\mathbb{R}^3$ to represent space points, and uses free vectors to

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represent image points. Then he can use vector algebra to carry out geometric computing. This algebraic representation is convenient for two-dimensional projective geometry, but not for three-dimensional one. The space representing image points depends neither on the retina plane nor on the optical center.

Bayro-Corrochano, Lasenby and Sommer use $\mathbb{R}^4$ for both two-dimensional and three-dimensional projective geometries[19, 17, 222]. They use a coordinate system $\{e_1, e_2, e_3, C\}$ of $\mathbb{R}^4$ to describe a pinhole camera, where the e's are points on the retina plane and C is the optical center. Both space points and image points are represented by vectors fixed at the origin of $\mathbb{R}^4$, the only difference is that an image point is in the space spanned by vectors e_1, e_2, e_3. This algebraic representation is convenient for projective geometric computations using the incidence algebra formulated in Clifford algebra. However, it always needs a coordinate system for the camera. The space representing image points depends only on the retina plane.

We noticed that none of these algebraic representations of image points is related to the optical center. By intuition, it is better to represent image points by vectors fixed at the optical center. The above-mentioned coordinate-free representations do not have this property.

Hestenes [113] proposed a technique called space-time split to realize the Clifford algebra of the Euclidean space in the Clifford algebra of the Minkowskii space. The technique is later generalized to projective split by Hestenes and Ziegler [118] for projective geometry. We find that a version of this technique offers us exactly what we need: three-dimensional linear spaces imbedded in a four-dimensional one, whose origins do not concur with that of the four-dimensional space but whose Clifford algebras are realized in that of the four-dimensional space.

Let C be a vector in $\mathbb{R}^4$. It represents either a space point or a point at infinity of the space. Let M be another vector in $\mathbb{R}^4$. The image of the space point or point at infinity M by a pinhole camera with optical center C can be described by $C \wedge M$. The image points can be represented by the three-dimensional space $C \wedge \mathbb{R}^4 = \{C \wedge X | X \in \mathbb{R}^4\}$. The Clifford algebra of the space $C \wedge \mathbb{R}^4$ can be realized in the Clifford algebra of $\mathbb{R}^4$ by the theorem of projective split proposed later in this chapter. The space representing image points depends only on the optical center. The representation is completely projective and completely coordinate-free.

Using this new representation and the version of Grassmann-Cayley algebra formulated by Hestenes and Ziegler [118] within Clifford algebra, we have reformulated camera modeling and calibration, epipolar and trifocal geometries, relations among epipoles, epipolar tensors and trifocal tensors. Remarkable simplifications and generalizations are obtained through the reformulation, both in conception and in application. In particular, we are to derive and generalize all known constraints on epipolar and trifocal tensors [76, 80, 81, 83] in a systematic way.

This chapter is arranged as follows: in section 17.2 we collect some necessary mathematical techniques, in particular the theorem of projective split in Grassmann-Cayley algebra. In sections 17.3 and 17.4 we reformulate camera modeling and calibration, and epipolar and trifocal geometries. In section 17.5 we derive and generalize the constraints on epipolar and trifocal tensors systematically.

17.2 Preparatory Mathematics

17.2.1 Dual Bases

According to Hestenes and Sobczyk [117], let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$ and $\{e_1^*, \ldots, e_n^*\}$ be the corresponding dual (or reciprocal) basis, then

$$\begin{aligned} e_i^* &= (-1)^{i-1}(e_1 \wedge \cdots \wedge \check{e}_i \wedge \cdots \wedge e_n)^{\sim}, \\ e_i &= (-1)^{i-1}(e_1^* \wedge \cdots \wedge \check{e}_i^* \wedge \cdots \wedge e_n^*)^{\sim}, \end{aligned} \tag{17.1}$$

for $1 \leq i \leq n$. Here "$\sim$" is the dual operator in $\mathcal{G}_n$ with respect to $e_1 \wedge \cdots \wedge e_n$.

The basis $\{e_1, \ldots, e_n\}$ induces a basis $\{e_{j_1} \wedge \cdots \wedge e_{j_s} | 1 \leq j_1 < \ldots < j_s \leq n\}$ for the s-vector subspace $\mathcal{G}_n^s$ of the Clifford algebra $\mathcal{G}_n$ of $\mathbb{R}^n$. We have

$$\begin{aligned} &(e_{j_1} \wedge \cdots \wedge e_{j_s})^* \\ &\quad = e_{j_s}^* \wedge \cdots \wedge e_{j_1}^* \\ &\quad = (-1)^{j_1 + \cdots + j_s + s(s+1)/2}(e_1 \wedge \cdots \wedge \check{e}_{j_1} \wedge \cdots \wedge \check{e}_{j_s} \wedge \cdots \wedge e_n)^{\sim}. \end{aligned} \tag{17.2}$$

Let $x \in \mathcal{G}_n^s$, then

$$\begin{aligned} x &= \sum_{1 \leq j_1 < \ldots < j_s \leq n} x \cdot (e_{j_1} \wedge \cdots \wedge e_{j_s})^* \;\; e_{j_1} \wedge \cdots \wedge e_{j_s} \\ &= \sum_{1 \leq j_1 < \ldots < j_s \leq n} (-1)^{j_1 + \cdots + j_s + s(s+1)/2} \; e_{j_1} \wedge \cdots \wedge e_{j_s} \\ &\qquad\qquad (e_1 \wedge \cdots \wedge \check{e}_{j_1} \wedge \cdots \wedge \check{e}_{j_s} \wedge \cdots \wedge e_n) \vee x. \end{aligned} \tag{17.3}$$

Let an invertible transformation T of $\mathbb{R}^n$ maps $\{e_1, \ldots, e_n\}$ to a basis $\{e'_1, \ldots, e'_n\}$. Let $T^* = (T^T)^{-1}$. Then T^* maps the dual basis $\{e_1^*, \ldots, e_n^*\}$ to the dual basis $\{e'^*_1, \ldots, e'^*_n\}$.

Any linear mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has a tensor representation in $\mathbb{R}^n \otimes \mathbb{R}^m$. Then

$$T = \sum_{i=1}^{n} e'_i \otimes e_i^*. \tag{17.4}$$

For example, let Π_n be the identity transformation of $\mathbb{R}^n$, then in tensor representation, $\Pi_n = \sum_{i=1}^{n} e_i \otimes e_i^*$ for any basis $\{e_1, \ldots, e_n\}$.

17.2.2 Projective and Affine Spaces

An n-dimensional real projective space $\mathbb{P}^n$ can be realized in the space $\mathbb{R}^{n+1}$, where a projective r-space is an $(r+1)$-dimensional linear subspace. In $\mathcal{G}_{n+1}$, a projective r-space is represented by an $(r+1)$-blade, and the representation is unique up to a nonzero scale. Throughout this chapter we use "$x \simeq y$" to denote that if x, y are scalars, they are equal up to a nonzero index-free scale, otherwise they are equal up to a nonzero scale.

An n-dimensional real affine space $\mathcal{A}^n$ can be realized in the space $\mathbb{R}^{n+1}$ as a hyperplane away from the origin. Let e_0 be the vector from the origin to the hyperplane and orthogonal to the hyperplane. When $e_0^2 = 1$, a vector $x \in \mathbb{R}^{n+1}$ is an affine point if and only if $x \cdot e_0 = 1$. An r-dimensional affine plane is the intersection of an $(r+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$ with $\mathcal{A}^n$, and can be represented by an $(r+1)$-blade of $\mathcal{G}_{n+1}$ representing the subspace.

The space of displacements of $\mathcal{A}^n$ is defined as $\overset{\infty}{\mathcal{A}^n} = \{x - y | x, y \in \mathcal{A}^n\}$. It is an n-dimensional linear subspace of $\mathbb{R}^{n+1}$. Any element of it is called a direction. When $\overset{\infty}{\mathcal{A}^n}$ is taken as an $(n-1)$-dimensional projective space, any element in it is called a point at infinity, and $\overset{\infty}{\mathcal{A}^n}$ is called the space at infinity of $\mathcal{A}^n$.

Let $I_n = e_0 \cdot I_{n+1}$. Then it represents the space $\overset{\infty}{\mathcal{A}^n}$. The mapping

$$\partial_{I_n} : \quad x \mapsto e_0 \cdot x = I_n \vee x, \text{ for } x \in \mathcal{G}_{n+1}, \tag{17.5}$$

maps $\mathcal{G}_{n+1}$ to $\mathcal{G}(\overset{\infty}{\mathcal{A}^n})$, called the boundary mapping. When I_n is fixed, ∂_{I_n} is often written as ∂. Geometrically, if I_{r+1} represents an r-dimensional affine space, then ∂I_r represents its space at infinity. For example, when x, y are both affine points, $\partial(x \wedge y) = y - x$ is the point at infinity of line xy.

Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of $\mathbb{R}^{n+1}$. If $e_{n+1} \in \mathcal{A}^n$, $e_1, \ldots, e_n \in \overset{\infty}{\mathcal{A}^n}$, the basis is called a Cartesian coordinate system of $\mathcal{A}^n$, written as $\{e_1, \ldots, e_n; e_{n+1}\}$. The affine point e_{n+1} is called the origin. Let $x \in \mathcal{A}^n$, then $x = e_{n+1} + \sum_{i=1}^{n} \lambda_i e_i$. $(\lambda_1, \ldots, \lambda_n)$ is called the Cartesian coordinates of x with respect to the basis.

Below we list some properties of the three-dimensional projective (or affine) space when described in $\mathcal{G}_4$.

- Two planes N, N' are identical if and only if $N \vee N' = 0$, where N, N' are 3-blades.
- A line L is on a plane N if and only if $L \vee N = 0$, where L is a 2-blade.
- Two lines L, L' are coplanar if and only if $L \vee L' = 0$, or equivalently, if and only if $L \wedge L' = 0$.
- A point A is on a plane N if and only if $A \vee N = 0$, or equivalently, if and only if $A \wedge N = 0$. Here A is a vector.

– A point A is on a line L if and only if $A \wedge L = 0$.
– Three planes N, N', N'' are concurrent if and only if $N \vee N' \vee N'' = 0$.
– For two lines L, L', $L \vee L' = L^{\sim} \vee L'^{\sim}$.
– For point A and plane N, $A \vee N = A^{\sim} \vee N^{\sim}$.

17.2.3 Projective Splits

The following is a modified version of the technique of projective split.

Definition 17.2.1. Let C be a blade in $\mathcal{G}_n$. The projective split P_C of $\mathcal{G}_n$ with respect to C is the following transformation: $x \mapsto C \wedge x$, for $x \in \mathcal{G}_n$.

Theorem 17.2.1. [Theorem of projective split in Grassmann-Cayley algebra [1]] Let C be an r-blade in $\mathcal{G}_n$. Let $C \wedge \mathcal{G}_n = \{C \wedge x | x \in \mathcal{G}_n\}$. Define in it two products "$\wedge_C$" and "$\vee_C$": for $x, y \in \mathcal{G}_n$,

$$\begin{aligned}(C \wedge x) \wedge_C (C \wedge y) &= C \wedge x \wedge y,\\ (C \wedge x) \vee_C (C \wedge y) &= (C \wedge x) \vee (C \wedge y),\end{aligned} \tag{17.6}$$

and define

$$(C \wedge x)^{\sim_C} = C \wedge (C \wedge x)^{\sim}. \tag{17.7}$$

Then vector space $C \wedge \mathcal{G}_n$ equipped with "$\wedge_C$", "$\vee_C$", "$\sim_C$" is a Grassmann-Cayley algebra isomorphic to $\mathcal{G}_{n-r}$, which is taken as a Grassmann-Cayley algebra.

Proof. Let $C \wedge \mathbb{R}^n = \{C \wedge x | x \in \mathbb{R}^n\}$. It is an $(n-r)$-dimensional vector space. By the linear isomorphism of $\{\lambda C | \lambda \in \mathbb{R}\}$ with $\mathbb{R}$, it can be verified that $(C \wedge \mathcal{G}_n, \wedge_C)$ is isomorphic to the Grassmann algebra generated by $C \wedge \mathbb{R}^n$. A direct computation shows that the composition of "$\sim_C$" with itself is the scalar multiplication by $(-1)^{n(n-1)/2} C^2$. That $C \wedge \mathcal{G}_n$ is a Grassmann-Cayley algebra follows from the identity

$$(C \wedge x)^{\sim_C} \vee_C (C \wedge y)^{\sim_C} = ((C \wedge x) \wedge_C (C \wedge y))^{\sim_C}, \tag{17.8}$$

which can be verified by the definitions (17.6) and (17.7).

[1] Theorem 17.2.1 can be generalized to the following one, which is nevertheless not needed in this chapter:

[Theorem of projective split in Clifford algebra] Let C be a blade in $\mathcal{G}_n$. The space $C \wedge \mathcal{G}_n$ equipped with the following outer product "$\wedge_C$" and inner product "$\cdot_C$" is a Clifford algebra isomorphic to $\mathcal{G}(C^{\sim})$:

$$\begin{aligned}(C \wedge x) \wedge_C (C \wedge y) &= C \wedge x \wedge y,\\ (x \wedge C) \cdot_C (C \wedge y) &= C^{-2}\, C \wedge ((x \wedge C) \cdot (C \wedge y)),\end{aligned}$$

for $x, y \in \mathcal{G}_n$.

Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{R}^n$. The projective split P_C can be written as the composition of the outer product by C and the identity transformation. It has the following tensor representation:

$$P_C = \sum_{s=0}^{n} \sum_{1 \le j_1 < \dots < j_s \le n} (C \wedge e_{j_1} \wedge \cdots \wedge e_{j_s}) \otimes (e_{j_1} \wedge \cdots \wedge e_{j_s})^*. \qquad (17.9)$$

For example, when C is a vector and P_C is restricted to $\mathbb{R}^n$, then

$$P_C = \sum_{i=1}^{n} (C \wedge e_i) \otimes e_i^*. \qquad (17.10)$$

In particular, when $\{e_1, \dots, e_{n-1}, C\}$ is a basis of $\mathbb{R}^n$, then

$$P_C = \sum_{i=1}^{n-1} (C \wedge e_i) \otimes e_i^*. \qquad (17.11)$$

When P_C is restricted to $\mathcal{G}_n^2$, then

$$P_C = \sum_{1 \le j_1 < j_2 \le n} (C \wedge e_{j_1} \wedge e_{j_2}) \otimes (e_{j_1} \wedge e_{j_2})^*. \qquad (17.12)$$

In particular, when $\{e_1, \dots, e_{n-1}, C\}$ is a basis of $\mathbb{R}^n$, then

$$P_C = - \sum_{1 \le j_1 < j_2 \le n-1} (C \wedge e_{j_1} \wedge e_{j_2}) \otimes (e_{j_2}^* \wedge e_{j_1}^*). \qquad (17.13)$$

When $n = 4$, we use the notation $i \prec i_1 \prec i_2$ to denote that i, i_1, i_2 is an even permutation of $1, 2, 3$. Let

$$\hat{e}_i = e_{i_1} \wedge e_{i_2}, \quad \hat{e}_i^* = e_{i_1}^* \wedge e_{i_2}^*. \qquad (17.14)$$

then

$$P_C = - \sum_{i=1}^{3} (C \wedge \hat{e}_i) \otimes \hat{e}_i^*. \qquad (17.15)$$

The following theorem establishes a connection between the projective split and the boundary mapping.

Theorem 17.2.2. When C is an affine point, the boundary mapping ∂ realizes an algebraic isomorphism between the Grassmann-Cayley algebras $C \wedge \mathcal{G}_{n+1}$ and $\mathcal{G}(\overset{\infty}{\mathcal{A}^n})$.

17.3 Camera Modeling and Calibration

17.3.1 Pinhole Cameras

According to Faugeras [77], a pinhole camera can be taken as a system that performs the perspective projection from $\mathbb{P}^3$ to $\mathbb{P}^2$ with respect to the optical point $C \in \mathbb{P}^3$. To describe this mapping algebraically, let $\{e_1, e_2, e_3; O\}$ be a fixed Cartesian coordinate system of $\mathcal{A}^3$, called the world coordinate system. Let $\{e_1^C, e_2^C, e_3^C, C\}$ be a basis of $\mathbb{R}^4$ satisfying $(e_1^C \wedge e_2^C \wedge e_3^C \wedge C)^{\sim} = 1$, called a camera projective coordinate system. When C is an affine point, let e_3^C be the vector from C to the origin O^C of the retina plane (or image plane), and let e_1^C, e_2^C be two vectors in the retina plane. Then $\{e_1^C, e_2^C, e_3^C; C\}$ is a Cartesian coordinate system of $\mathcal{A}^3$, called a camera affine coordinate system.

Let M be a point or point at infinity of $\mathcal{A}^3$, and let m^C be its image. Then M can be represented by its homogeneous coordinates which is a 4×1 matrix, and m^C can be represented by its homogeneous coordinates which is a 3×1 matrix. The perspective projection can then be represented by a 3×4 matrix.

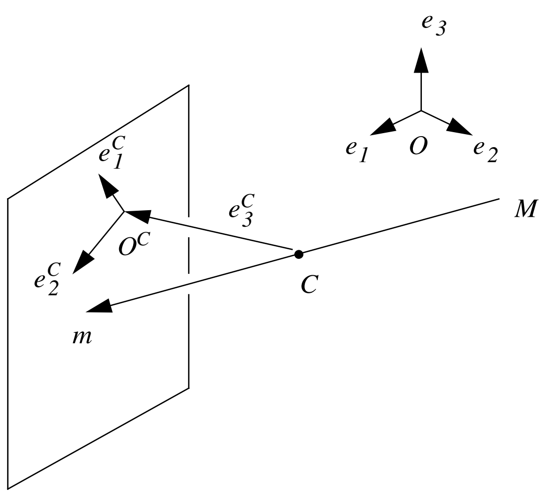

Fig. 17.1. A pinhole camera.

In our approach, we describe a pinhole camera with optical center C, which is either an affine point or a point at infinity of $\mathcal{A}^3$, as a system performing the projective split of $\mathcal{G}_4$ with respect to $C \in \mathbb{R}^4$.

To see how this representation works, we first derive the matrix of the project split P_C restricted to $\mathbb{R}^4$. We consider the case when the camera coordinate system $\{e_1^C, e_2^C, e_3^C, C\}$ is affine. According to (17.11),

$$P_C = \sum_{i=1}^{3} (C \wedge e_i^C) \otimes e_i^{C*}. \tag{17.1}$$

In the camera coordinate system, let the coordinates of e_i, $i = 1, 2, 3$, and O, be $(e_{i1}, e_{i2}, e_{i3}, 0) = (\mathbf{e}_i^T, 0)$, and $(O_1, O_2, O_3, 1) = (-\mathbf{c}^T, 1)$, respectively. Here $\mathbf{e}_i$ and $\mathbf{c}$ represent 3×1 matrices. The following matrix changes $\{e_1^C, e_2^C, e_3^C, C\}$ to $\{e_1, e_2, e_3, O\}$:

$$\begin{pmatrix} \mathbf{e}_1^T & 0 \\ \mathbf{e}_2^T & 0 \\ \mathbf{e}_3^T & 0 \\ -\mathbf{c}^T & 1 \end{pmatrix}. \tag{17.2}$$

Its transpose changes $\{e_1^*, e_2^*, e_3^*, O^*\}$ to $\{e_1^{C*}, e_2^{C*}, e_3^{C*}, C^*\}$. Substituting e_i^{C*}, $i = 1, 2, 3$ expressed by e_1^*, e_2^*, e_3^*, O^* into (17.1), we get the matrix of P_C:

$$\mathbf{P}_C = (\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3 \;\; -\mathbf{c}). \tag{17.3}$$

When $C = O$, $e_1^C = e_1$, $e_2^C = e_2$ and $e_3^C = -fe_3$, where f is the focal length of the camera,

$$\mathbf{P}_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/f & 0 \end{pmatrix}, \tag{17.4}$$

which is the standard perspective projection matrix. This justifies the representation of the perspective projection by P_C and the representation of image points by vectors in $C \wedge \mathbb{R}^4$.

In the case when the camera coordinate system is projective, let the 4×1 matrices $\mathbf{e}_i^{C*}$, $i = 1, 2, 3$ represent the coordinates of e_i^{C*} with respect to $\{e_1^*, e_2^*, e_3^*, O^*\}$. By (17.1),

$$\mathbf{P}_C = (\mathbf{e}_1^{C*} \;\; \mathbf{e}_2^{C*} \;\; \mathbf{e}_3^{C*})^T. \tag{17.5}$$

Below we derive the matrix of P_C restricted to $\mathcal{G}_4^2$. Let

$$\hat{\mathbf{e}}_i^{C*} = \mathbf{e}_{i_1}^{C*} \times \mathbf{e}_{i_2}^{C*}, \tag{17.6}$$

where $i \prec i_1 \prec i_2$. It represents the coordinates of $\hat{e}_i^{C*}$ with respect to the basis of $\mathcal{G}_4^2$ induced by $\{e_1^*, e_2^*, e_3^*, O^*\}$. According to (17.15), the matrix of P_C is

$$\mathbf{P}_C = -(\hat{\mathbf{e}}_1^{C*} \;\; \hat{\mathbf{e}}_2^{C*} \;\; \hat{\mathbf{e}}_3^{C*})^T. \tag{17.7}$$

17.3.2 Camera Constraints

It is clear that as long as $\det(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) \neq 0$, the matrix $\mathbf{P}_C = (\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3 - \mathbf{c})$ represents a perspective projection. When there is further information on the pinhole camera, for example vectors e_1^C, e_2^C of the camera affine coordinate system are perpendicular, then $\mathbf{P}_C$ needs to satisfy additional equality constraints in order to represent the perspective projection carried out by such a camera.

Let "$\sim_3$" represent the dual in $\mathcal{G}(\overset{\infty}{\mathcal{A}}{}^3)$. Let the dual bases of $\{e_1, e_2, e_3\}$ and $\{e_1^C, e_2^C, e_3^C\}$ in $\overset{\infty}{\mathcal{A}}{}^3$ be $\{e_1^{*_3}, e_2^{*_3}, e_3^{*_3}\}$ and $\{e_1^{C*_3}, e_2^{C*_3}, e_3^{C*_3}\}$, respectively. Then

$$
\begin{aligned}
e_1^C &= (e_2^{C*_3} \wedge e_3^{C*_3})^{\sim_3} = e_2^{C*_3} \times e_3^{C*_3}, \\
e_2^C &= (e_3^{C*_3} \wedge e_1^{C*_3})^{\sim_3} = e_3^{C*_3} \times e_1^{C*_3},
\end{aligned}
\tag{17.8}
$$

where "$\times$" is the cross product in vector algebra. The perpendicularity constraint can be represented by

$$
e_1^C \cdot e_2^C = (e_2^{C*_3} \times e_3^{C*_3}) \cdot (e_3^{C*_3} \times e_1^{C*_3}) = 0. \tag{17.9}
$$

Let the 3×1 matrix $\mathbf{e}_i^{C*_3}$ represent the coordinates of $e_i^{C*_3}$ with respect to $\{e_1^{*_3}, e_2^{*_3}, e_3^{*_3}\}$. Under the assumption that $\{e_1, e_2, e_3\}$ is an orthonormal basis, $e_i^{C*_3} \cdot e_j^{C*_3} = \mathbf{e}_i^{C*_3} \cdot \mathbf{e}_j^{C*_3}$ for any $1 \leq i, j \leq 3$. Then (17.9) is changed to

$$
(\mathbf{e}_2^{C*_3} \times \mathbf{e}_3^{C*_3}) \cdot (\mathbf{e}_3^{C*_3} \times \mathbf{e}_1^{C*_3}) = 0, \tag{17.10}
$$

which is a constraint on $\mathbf{P}_C$ because

$$
(\mathbf{e}_1^{C*_3}\ \ \mathbf{e}_2^{C*_3}\ \ \mathbf{e}_3^{C*_3}) = (\mathbf{e}_1\ \ \mathbf{e}_2\ \ \mathbf{e}_3)^T. \tag{17.11}
$$

17.3.3 Camera Calibration

Let M be a space point or point at infinity, m^C be its image in the retina plane. Assume that m^C is a point, and has homogeneous coordinates $(u, v, 1)$ in the Cartesian coordinate system of the retina plane. Let the 4×1 matrix $\mathbf{M}$ represent the homogeneous coordinates of M in the world coordinate system. Then

$$
(u\ \ v\ \ 1)^T \simeq \mathbf{P}_C \mathbf{M} = (\mathbf{e}_1^{C*} \cdot \mathbf{M}\quad \mathbf{e}_2^{C*} \cdot \mathbf{M}\quad \mathbf{e}_3^{C*} \cdot \mathbf{M})^T, \tag{17.12}
$$

which can be written as two scalar equations:

$$
(\mathbf{e}_1^{C*} - u\mathbf{e}_3^{C*}) \cdot \mathbf{M} = 0, \quad (\mathbf{e}_2^{C*} - v\mathbf{e}_3^{C*}) \cdot \mathbf{M} = 0. \tag{17.13}
$$

The matrix $\mathbf{P}_C = (\mathbf{e}_1^{C*}\ \mathbf{e}_2^{C*}\ \mathbf{e}_3^{C*})^T$ can be taken as a vector in the space $\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$ equipped with the induced inner product from $\mathbb{R}^4$. By this inner product, (17.13) can be written as

$$(\mathbf{M} \quad 0 \ -u\mathbf{M})^T \cdot \mathbf{P}_C = 0, \quad (0 \quad \mathbf{M} \ -v\mathbf{M})^T \cdot \mathbf{P}_C = 0. \tag{17.14}$$

Given $\mathbf{M}_i$ and (u_i, v_i) for $i = 1, \ldots, 6$, there are 12 equations of the forms in (17.14). If there is no camera constraint, then since a 3×4 matrix representing a perspective projection has 11 free parameters, $\mathbf{P}_C$ can be solved from the 12 equations if and only if the determinant of the coefficient matrix $\mathbf{A}$ of these equations is zero, i. e.,

$$\Lambda_{i=1}^6(\mathbf{M}_i \quad 0 \ -u_i\mathbf{M}_i) \wedge \Lambda_{i=1}^6(0 \quad \mathbf{M}_i \ -v_i\mathbf{M}_i) = 0, \tag{17.15}$$

where the outer products are in the Clifford algebra generated by $\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$. Expanding the left-hand side of (17.15), and changing outer products into determinants, we get

$$\sum_{\sigma,\tau} \epsilon(\sigma)\epsilon(\tau)u_{\sigma(1)}u_{\sigma(2)}v_{\tau(1)}v_{\tau(2)} \det(\mathbf{M}_{\sigma(1)} \ \mathbf{M}_{\sigma(2)} \ \mathbf{M}_{\tau(1)} \ \mathbf{M}_{\tau(2)}) \\ \det(\mathbf{M}_{\sigma(i)})_{i=3..6} \det(\mathbf{M}_{\tau(j)})_{j=3..6} = 0, \tag{17.16}$$

where σ, τ are any permutations of $1, \ldots, 6$ by moving two elements to the front of the sequence, and $\epsilon(\sigma), \epsilon(\tau)$ are the signs of permutation.

For experimental data, (17.16) is not necessarily satisfied because of errors in measurements.

17.4 Epipolar and Trifocal Geometries

17.4.1 Epipolar Geometry

There is no much difference between our algebraic description of the pinhole camera and others if there is only one fixed camera involved, because the underlying Grassmann-Cayley algebras are isomorphic. Let us reformulate the epipolar geometry of two cameras with optical centers C, C' respectively.

The image of C' in camera C is $E^{CC'} = C \wedge C'$, called the epipole of C' in camera C. Similarly, the image of C in camera C' is $E^{C'C} = C' \wedge C$, called the epipole of C in camera C'. An image line passing through the epipole in camera C (or C') is called an epipolar line with respect to C' (or C). Algebraically, an epipolar line is a vector in

$$C \wedge C' \wedge \mathbb{R}^4 = (C \wedge \mathcal{G}_4^2) \cap (C' \wedge \mathcal{G}_4^2). \tag{17.1}$$

An epipolar line $C \wedge C' \wedge M$ corresponds to a unique epipolar line $C' \wedge C \wedge M$, and vice versa.

Let there be two camera projective coordinate systems in the two cameras respectively: $\{e_1^C, e_2^C, e_3^C, C\}$ and $\{e_1^{C'}, e_2^{C'}, e_3^{C'}, C'\}$. Using the relations

$$(C \wedge e_i^C) \vee (C \wedge \hat{e}_i^C) = -C, \text{ for } 1 \le i \le 3, \tag{17.2}$$

and

$$(C \wedge \hat{e}_{i_1}^C) \vee (C \wedge \hat{e}_{i_2}^C) = C \wedge \hat{e}_i^C, \text{ for } i \prec i_1 \prec i_2, \tag{17.3}$$

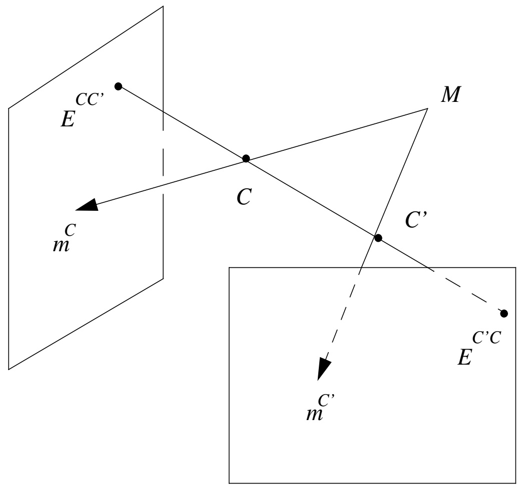

Fig. 17.2. Epipolar geometry.

we get the coordinates of epipole $E^{CC'}$:

$$\begin{aligned}\mathbf{E}^{CC'} &= ((C \wedge \hat{e}_i^C) \vee C')_{i=1..3} \\ &= ((C \wedge \hat{e}_i^C)^{\sim} \vee C'^{\sim})_{i=1..3} \\ &= ((e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*}) \vee e_i^{C*})_{i=1..3}.\end{aligned} \tag{17.4}$$

The following tensor in $(C \wedge \mathbb{R}^4) \otimes (C' \wedge \mathbb{R}^4)$ is called the epipolar tensor decide by C, C':

$$F^{CC'}(m^C, m^{C'}) = m^C \vee m^{C'}. \tag{17.5}$$

Let $m^C \in C \wedge \mathbb{R}^4$, $m^{C'} \in C' \wedge \mathbb{R}^4$. They are images of the same space point or point at infinity if and only if $F^{CC'}(m^C, m^{C'}) = 0$. This equality is called the epipolar constraint between m^C and $m^{C'}$.

In matrix form, with respect to the bases $\{C \wedge e_1^C, C \wedge e_2^C, C \wedge e_3^C\}$ and $\{C' \wedge e_1^{C'}, C' \wedge e_2^{C'}, C' \wedge e_3^{C'}\}$, $F^{CC'}$ can be represented by

$$\begin{aligned}\mathbf{F}^{CC'} &= ((C \wedge e_i^C) \vee (C' \wedge e_j^{C'}))_{i,j=1..3} \\ &= ((C \wedge e_i^C)^{\sim} \vee (C' \wedge e_j^{C'})^{\sim})_{i,j=1..3} \\ &= (\hat{e}_i^{C*} \vee \hat{e}_j^{C'*})_{i,j=1..3}.\end{aligned} \tag{17.6}$$

(17.6) is called the fundamental matrix.

The epipolar tensor induces a linear mapping $F^{C;C'}$ from $C \wedge \mathbb{R}^4$ to $(C' \wedge \mathbb{R}^4)^* = C' \wedge \mathcal{G}_4^2$, called the epipolar transformation from camera C to camera C':

$$F^{C;C'}(m^C) = C' \wedge m^C. \tag{17.7}$$

Similarly, it induces an epipolar transformation from camera C' to camera C as follows:

$$F^{C';C}(m^{C'}) = C \wedge m^{C'}. \tag{17.8}$$

Both transformations are just projective splits.

The kernel of $F^{C;C'}$ is the one-dimensional subspace of $C \wedge \mathbb{R}^4$ represented by $C \wedge C'$, the range of $F^{C;C'}$ is the two-dimensional space $C' \wedge C \wedge \mathbb{R}^4$. In geometric language, $F^{C;C'}$ maps the epipole of C' to zero, and maps any other point in camera C to an epipolar line with respect to C.

Furthermore, we have the following conclusion:

Proposition 17.4.1. Let L^C be an epipolar line in camera C. If its dual is mapped to epipolar line $L^{C'}$ in camera C' by $F^{C;C'}$, then the dual of $L^{C'}$ is mapped back to L^C by $F^{C';C}$.

The proof follows from the identity that for any vector $M \in \mathbb{R}^4$,

$$C \wedge (C' \wedge (C \wedge C' \wedge M)^{\sim_C})^{\sim_{C'}} \simeq C \wedge C' \wedge M. \tag{17.9}$$

17.4.2 Trifocal Geometry

Let there be three cameras with optical centers C, C', C'' respectively. Let M be a space point or point at infinity. Its images $C \wedge M$, $C' \wedge M$ and $C'' \wedge M$ in the three cameras must satisfy pairwise epipolar constraints. Let us consider the inverse problem: If there are three image points $m^C, m^{C'}, m^{C''}$ in the three cameras respectively, they satisfy the pairwise epipolar constraints, is it true that they are images of the same space point or point at infinity?

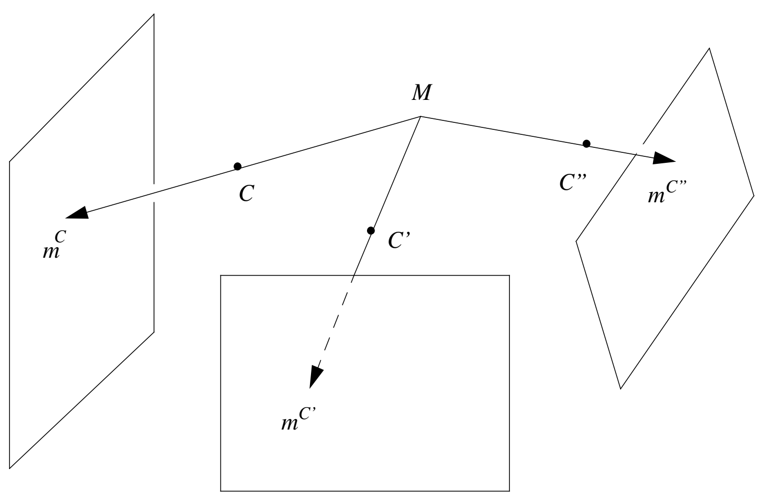

Fig. 17.3. Point correspondence in three cameras.

A simple counter-example shows that the epipolar constraints are not enough. When the 2-blades $m^C, m^{C'}, m^{C''}$ belong to $\mathcal{G}(C \wedge C' \wedge C'')$, the epipolar constraints are always satisfied, but the blades do not necessarily share a common vector.

Assume that the epipolar constraint between $m^{C'}$ and $m^{C''}$ is satisfied. Let M be the intersection of the two lines $m^{C'}$ and $m^{C''}$ in $\mathbb{P}^3$. Then m^C represents the image of M in camera C if and only if $m^C \wedge M = 0$, or equivalently,

$$m^C \vee (M \wedge x) = 0, \text{ for any } x \in \mathbb{R}^4. \tag{17.10}$$

When C', C'', M are not collinear, since

$$M \wedge \mathbb{R}^4 = (C' \wedge M \wedge \mathbb{R}^4) \vee (C'' \wedge M \wedge \mathbb{R}^4), \tag{17.11}$$

(17.10) can be written as

$$m^C \vee (m^{C'} \wedge_{C'} m_0^{C'}) \vee (m^{C''} \wedge_{C''} m_0^{C''}) = 0, \tag{17.12}$$

for any image points $m_0^{C'}, m_0^{C''}$ in cameras C', C'' respectively. When C', C'', M are collinear, since $m^{C'} \simeq m^{C''}$, (17.12) is equivalent to the epipolar constraint between m^C and $m^{C'}$. So the constraint (17.12) must be satisfied for $m^C, m^{C'}, m^{C''}$ to be images of the same space point or point at infinity.

Definition 17.4.1. The following tensor in $(C \wedge \mathbb{R}^4) \otimes (C' \wedge \mathcal{G}_4^2) \otimes (C'' \wedge \mathcal{G}_4^2)$ is called the trifocal tensor [105, 106, 214] of camera C with respect to cameras C', C'':

$$T(m^C, L^{C'}, L^{C''}) = m^C \vee L^{C'} \vee L^{C''}, \tag{17.13}$$

where $m^C \in C \wedge \mathbb{R}^4$, $L^{C'} \in C' \wedge \mathcal{G}_4^2$, $L^{C''} \in C'' \wedge \mathcal{G}_4^2$.

Two other trifocal tensors can be defined by interchanging C with C', C'' respectively:

$$\begin{aligned} T'(m^{C'}, L^C, L^{C''}) &= m^{C'} \vee L^C \vee L^{C''}, \\ T''(m^{C''}, L^C, L^{C'}) &= m^{C''} \vee L^C \vee L^{C'}. \end{aligned} \tag{17.14}$$

In this section we discuss T only. Let $\{e_1^C, e_2^C, e_3^C, C\}$, $\{e_1^{C'}, e_2^{C'}, e_3^{C'}, C'\}$, $\{e_1^{C''}, e_2^{C''}, e_3^{C''}, C''\}$ be camera projective coordinate systems of the three cameras respectively. Then T has the following component representation:

$$\begin{aligned} \mathbf{T} &= ((C \wedge e_i^C) \vee (C' \wedge \hat{e}_j^{C'}) \vee (C'' \wedge \hat{e}_k^{C''}))_{i,j,k=1..3} \\ &= ((C \wedge e_i^C)^\sim \vee ((C' \wedge \hat{e}_j^{C'})^\sim \wedge (C'' \wedge \hat{e}_k^{C''})^\sim))_{i,j,k=1..3} \\ &= (\hat{e}_i^{C*} \vee (e_j^{C'*} \wedge e_k^{C''*}))_{i,j,k=1..3} \\ &= (-(\hat{e}_i^{C*} \wedge e_j^{C'*}) \vee e_k^{C''*})_{i,j,k=1..3}. \end{aligned} \tag{17.15}$$

The trifocal tensor T induces three trifocal transformations:

1. The mapping $T^C : (C' \wedge \mathcal{G}_4^2) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow (C \wedge \mathbb{R}^4)^* = C \wedge \mathcal{G}_4^2$ is defined as
$$T^C(L^{C'}, L^{C''}) = C \wedge (L^{C'} \vee L^{C''}). \tag{17.16}$$
When $L^{C'}$ is fixed, T^C induces a linear mapping $T^{CC'}_{L^{C'}} : C'' \wedge \mathcal{G}_4^2 \longrightarrow C \wedge \mathcal{G}_4^2$:
$$T^{CC'}_{L^{C'}}(L^{C''}) = C \wedge (L^{C'} \vee L^{C''}). \tag{17.17}$$
If $L^{C'}$ is an epipolar line with respect to C, the kernel of $T^{CC'}_{L^{C'}}$ is all epipolar lines with respect to C, the range is the epipolar line represented by $L^{C'}$; else if $L^{C'}$ is an epipolar line with respect to C'', the kernel is the epipolar line represented by $L^{C'}$, the range is all epipolar lines with respect to C''. For other cases, the kernel is zero.
Geometrically, when $T^C(L^{C'}, L^{C''}) \neq 0$, then $L^{C'} \vee L^{C''}$ represents a line or line at infinity L of $\mathcal{A}^3$, both $L^{C'}$ and $L^{C''}$ are images of L. $T^C(L^{C'}, L^{C''})$ is just the image of L in camera C.
2. The mapping $T^{C'} : (C \wedge \mathbb{R}^4) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow (C' \wedge \mathcal{G}_4^2)^* = C' \wedge \mathbb{R}^4$ is defined as
$$T^{C'}(m^C, L^{C''}) = C' \wedge (m^C \vee L^{C''}). \tag{17.18}$$
When m^C is fixed, $T^{C'C}$ induces a linear mapping $T^{C'C}_{m^C} : C'' \wedge \mathcal{G}_4^2 \longrightarrow C' \wedge \mathbb{R}^4$:
$$T^{C'C}_{m^C}(L^{C''}) = C' \wedge (m^C \vee L^{C''}). \tag{17.19}$$
If m^C is the epipole of C'', the kernel of $T^{C'C}_{m^C}$ is all epipolar lines with respect to C, the range is the epipole of C''. For other cases, the kernel is the epipolar line $C'' \wedge m^C$, the range is the two-dimensional subspace of $C' \wedge \mathbb{R}^4$ represented by $C' \wedge m^C$.
Geometrically, when $T^{C'}(m^C, L^{C''}) \neq 0$, then $m^C \vee L^{C''}$ represents a point or point at infinity M of $\mathcal{A}^3$, m^C is its image in camera C, and $L^{C''}$ is the image of a space line or line at infinity passing through M. $T^{C'}(m^C, L^{C''})$ is just the image of M in camera C'.
3. The mapping $T^{C''} : (C \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}_4^2) \longrightarrow (C'' \wedge \mathcal{G}_4^2)^* = C'' \wedge \mathbb{R}^4$ is defined as
$$T^{C''}(m^C, L^{C'}) = C'' \wedge (m^C \vee L^{C'}). \tag{17.20}$$

We prove below two propositions in [81, 83] using the above reformulation of trifocal tensors.

Proposition 17.4.2. Let $L^{C'}$ be an epipolar line in camera C' with respect to C and L^C be the corresponding epipolar line in camera C. Then for any line $L^{C''}$ in camera C'' which is not the epipolar line with respect to C, $T^C(L^{C'}, L^{C''}) \simeq L^C$.

Proof. The hypotheses are $L^{C'} \simeq L^C$, $C \vee L^{C''} \neq 0$. Using the formula that for any $C \in \mathbb{R}^4$, $A_3, B_3 \in \mathcal{G}_4^3$,

$$C \wedge (A_3 \vee B_3) = (C \vee B_3)A_3 - (C \vee A_3)B_3, \tag{17.21}$$

we get

$$T^C(L^{C'}, L^{C''}) \simeq C \wedge (L^C \vee L^{C''}) = (C \vee L^{C''})L^C - (C \vee L^C)L^{C''} \simeq L^C.$$

Proposition 17.4.3. Let $m^{C'}, m^{C''}$ be images of the point or point at infinity M in cameras C', C'' respectively. Let $L^{C'}$ be an image line passing through $m^{C'}$ but not through $E^{C'C}$. Let $L^{C''}$ be an image line passing through $m^{C''}$ but not through $E^{C''C'}$. Then the intersection of $T^C(L^{C'}, L^{C''})$ with the epipolar line $C \wedge m^{C'}$ is the image of M in camera C.

Proof. The hypotheses are $M \vee L^{C'} = M \vee L^{C''} = 0$, $C \vee L^{C'} \neq 0$, $C' \vee L^{C''} \neq 0$. So

$$\begin{aligned}
&T^C(L^{C'}, L^{C''}) \vee (C \wedge m^{C'}) \\
&\quad = (C \wedge (L^{C'} \vee L^{C''})) \vee (C \wedge C' \wedge M) \\
&\quad = ((C \wedge C') \vee L^{C'} \vee L^{C''})(C \wedge M) - ((C \wedge M) \vee L^{C'} \vee L^{C''})(C \wedge C') \\
&\quad = -(C \vee L^{C'})(C' \vee L^{C''})(C \wedge M) \\
&\quad \simeq C \wedge M.
\end{aligned}$$

17.5 Relations among Epipoles, Epipolar Tensors, and Trifocal Tensors of Three Cameras

Consider the following 9 vectors of $\mathbb{R}^4$:

$$ES = \{e_i^{C*}, e_j^{C'*}, e_k^{C''*} | 1 \leq i, j, k \leq 3\}. \tag{17.1}$$

According to (17.4), (17.6) and (17.15), by interchanging among C, C', C'' any of the epipoles, epipolar tensors and trifocal tensors of the three cameras has its components represented as a determinant of 4 vectors in ES. For example,

$$\begin{aligned}
E_i^{CC'} &= (e_i^{C*} \wedge e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*})^{\sim}; \\
F_{ij}^{CC'} &= (\hat{e}_i^{C*} \wedge \hat{e}_j^{C'*})^{\sim}; \\
T_{ijk} &= (\hat{e}_i^{C*} \wedge e_j^{C'*} \wedge e_k^{C''*})^{\sim}.
\end{aligned} \tag{17.2}$$

Conversely, any determinant of 4 vectors in ES equals a component of one of the epipoles, epipolar tensors and trifocal tensors up to an index-free scale. Since the only constraint on the 9 vectors is that they are all in $\mathbb{R}^4$, theoretically all relations among the epipoles, epipolar tensors and trifocal tensors can be established by manipulating in the algebra of determinants of vectors in ES using the following Cramer's rule [76, 80]:

$$\begin{aligned}(x_2 \wedge x_3 \wedge x_4 \wedge x_5)^{\sim} x_1 = (x_1 \wedge x_3 \wedge x_4 \wedge x_5)^{\sim} x_2 - (x_1 \wedge x_2 \wedge x_4 \wedge x_5)^{\sim} x_3 \\ +(x_1 \wedge x_2 \wedge x_3 \wedge x_5)^{\sim} x_4 - (x_1 \wedge x_2 \wedge x_3 \wedge x_4)^{\sim} x_5,\end{aligned} \tag{17.3}$$

where the x's are vectors in $\mathbb{R}^4$.

In practice, however, we can only select a few expressions from the algebra of determinants and make manipulations, and it is difficult to make the selection. In this section we propose a different approach. Instead of considering the algebra of determinants directly, we consider the set of meets of different blades, each blade being an outer product of vectors in ES. Since the meet operator is associative and anti-commutative in the sense that

$$A_r \vee B_s = (-1)^{rs} B_s \vee A_r, \tag{17.4}$$

for $A_r \in \mathcal{G}_4^r$ and $B_s \in \mathcal{G}_4^s$, for the same expression of meets we can have a variety of expansions. Then we can obtain various equalities on determinants of vectors in ES, which may be changed into equalities, or equalities up to an index-free constant, on components of the epipoles, epipolar tensors and trifocal tensors.

It appears that we need only 7 expressions of meets to derive and further generalize all the known constraints on epipolar and trifocal tensors.

It should be reminded that in this chapter we always use the notation $i \prec i_1 \prec i_2$ to denote that i, i_1, i_2 is an even permutation of $1, 2, 3$.

17.5.1 Relations on Epipolar Tensors

Consider the following expression:

$$Fexp = (e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.5}$$

It is the dual of the blade $C' \wedge C \wedge C''$.

Expanding $Fexp$ from left to right, we get

$$\begin{aligned} Fexp &= \sum_{i,k=1}^{3} ((e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*}) \vee e_i^{C*})(\hat{e}_i^{C*} \vee \hat{e}_k^{C''*}) e_k^{C''*} \\ &= \sum_{i,k=1}^{3} E_i^{CC'} F_{ik}^{CC''} e_k^{C''*}. \end{aligned}$$

Expanding $Fexp$ from right to left, we get

$$\begin{aligned} Fexp = \sum_{k=1}^{3} \Big(&((e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*}) \vee e_{k_2}^{C''*})((e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*}) \vee e_{k_1}^{C''*}) \\ &- ((e_1^{C'*} \wedge e_2^{C'*} \wedge e_3^{C'*}) \vee e_{k_1}^{C''*})((e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*}) \vee e_{k_2}^{C''*}) \Big)\ e_k^{C''*} \\ = \sum_{k=1}^{3} &(E_{k_2}^{C''C'} E_{k_1}^{C''C} - E_{k_1}^{C''C'} E_{k_2}^{C''C}) e_k^{C''*}, \end{aligned}$$

where $k \prec k_1 \prec k_2$. So for any $1 \le i \le 3$,

$$\sum_{k=1}^{3} E_i^{CC'} F_{ik}^{CC''} \simeq K_k^{C''CC'}, \tag{17.6}$$

where $K_k^{C''CC'} = E_{k_1}^{C''C} E_{k_2}^{C''C'} - E_{k_2}^{C''C} E_{k_1}^{C''C'}$.

(17.6) is a fundamental relation on the epipolar tensor $F^{CC''}$ and the epipoles. In matrix form, it can be written as

$$(\mathbf{F}^{CC''})^T \mathbf{E}^{CC'} \simeq \mathbf{E}^{C''C} \times \mathbf{E}^{C''C'}; \tag{17.7}$$

in Grassmann-Cayley algebra, it can be written as

$$C'' \wedge (C \wedge C') \simeq (C'' \wedge C) \wedge_{C''} (C'' \wedge C'). \tag{17.8}$$

Geometrically, it means that the epipolar line in camera C'' with respect to both C and C' is the image line connecting the two epipoles $E^{C''C}$ and $E^{C''C'}$. One should notice the obvious advantage of Grassmann-Cayley algebraic representation in geometric interpretation.

Since $\mathbf{E}^{C''C} \times \mathbf{E}^{C''C'}$ is orthogonal to $\mathbf{E}^{C''C'}$, an immediate corollary is

$$(\mathbf{E}^{CC'})^T \mathbf{F}^{CC''} \mathbf{E}^{C''C'} = 0, \tag{17.9}$$

which is equivalent to $(C \wedge C') \vee (C'' \wedge C') = 0$. Geometrically, it means that the two epipoles $E^{C''C}$ and $E^{C''C'}$ satisfy the epipolar constraint.

17.5.2 Relations on Trifocal Tensors I

The first idea to derive relations on trifocal tensors is very simple: if the tensor $(T_{ijk})_{i,j,k=1..3}$ is given, then expanding

$$(\hat{e}_i^{C*} \wedge e_j^{C'*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}) \tag{17.10}$$

gives a 2-vector of the $e^{C''*}$'s whose coefficients are known. Similarly, expanding

$$Texp_1 = (\hat{e}_i^{C*} \wedge e_{j_1}^{C'*}) \vee (\hat{e}_i^{C*} \wedge e_{j_2}^{C'*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.11}$$

from right to left gives a vector of the $e^{C''*}$'s whose coefficients are known. Expanding $Texp_1$ from left to right, we get a vector of the $e^{C''*}$'s whose coefficients depend on epipolar tensors. By comparing the coefficients of the $e^{C''*}$'s, we get a relation on T and epipolar tensors.

Assume that $j \prec j_1 \prec j_2$. Expanding $Texp_1$ from left to right, we get

$$\begin{aligned} Texp_1 &= -(\hat{e}_i^{C*} \wedge e_{j_1}^{C'*} \wedge e_{j_2}^{C'*})^\sim \sum_{k=1}^{3} (\hat{e}_i^{C*} \vee \hat{e}_k^{C''*})\, e_k^{C''*} \\ &= -\sum_{k=1}^{3} F_{ij}^{CC'} F_{ik}^{CC''} e_k^{C''*}. \end{aligned}$$

Expanding $Texp_1$ from right to left, we get

$$\begin{aligned} Texp_1 &= \sum_{k=1}^{3} \Big(((\hat{e}_i^{C*} \wedge e_{j_1}^{C'*}) \vee e_{k_2}^{C''*})((\hat{e}_i^{C*} \wedge e_{j_2}^{C'*}) \vee e_{k_1}^{C''*}) \\ &\qquad - ((\hat{e}_i^{C*} \wedge e_{j_1}^{C'*}) \vee e_{k_1}^{C''*})((\hat{e}_i^{C*} \wedge e_{j_2}^{C'*}) \vee e_{k_2}^{C''*}) \Big)\; e_k^{C''*} \\ &= \sum_{k=1}^{3} (T_{ij_1k_2} T_{ij_2k_1} - T_{ij_1k_1} T_{ij_2k_2})\, e_k^{C''*}, \end{aligned}$$

where $k \prec k_1 \prec k_2$. So

$$F_{ij}^{CC'} F_{ik}^{CC''} = t_{ijk}^{C}, \tag{17.12}$$

where

$$t_{ijk}^{C} = T_{ij_1k_1} T_{ij_2k_2} - T_{ij_1k_2} T_{ij_2k_1}. \tag{17.13}$$

Proposition 17.5.1. For any $1 \le i, j, k \le 3$,

$$F_{ij}^{CC'} F_{ik}^{CC''} \simeq t_{ijk}^{C}. \tag{17.14}$$

Corollary 17.5.1. Let $1 \le i, j_1, j_2, k_1, k_2 \le 3$, then

$$\frac{F_{ij_1}^{CC'}}{F_{ij_2}^{CC'}} = \frac{t_{ij_1k}^{C}}{t_{ij_2k}^{C}}, \text{ for any } 1 \le k \le 3; \tag{17.15}$$

$$\frac{F_{ik_1}^{CC''}}{F_{ik_2}^{CC''}} = \frac{t_{ijk_1}^{C}}{t_{ijk_2}^{C}}, \text{ for any } 1 \le j \le 3; \tag{17.16}$$

$$\frac{t_{ij_1k_1}^{C}}{t_{ij_1k_2}^{C}} = \frac{t_{ij_2k_1}^{C}}{t_{ij_2k_2}^{C}}. \tag{17.17}$$

Notice that (17.17) is a constraint of degree 4 on T.

To understand relation (17.14) geometrically, we first express it in terms of Grassmann-Cayley algebra. When $C \wedge e_i^C$ is fixed, T induces a linear mapping $T_i^{C''C} : C' \wedge \mathcal{G}_4^2 \longrightarrow C'' \wedge \mathbb{R}^4$ by

$$T_i^{C''C}(L^{C'}) = C'' \wedge ((C \wedge e_i^C) \vee L^{C'}). \tag{17.18}$$

The matrix of $T_i^{C''C}$ is $(-T_{ijk})_{j,k=1..3}^{T}$.

Define a linear mapping $t_i^{C''C} : C' \wedge \mathbb{R}^4 \longrightarrow C'' \wedge \mathcal{G}_4^2$ as follows: let $m^{C'} \in C' \wedge \mathbb{R}^4$ and $m^{C'} = L_1^{C'} \vee L_2^{C'}$, where $L_1^{C'}, L_2^{C'} \in C' \wedge \mathcal{G}_4^2$, then

$$t_i^{C''C}(m^{C'}) = T_i^{C''C}(L_1^{C'}) \wedge_{C''} T_i^{C''C}(L_2^{C'}). \tag{17.19}$$

We need to prove that this mapping is well-defined. Using the formula that for any 2-blade $C_2 \in \mathcal{G}_4^2$ and 3-blades $A_3, B_3 \in \mathcal{G}_4^3$,

$$(C_2 \vee A_3) \wedge (C_2 \vee B_3) = -(A_3 \vee B_3 \vee C_2)C_2, \tag{17.20}$$

we get

$$\begin{aligned} t_i^{C''C}(m^{C'}) &= C'' \wedge ((C \wedge e_i^C) \vee L_1^{C'}) \wedge ((C \wedge e_i^C) \vee L_2^{C'}) \\ &= -L_1^{C'} \vee L_2^{C'} \vee (C \wedge e_i^C) \;\; C'' \wedge C \wedge e_i^C \\ &= -m^{C'} \vee (C \wedge e_i^C) \;\; C'' \wedge C \wedge e_i^C \\ &= \sum_{k=1}^{3} m^{C'} \vee (C \wedge e_i^C) \;\; (C'' \wedge C \wedge e_i^C \wedge e_k^{C''})^{\sim} \; C'' \wedge \hat{e}_k^{C''}. \end{aligned} \tag{17.21}$$

So $t_i^{C''C}$ is well-defined. Let $j \prec j_1 \prec j_2$ and $k \prec k_1 \prec k_2$, then since

$$\begin{aligned} t_i^{C''C}(C' \wedge e_j^{C'}) &= T_i^{C''C}(C' \wedge e_j^{C'} \wedge e_{j_1}^{C'}) \wedge_{C''} T_i^{C''C}(C' \wedge e_j^{C'} \wedge e_{j_2}^{C'}) \\ &= -\left(\sum_{k_2=1}^{3} T_{ij_2k_2} C'' \wedge e_{k_2}^{C''}\right) \wedge_{C''} \left(\sum_{k_1=1}^{3} T_{ij_1k_1} C'' \wedge e_{k_1}^{C''}\right) \\ &= \sum_{k=1}^{3} (T_{ij_1k_1}T_{ij_2k_2} - T_{ij_1k_2}T_{ij_2k_1})\, C'' \wedge \hat{e}_k^{C''}, \end{aligned}$$

the matrix of $t_i^{C''C}$ is $(t_{ijk})^T_{j,k=1..3}$.

So (17.14) is equivalent to

$$T_i^{C''C}(L_1^{C'}) \wedge_{C''} T_i^{C''C}(L_2^{C'}) = -m^{C'} \vee (C \wedge e_i^C) \;\; C'' \wedge C \wedge e_i^C. \tag{17.22}$$

Geometrically, $T_i^{C''C}$ maps an image line in camera C' to an image point on the epipolar line $C'' \wedge C \wedge e_i^C$ in camera C''. (17.22) says that the image line connecting the two image points $T_i^{C''C}(L_1^{C'})$ and $T_i^{C''C}(L_2^{C'})$ in camera C'' is just the epipolar line $C'' \wedge C \wedge e_i^C$. This is the geometric interpretation of (17.14).

17.5.3 Relations on Trifocal Tensors II

Now we let the two $\hat{e}^{C*}$'s in $Texp_1$ be different, and let the two $e^{C'*}$ be the same, i. e., we consider the expression

$$Texp_2 = (e_{i_1}^{C*} \wedge e_i^{C*} \wedge e_j^{C'*}) \vee (e_{i_2}^{C*} \wedge e_i^{C*} \wedge e_j^{C'*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.23}$$

Assume that $i \prec i_1 \prec i_2$. Expanding $Texp_2$ from left to right, we get

$$\begin{aligned} Texp_2 &= -(e_i^{C*} \wedge e_{i_1}^{C*} \wedge e_{i_2}^{C*} \wedge e_j^{C'*})^{\sim} \left(\sum_{k=1}^{3} ((e_i^{C*} \wedge e_j^{C'*}) \vee \hat{e}_k^{C''*}) e_k^{C''*} \right) \\ &= \sum_{k=1}^{3} E_j^{C'C} T''_{kij} e_k^{C''*}. \end{aligned}$$

Expanding $Texp_2$ from right to left, we get

$$\begin{aligned} Texp_2 = &- \sum_{k=1}^{3} \Big(((\hat{e}_{i_2}^{C*} \wedge e_j^{C'*}) \vee e_{k_2}^{C''*})((\hat{e}_{i_1}^{C*} \wedge e_j^{C'*}) \vee e_{k_1}^{C''*}) \\ &\qquad - ((\hat{e}_{i_2}^{C*} \wedge e_j^{C'*}) \vee e_{k_1}^{C''*})((\hat{e}_{i_1}^{C*} \wedge e_j^{C'*}) \vee e_{k_2}^{C''*}) \Big) \, e_k^{C''*} \\ = &- \sum_{k=1}^{3} (T_{i_1 j k_1} T_{i_2 j k_2} - T_{i_1 j k_2} T_{i_2 j k_1}) e_k^{C''*}, \end{aligned}$$

where $k \prec k_1 \prec k_2$. So

$$-E_j^{C'C} T''_{kij} = t_{ijk}^{C'}, \tag{17.24}$$

where

$$t_{ijk}^{C'} = T_{i_1 j k_1} T_{i_2 j k_2} - T_{i_1 j k_2} T_{i_2 j k_1}. \tag{17.25}$$

Proposition 17.5.2. For any $1 \le i, j, k \le 3$,

$$E_j^{C'C} T''_{kij} \simeq t_{ijk}^{C'}. \tag{17.26}$$

Corollary 17.5.2. For any $1 \le i, i_1, i_2, j, k, k_1, k_2 \le 3$,

$$\frac{T''_{ki_1j}}{T''_{ki_2j}} = \frac{t_{i_1jk}^{C'}}{t_{i_2jk}^{C'}}; \quad \frac{T''_{k_1ij}}{T''_{k_2ij}} = \frac{t_{k_1ij}^{C'}}{t_{k_2ij}^{C'}}. \tag{17.27}$$

Same as before, to understand relation (17.26) geometrically, we first express it in terms of Grassmann-Cayley algebra. When $C' \wedge \hat{e}_j^{C'}$ is fixed, T induces a linear mapping $T_j^{CC'} : C'' \wedge \mathcal{G}_4^2 \longrightarrow C \wedge \mathcal{G}_4^2$ by

$$T_j^{CC'}(L^{C''}) = C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L^{C''}), \tag{17.28}$$

whose matrix is $(-T_{ijk})_{i,k=1..3}$. T'' also induces a linear mapping ${T''}_j^{CC'}$: $C'' \wedge \mathbb{R}^4 \longrightarrow C \wedge \mathbb{R}^4$ by

$${T''}_j^{CC'}(m^{C''}) = C \wedge (m^{C''} \vee (C' \wedge \hat{e}_j^{C'})), \tag{17.29}$$

whose matrix is $(-T''_{kij})^T_{k,i=1..3}$. Define a linear mapping $t_j^{CC'} : C'' \wedge \mathbb{R}^4 \longrightarrow C \wedge \mathbb{R}^4$ as follows: let $m^{C''} \in C'' \wedge \mathbb{R}^4$ and $m^{C''} = L_1^{C''} \vee L_2^{C''}$, where $L_1^{C''}, L_2^{C''} \in C'' \wedge \mathcal{G}_4^2$, then

$$t_j^{CC'}(m^{C''}) = T_j^{CC'}(L_1^{C''}) \vee T_j^{CC'}(L_2^{C''}). \tag{17.30}$$

We need to prove that this mapping is well-defined. Using the formula that for any $C \in \mathbb{R}^4$ and $A_3, B_3 \in \mathcal{G}_4^3$,

$$(C \wedge (A_3 \vee B_3)) \vee B_3 = -(B_3 \vee C)(A_3 \vee B_3), \tag{17.31}$$

we get

$$\begin{aligned} t_j^{CC'}(m^{C''}) &= \Big(C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L_1^{C''})\Big) \vee \Big(C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L_2^{C''})\Big) \\ &= -C \wedge \Big(C \wedge \Big((C' \wedge \hat{e}_j^{C'}) \vee L_1^{C''}\Big) \vee (C' \wedge \hat{e}_j^{C'}) \vee L_2^{C''}\Big) \\ &= (C' \wedge \hat{e}_j^{C'}) \vee C \;\; C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L_1^{C''} \vee L_2^{C''}) \\ &= (C' \wedge \hat{e}_j^{C'}) \vee C \;\; C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee m^{C''}) \\ &= E_j^{C'C} \;\; T''^{CC'}_j(m^{C''}). \end{aligned} \tag{17.32}$$

So $t_j^{CC'}$ is well-defined. Using (17.30), it can be verified that the matrix of $t_j^{CC'}$ is $(t_{ijk}^{C'})_{i,k=1..3}$.

Thus, (17.26) is equivalent to

$$T_j^{CC'}(L_1^{C''}) \vee T_j^{CC'}(L_2^{C''}) = (C' \wedge \hat{e}_j^{C'}) \vee C \;\; C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L_1^{C''} \vee L_2^{C''}). \tag{17.33}$$

Geometrically, $T_j^{CC'}$ maps an image line $L^{C''}$ in camera C'' to the image line in camera C, which is the image of the space line on both planes $C' \wedge \hat{e}_j^{C'}$ and $L^{C''}$. (17.33) says that the intersection of the two image lines $T_j^{CC'}(L_1^{C''})$ and $T_j^{CC'}(L_2^{C''})$ is just the image of the intersection of the plane $C' \wedge \hat{e}_j^{C'}$ with the line $L_1^{C''} \vee L_2^{C''}$ in the space. This is the geometric interpretation of (17.26).

17.5.4 Relations on Trifocal Tensors III

Consider the following expression obtained by changing one of the $e^{C'*}$'s in $Texp_1$ to an $e^{C''*}$:

$$Texp_3 = (\hat{e}_i^{C*} \wedge e_j^{C'*}) \vee (\hat{e}_i^{C*} \wedge e_k^{C''*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.34}$$

Expanding $Texp_3$ from left to right, we get

$$\begin{aligned} Texp_3 &= (\hat{e}_i^{C*} \wedge e_j^{C'*}) \vee e_k^{C''*} \ \left(\sum_{l=1}^{3} (\hat{e}_i^{C*} \vee \hat{e}_l^{C''*}) \ e_l^{C''*} \right) \\ &= - \sum_{l=1}^{3} T_{ijk} F_{il}^{CC''} e_l^{C''*}. \end{aligned}$$

Expanding $Texp_3$ from right to left, we get

$$\begin{aligned} Texp_3 &= (\hat{e}_i^{C*} \wedge e_j^{C'*}) \vee \Big((\hat{e}_i^{C*} \wedge e_k^{C''*} \wedge e_{k_1}^{C''*})^{\sim} e_{k_2}^{C''*} \wedge e_k^{C''*} \\ &\qquad - (\hat{e}_i^{C*} \wedge e_k^{C''*} \wedge e_{k_2}^{C''*})^{\sim} e_{k_1}^{C''*} \wedge e_k^{C''*} \Big) \\ &= (T_{ijk_1} F_{ik_1}^{CC''} + T_{ijk_2} F_{ik_2}^{CC''}) e_k^{C''*} \\ &\quad - T_{ijk} F_{ik_1}^{CC''} e_{k_1}^{C''*} - T_{ijk} F_{ik_2}^{CC''} e_{k_2}^{C''*}, \end{aligned}$$

where $k \prec k_1 \prec k_2$.

Proposition 17.5.3. For any $1 \leq i, j \leq 3$,

$$\sum_{k=1}^{3} T_{ijk} F_{ik}^{CC''} = 0. \tag{17.35}$$

By (17.16), $F_{ik}^{CC''} = F_{i1}^{CC''} t_{ijk}^{C} / t_{ij1}^{C}$. So (17.35) is equivalent to

$$\sum_{k=1}^{3} T_{ijk} t_{ijk}^{C} = \det(T_{ijk})_{j,k=1..3} = 0, \tag{17.36}$$

for any $1 \leq i, j \leq 3$. (17.36) can also be obtained directly by expanding the following expression:

$$(\hat{e}_i^{C*} \wedge e_1^{C'*}) \vee (\hat{e}_i^{C*} \wedge e_2^{C'*}) \vee (\hat{e}_i^{C*} \wedge e_3^{C'*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.37}$$

Expanding from left to right, (17.37) gives zero; expanding from right to left, it gives $\det(T_{ijk})_{j,k=1..3}$.

To understand (17.36) geometrically, we check the dual form of (17.37), which is

$$\begin{aligned} & C'' \wedge ((C \wedge e_i^{C}) \vee (C' \wedge \hat{e}_1^{C'})) \\ & \wedge ((C \wedge e_i^{C}) \vee (C' \wedge \hat{e}_2^{C'})) \\ & \wedge ((C \wedge e_i^{C}) \vee (C' \wedge \hat{e}_3^{C'})). \end{aligned} \tag{17.38}$$

(17.38) equals zero because the intersections of a line with three planes are always collinear.

Interchanging C with C'', we get $\det(T''_{kij})_{i,j=1..3} = 0$ for any $1 \leq k \leq 3$. By (17.26), we have

$$\det(t_{ijk}^{C'})_{i,j=1..3} = 0. \tag{17.39}$$

A similar constraint can be obtained by interchanging C and C'.

(17.37) can be generalized to the following one:

$$\begin{aligned}&\Big((\sum_{i=1}^{3}\lambda_i \hat{e}_i^{C*}) \wedge e_1^{C'*}\Big) \vee \Big((\sum_{i=1}^{3}\lambda_i \hat{e}_i^{C*}) \wedge e_2^{C'*}\Big)\\ &\qquad \vee \Big((\sum_{i=1}^{3}\lambda_i \hat{e}_i^{C*}) \wedge e_3^{C'*}\Big) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}).\end{aligned} \tag{17.40}$$

where the λ's are indeterminants. (17.40) equals zero when expanded from the left, and equals a polynomial of the λ's when expanded from the right. The coefficients of the polynomial are expressions of the T_{ijk}'s. Thus, we get 10 constraints of degree 3 on T, called the rank constraints by Faugeras and Papadopoulo [81, 83].

17.5.5 Relations on Trifocal Tensors IV

Now, we let the two $\hat{e}^{C*}$'s in $Texp_3$ be different. Consider

$$Texp_4 = (e_i^{C*} \wedge e_{i_1}^{C*} \wedge e_j^{C'*}) \vee (e_i^{C*} \wedge e_{i_2}^{C*} \wedge e_k^{C''*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.41}$$

Assume that $i \prec i_1 \prec i_2$. Expanding $Texp_4$ from left to right, we get

$$\begin{aligned}Texp_4 &= (e_i^{C*} \wedge e_{i_1}^{C*} \wedge e_{i_2}^{C*} \wedge e_j^{C'*})^{\sim}(e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*} \wedge e_i^{C*})^{\sim} e_k^{C''*}\\ &\qquad -(e_i^{C*} \wedge e_{i_1}^{C*} \wedge e_j^{C'*} \wedge e_k^{C''*})^{\sim}\\ &\qquad \Big(\sum_{l=1}^{3}((e_i^{C*} \wedge e_{i_2}^{C*}) \vee \hat{e}_l^{C''*})e_l^{C''*}\Big)\\ &= (E_j^{C'C}E_i^{CC''} + T_{i_2jk}F_{i_1k}^{CC''})e_k^{C''*}\\ &\qquad + T_{i_2jk}F_{i_1k_1}^{CC''}e_{k_1}^{C''*} + T_{i_2jk}F_{i_1k_2}^{CC''}e_{k_2}^{C''*}.\end{aligned}$$

Expanding $Texp_4$ from right to left, we get

$$\begin{aligned}Texp_4 &= ((e_i^{C*} \wedge e_{i_1}^{C*}) \vee (e_j^{C'*} \wedge e_k^{C''*}))(\hat{e}_{i_1}^{C*} \vee \hat{e}_{k_1}^{C''*})e_{k_1}^{C''*}\\ &\qquad + ((e_i^{C*} \wedge e_{i_1}^{C*}) \vee (e_j^{C'*} \wedge e_k^{C''*}))(\hat{e}_{i_1}^{C*} \vee \hat{e}_{k_2}^{C''*})e_{k_2}^{C''*}\\ &\qquad - \Big(((e_i^{C*} \wedge e_{i_1}^{C*}) \vee (e_j^{C'*} \wedge e_{k_1}^{C''*}))(\hat{e}_{i_1}^{C*} \vee \hat{e}_{k_1}^{C''*})\\ &\qquad\quad + ((e_i^{C*} \wedge e_{i_1}^{C*}) \vee (e_j^{C'*} \wedge e_{k_2}^{C''*}))(\hat{e}_{i_1}^{C*} \vee \hat{e}_{k_2}^{C''*})\Big)\ e_k^{C''*}\\ &= T_{i_2jk}F_{i_1k_1}^{CC''}e_{k_1}^{C''*} + T_{i_2jk}F_{i_1k_2}^{CC''}e_{k_2}^{C''*}\\ &\qquad - (T_{i_2jk_1}F_{i_1k_1}^{CC''} + T_{i_2jk_2}F_{i_1k_2}^{CC''})e_k^{C''*},\end{aligned}$$

where $k \prec k_1 \prec k_2$. So

$$\sum_{k=1}^{3} T_{i_2jk} F_{i_1k}^{CC''} = -E_j^{C'C} E_i^{CC''}. \tag{17.42}$$

Interchanging i_1, i_2 in $Texp_4$, we obtain

$$\sum_{k=1}^{3} T_{i_1jk} F_{i_2k}^{CC''} = E_j^{C'C} E_i^{CC''}. \tag{17.43}$$

Proposition 17.5.4. For any $1 \le i, j \le 3$,

$$E_j^{C'C} E_i^{CC''} \simeq W_{ij}, \tag{17.44}$$

where $W_{ij} = \sum_{k=1}^{3} T_{i_1jk} F_{i_2k}^{CC''}$.

From (17.36), (17.42) and (17.43) we get

Proposition 17.5.5. For any $1 \le i_1, i_2, j \le 3$,

$$\sum_{k=1}^{3} (T_{i_1jk} F_{i_2k}^{CC''} + T_{i_2jk} F_{i_1k}^{CC''}) = 0. \tag{17.45}$$

In fact, (17.44) can be proved by direct computation:

$$\begin{aligned} W_{ij} &= \sum_{k=1}^{3} (C \wedge e_{i_1}^{C}) \vee (C' \wedge \hat{e}_j^{C'}) \vee (C'' \wedge \hat{e}_k^{C''}) \ \ (C \wedge e_{i_2}^{C}) \vee (C'' \wedge e_k^{C''}) \\ &= -\left(C'' \wedge \left(\sum_{k=1}^{3} (C \wedge e_{i_2}^{C}) \vee (C'' \wedge e_k^{C''}) \ \hat{e}_k^{C''}\right)\right) \vee (C \wedge e_{i_1}^{C}) \vee (C' \wedge \hat{e}_j^{C'}) \\ &= (C'' \wedge C \wedge e_{i_2}^{C}) \vee (C \wedge e_{i_1}^{C}) \vee (C' \wedge \hat{e}_j^{C'}) \\ &= (C'' \wedge C \wedge e_{i_1}^{C} \wedge e_{i_2}^{C})^{\sim} (C \wedge C' \wedge \hat{e}_j^{C'})^{\sim} \\ &= E_i^{CC''} E_j^{C'C}. \end{aligned}$$

So (17.44) is equivalent to

$$\begin{aligned} (C'' \wedge C \wedge e_{i_2}^{C}) \vee (C \wedge e_{i_1}^{C}) \vee (C' \wedge \hat{e}_j^{C'}) = \\ (C'' \wedge C \wedge e_{i_1}^{C} \wedge e_{i_2}^{C})^{\sim} (C \wedge C' \wedge \hat{e}_j^{C'})^{\sim}; \end{aligned} \tag{17.46}$$

(17.45) is equivalent to the anti-symmetry of $C'' \wedge C \wedge e_{i_1}^{C} \wedge e_{i_2}^{C}$ with respect to $e_{i_1}^{C}$ and $e_{i_2}^{C}$.

Define

$$u^{C''C}_{i_1 i_2 j_1 j_2} = \sum_{k=1}^{3} t^{C}_{i_1 j_1 k} T_{i_2 j_2 k} \tag{17.47}$$

for $1 \leq i_1, i_2, j_1, j_2 \leq 3$. By (17.12), (17.36), (17.42) and (17.43),

$$\begin{aligned} u^{C''C}_{i_1 i_2 j_1 j_2} &= \sum_{k=1}^{3} F^{CC'}_{i_1 j_1} F^{CC''}_{i_1 k} T_{i_2 j_2 k} \\ &= \begin{cases} 0, & \text{if } i_1 = i_2; \\ -F^{CC'}_{i_1 j_1} E^{C'C}_{j_2} E^{CC''}_{i}, & \text{if } i \prec i_1 \prec i_2; \\ F^{CC'}_{i_1 j_1} E^{C'C}_{j_2} E^{CC''}_{i}, & \text{if } i \prec i_2 \prec i_1. \end{cases} \end{aligned} \tag{17.48}$$

Two corollaries can be drawn immediately:

Corollary 17.5.3. 1. For any $1 \leq i_l, j_l \leq 3$, where $1 \leq k \leq 4$,

$$\frac{u^{C''C}_{i_1 i_2 j_1 j_2}}{u^{C''C}_{i_1 i_2 j_1 j_3}} = \frac{u^{C''C}_{i_3 i_4 j_4 j_2}}{u^{C''C}_{i_3 i_4 j_4 j_3}} = \frac{E^{C'C}_{j_2}}{E^{C'C}_{j_3}}. \tag{17.49}$$

2. Let $i \prec i_1 \prec i_2$. Then for any $1 \leq j_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{C''C}_{i_1 i_2 j_1 j_2}}{u^{C''C}_{i_1 i j_1 j_2}} = \frac{u^{C''C}_{i_1 i_2 j_3 j_4}}{u^{C''C}_{i_1 i j_3 j_4}} = -\frac{E^{CC''}_{i}}{E^{CC''}_{i_2}}. \tag{17.50}$$

Corollary 17.5.4. 1. For any $1 \leq i_1, i_2, j_1 \leq 3$ where $i_1 \neq i_2$,,

$$\mathbf{E}^{C'C} \simeq (u^{C''C}_{i_1 i_2 j_1 j_2})_{j_2 = 1..3}. \tag{17.51}$$

2. For any $1 \leq j_1, j_2 \leq 3$,

$$\mathbf{E}^{CC''} \simeq (u^{C''C}_{23 j_1 j_2} u^{C''C}_{32 j_1 j_2}, -u^{C''C}_{23 j_1 j_2} u^{C''C}_{31 j_1 j_2}, -u^{C''C}_{21 j_1 j_2} u^{C''C}_{32 j_1 j_2})^T. \tag{17.52}$$

Now we explain (17.48) in terms of Grassmann-Cayley algebra. We have defined two mappings $T^{C''C}_i$ and $t^{C''C}_i$ in (17.18) and (17.19), whose matrices are $(-T_{ijk})_{j,k=1..3}$ and $(t^{C}_{ijk})_{j,k=1..3}$ respectively. By the definition of $u^{C''C}_{i_1 i_2 j_1 j_2}$,

$$u^{C''C}_{i_1 i_2 j_1 j_2} C'' = t^{C''C}_{i_1}(C' \wedge e^{C'}_{j_1}) \vee T^{C''C}_{i_2}(C' \wedge \hat{e}^{C'}_{j_2}). \tag{17.53}$$

Expanding the right-hand side of (17.53), we get

$$u^{C''C}_{i_1 i_2 j_1 j_2} = U^{C''C}(C \wedge e^{C}_{i_1}, C \wedge e^{C}_{i_2}, C' \wedge e^{C'}_{j_1}, C' \wedge \hat{e}^{C'}_{j_2}), \tag{17.54}$$

where $U^{C''C} : (C \wedge \mathbb{R}^4) \times (C \wedge \mathbb{R}^4) \times (C' \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}^2_4) \longrightarrow \mathbb{R}$ is defined by

$$U^{C''C}(m^C_1, m^C_2, m^{C'}, L^{C'}) = -m^C_1 \vee m^{C'} \quad C \vee L^{C'} \quad C'' \vee (m^C_1 \wedge_C m^C_2). \tag{17.55}$$

(17.54) is (17.48) in Grassmann-Cayley algebraic form. It means that the $u^{C''C}$'s are components of the mapping $U^{C''C}$.

Notice that (17.49) is a group of degree 6 constraints on T. It is closely related to Faugeras and Mourrain's first group of degree 6 constraints:

$$\begin{aligned}|\mathbf{T}_{k_1k_2\cdot}\ \mathbf{T}_{k_1l_2\cdot}\ \mathbf{T}_{l_1l_2\cdot}||\mathbf{T}_{k_1k_2\cdot}\ \mathbf{T}_{l_1k_2\cdot}\ \mathbf{T}_{l_1l_2\cdot}|\\ = |\mathbf{T}_{l_1k_2\cdot}\ \mathbf{T}_{k_1l_2\cdot}\ \mathbf{T}_{l_1l_2\cdot}||\mathbf{T}_{k_1k_2\cdot}\ \mathbf{T}_{l_1k_2\cdot}\ \mathbf{T}_{k_1l_2\cdot}|,\end{aligned} \tag{17.56}$$

where $\mathbf{T}_{k_1k_2\cdot} = (T_{k_1k_2k})_{k=1..3}$.

It is difficult to find the symmetry of the indices in (17.56), so we first express (17.56) in terms of Grassmann-Cayley algebra. Using the fact that $-\mathbf{T}_{k_1k_2\cdot}$ is the coordinates of $C'' \wedge ((C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{k_2}))$, we get

$$\begin{aligned}&|\mathbf{T}_{k_1k_2\cdot}\ \mathbf{T}_{k_1l_2\cdot}\ \mathbf{T}_{l_1l_2\cdot}|^{\sim}\\ &= \Big(C'' \wedge \Big((C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{k_2})\Big)\Big)\\ &\qquad \wedge_{C''} \Big(C'' \wedge \Big((C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{l_2})\Big)\Big)\\ &\qquad \wedge_{C''} \Big(C'' \wedge \Big((C \wedge e^C_{l_1}) \vee (C' \wedge \hat{e}^{C'}_{l_s2})\Big)\Big)\\ &= C'' \wedge \Big((C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{k_2})\Big) \wedge \Big((C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{l_2})\Big)\\ &\qquad \wedge \Big((C \wedge e^C_{l_1}) \vee (C' \wedge \hat{e}^{C'}_{l_2})\Big).\end{aligned}$$

By formula (17.20),

$$\begin{aligned}|\mathbf{T}_{k_1k_2\cdot}\ \mathbf{T}_{k_1l_2\cdot}\ \mathbf{T}_{l_1l_2\cdot}| &= -(C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{l_2}) \vee (C' \wedge \hat{e}^{C'}_{k_2})\\ &\qquad (C'' \wedge C \wedge e^C_{k_1}) \vee (C \wedge e^C_{l_1}) \vee (C' \wedge \hat{e}^{C'}_{l_2})\\ &= -\,(C \wedge e^C_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{k_2}) \vee (C' \wedge \hat{e}^{C'}_{l_2})\\ &\qquad C'' \vee (C \wedge e^C_{k_1} \wedge e^C_{l_1}) \quad C \vee (C' \wedge \hat{e}^{C'}_{l_2}).\end{aligned} \tag{17.57}$$

Define a mapping $V^{C''} : (C \wedge \mathbb{R}^4) \times (C \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}^2_4) \times (C' \wedge \mathcal{G}^2_4) \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned}V^{C''}(m^C_1, m^C_2, L^{C'}_1, L^{C'}_2) = -(C'' \vee (m^C_1 \wedge_C m^C_2))\\ (C \vee L^{C'}_2)(m^C_1 \vee L^{C'}_1 \vee L^{C'}_2).\end{aligned} \tag{17.58}$$

Let

$$v^{C''}_{k_1l_1k_2l_2} = V^{C''}(C' \wedge \hat{e}^{C'}_{k_1},\ C' \wedge \hat{e}^{C'}_{l_1},\ C'' \wedge \hat{e}^{C''}_{k_2},\ C'' \wedge \hat{e}^{C''}_{l_2}). \tag{17.59}$$

By (17.57),

$$|\mathbf{T}_{k_1k_2.}\ \mathbf{T}_{k_1l_2.}\ \mathbf{T}_{l_1l_2.}| = v^{C''}_{k_1l_1k_2l_2}. \tag{17.60}$$

Similarly, we can get

$$\begin{aligned}
|\mathbf{T}_{k_1k_2.}\ \mathbf{T}_{l_1k_2.}\ \mathbf{T}_{l_1l_2.}| &= v^{C''}_{l_1k_1l_2k_2},\\
|\mathbf{T}_{l_1k_2.}\ \mathbf{T}_{k_1l_2.}\ \mathbf{T}_{l_1l_2.}| &= v^{C''}_{l_1k_1k_2l_2},\\
|\mathbf{T}_{k_1k_2.}\ \mathbf{T}_{l_1k_2.}\ \mathbf{T}_{k_1l_2.}| &= v^{C''}_{k_1l_1l_2k_2}.
\end{aligned} \tag{17.61}$$

So (17.56) is equivalent to

$$\frac{v^{C''}_{k_1l_1k_2l_2}}{v^{C''}_{k_1l_1l_2k_2}} = \frac{v^{C''}_{l_1k_1k_2l_2}}{v^{C''}_{l_1k_1l_2k_2}}, \tag{17.62}$$

which is simpler than (17.56) in appearance. By (17.58), (17.59), in Grassmann-Cayley algebra, (17.62) is just the following identity:

$$\begin{aligned}
&\frac{C''\vee(m^C_1\wedge_C m^C_2)\ \ C\vee L^{C'}_2\ \ m^C_1\vee L^{C'}_1\vee L^{C'}_2}{C''\vee(m^C_1\wedge_C m^C_2)\ \ C\vee L^{C'}_1\ \ m^C_1\vee L^{C'}_2\vee L^{C'}_1}\\
&\quad= \frac{C''\vee(m^C_2\wedge_C m^C_1)\ \ C\vee L^{C'}_2\ \ m^C_2\vee L^{C'}_1\vee L^{C'}_2}{C''\vee(m^C_2\wedge_C m^C_1)\ \ C\vee L^{C'}_1\ \ m^C_2\vee L^{C'}_2\vee L^{C'}_1},
\end{aligned} \tag{17.63}$$

for any $m^C_1, m^C_2 \in C\wedge\mathbb{R}^4$, $L^{C'}_1, L^{C'}_2 \in C'\wedge\mathcal{G}^2_4$.

By (17.58), we have

$$v^{C''}_{i_1i_2j_1j_2} = \begin{cases} 0, & \text{if } i_1=i_2 \text{ or } j_1=j_2;\\ -F^{CC'}_{i_1j}E^{C'C}_{j_2}E^{CC''}_i, & \text{if } i\prec i_1\prec i_2 \text{ and } j\prec j_1\prec j_2,\\ & \text{or } i\prec i_2\prec i_1 \text{ and } j\prec j_2\prec j_1;\\ F^{CC'}_{i_1j}E^{C'C}_{j_2}E^{CC''}_i, & \text{if } i\prec i_1\prec i_2 \text{ and } j\prec j_2\prec j_1,\\ & \text{or } i\prec i_2\prec i_1 \text{ and } j\prec j_1\prec j_2. \end{cases} \tag{17.64}$$

Corollary 17.5.5. 1. For any $1\le i_l, j_1, j_2\le 3$ where $1\le l\le 4$,

$$\frac{v^{C''}_{i_1i_2j_1j_2}}{v^{C''}_{i_1i_2j_2j_1}} = \frac{v^{C''}_{i_3i_4j_1j_2}}{v^{C''}_{i_3i_4j_2j_1}}. \tag{17.65}$$

2. Let $i\prec i_1\prec i_2$. Then for any $1\le j_l\le 3$ where $1\le l\le 4$,

$$\frac{v^{C''}_{i_1i_2j_1j_2}}{v^{C''}_{i_1ij_1j_2}} = \frac{v^{C''}_{i_1i_2j_3j_4}}{v^{C''}_{i_1ij_3j_4}}. \tag{17.66}$$

(17.56) is a special case of (17.65) where $i_3 = i_2, i_4 = i_1$. Comparing $U^{C''C}$ with $V^{C''}$, we get

$$V^{C''}(m_1^C, m_2^C, L_1^{C'}, L_2^{C'}) = U^{C''C}(m_1^C, m_2^C, L_1^{C'} \vee L_2^{C'}, L_2^{C'}). \tag{17.67}$$

It appears that we have generalized Faugeras and Mourrain's first group of degree-six constraints furthermore by $U^{C''C}$, because (17.50) is equivalent to (17.66), while (17.65) is a special case of (17.49) where $j_4 = j_1$. An explanation for this phenomenon is that the variables in $V^{C''}$ are less separated than those in $U^{C''C}$, so there are less constraints on T that come from $V^{C''}$ than from $U^{C''C}$.

Interchanging C' and C'' in (17.56), we get Faugeras and Mourrain's second group of degree 6 constraints:

$$\begin{aligned} &|\mathbf{T}_{k_1.k_2}\ \mathbf{T}_{k_1.l_2}\ \mathbf{T}_{l_1.l_2}||\mathbf{T}_{k_1.k_2}\ \mathbf{T}_{l_1.k_2}\ \mathbf{T}_{l_1.l_2}| \\ &\quad = |\mathbf{T}_{l_1.k_2}\ \mathbf{T}_{k_1.l_2}\ \mathbf{T}_{l_1.l_2}||\mathbf{T}_{k_1.k_2}\ \mathbf{T}_{l_1.k_2}\ \mathbf{T}_{k_1.l_2}|, \end{aligned} \tag{17.68}$$

where $\mathbf{T}_{k_1.k_2} = (T_{k_1 k k_2})_{k=1..3}$.

This group of constraints can be generalized similarly.

17.5.6 Relations on Trifocal Tensors V

Consider the following expression:

$$Texp_5 = (e_i^{C*} \wedge e_{i_1}^{C*} \wedge e_j^{C'*}) \vee (e_{i_2}^{C*} \wedge e_j^{C'*} \wedge e_k^{C''*}) \vee (e_1^{C''*} \wedge e_2^{C''*} \wedge e_3^{C''*}). \tag{17.69}$$

Assume that $i \prec i_1 \prec i_2$. Expanding $Texp_5$ from left to right, we get

$$Texp_5 = -E_j^{C'C} E_j^{C'C''} e_k^{C''*} - T_{i_2jk} \sum_{l=1}^{3} T''_{li_2j} e_l^{C''*}.$$

Expanding $Texp_4$ from right to left, we get

$$\begin{aligned} Texp_5 = &-T''_{k_2i_2j} T_{i_2jk} e_{k_2}^{C''*} - T''_{k_1i_2j} T_{i_2jk} e_{k_1}^{C''*} \\ &+(T''_{k_2i_2j} T_{i_2jk_2} + T''_{k_1i_2j} T_{i_2jk_1}) e_k^{C''*}, \end{aligned}$$

where $k \prec k_1 \prec k_2$. So

$$\sum_{k=1}^{3} T''_{ki_2j} T_{i_2jk} = -E_j^{C'C} E_j^{C'C''}. \tag{17.70}$$

Proposition 17.5.6. For any $1 \le i, j \le 3$,

$$\sum_{k=1}^{3} T''_{kij} T_{ijk} \simeq E_j^{C'C} E_j^{C'C''}. \tag{17.71}$$

Using the relation (17.26), we get

$$\sum_{k=1}^{3} t_{ijk}^{C'} T_{ijk} = \det(T_{ijk})_{i,k=1..3} = (E_j^{C'C})^2 E_j^{C'C''}. \tag{17.72}$$

Corollary 17.5.6. For any $1 \leq i_1, i_2, j_1 \leq 3$ where $i_1 \neq i_2$,

$$\mathbf{E}^{C'C''} \simeq \left(\frac{\det(T_{ijk})_{i,k=1..3}}{(u_{i_1 i_2 j_1 j}^{C''C})^2} \right)_{j=1..3}. \tag{17.73}$$

17.5.7 Relations on Trifocal Tensors VI

The second idea of deriving relations on trifocal tensors is as follows: if the tensor $(T_{ijk})_{i,j,k=1..3}$ is given, then expanding

$$(e_j^{C'*} \wedge e_k^{C''*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*}) \tag{17.74}$$

gives a vector of the e^{C*}'s whose coefficients are known. Similarly, expanding

$$\begin{aligned} &(e_j^{C'*} \wedge e_j^{C'*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*} \wedge e_j^{C'*}), \\ &(e_j^{C'*} \wedge e_k^{C''*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*} \wedge e_k^{C''*}) \end{aligned} \tag{17.75}$$

gives two 2-vectors of the $e_j^{C'*} \wedge e_i^{C*}$'s and the $e_k^{C''*} \wedge e_i^{C*}$'s respectively, whose coefficients are known. The meet of two such 2-vectors, i. e.,

$$\begin{aligned} Texp_6 = &\Big((e_{j_1}^{C'*} \wedge e_{k_1}^{C''*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*} \wedge e_{j_1}^{C'*}) \Big) \\ &\vee \Big((e_{j_2}^{C'*} \wedge e_{k_2}^{C''*}) \vee (e_1^{C*} \wedge e_2^{C*} \wedge e_3^{C*} \wedge e_{k_2}^{C''*}) \Big) \end{aligned} \tag{17.76}$$

is an expression of the T_{ijk}'s. Expanding the meets differently, we get a relation on T, epipoles and epipolar tensors.

Assume that $j \prec j_1 \prec j_2$ and $k \prec k_1 \prec k_2$. Expanding $Texp_6$ according to its parentheses, we get

$$\begin{aligned} Texp_6 &= \left(\sum_{i_1=1}^{3} (e_{j_1}^{C'*} \wedge e_{k_1}^{C''*}) \vee \hat{e}_{i_1}^{C*} \quad e_{i_1}^{C*} \wedge e_{j_1}^{C'*} \right) \\ &\quad \vee \left(\sum_{i_2=1}^{3} (e_{j_2}^{C'*} \wedge e_{k_2}^{C''*}) \vee \hat{e}_{i_2}^{C*} \quad e_{i_2}^{C*} \wedge e_{k_2}^{C''*} \right) \\ &= \sum_{i=1}^{3} (-T_{i_1 j_1 k_1} T_{i_2 j_2 k_2} + T_{i_2 j_1 k_1} T_{i_1 j_2 k_2}) T_{i j_1 k_2}, \end{aligned}$$

where $i \prec i_1 \prec i_2$. Using the fact that the meet of a 4-vector with any multivector in $\mathcal{G}_4$ is a scalar multiplication of the multivector by the dual of the 4-vector, we get

$$\begin{aligned} Texp_6 = (e^{C'*}_{j_1} \wedge e^{C''*}_{k_1}) \vee (e^{C'*}_{j_2} \wedge e^{C''*}_{k_2}) \vee (e^{C*}_1 \wedge e^{C*}_2 \wedge e^{C*}_3 \wedge e^{C'*}_{j_1}) \\ \vee (e^{C*}_1 \wedge e^{C*}_2 \wedge e^{C*}_3 \wedge e^{C''*}_{k_2}) \\ = -F^{C'C''}_{jk} E^{C'C}_{j_1} E^{C''C}_{k_2}. \end{aligned}$$

So

$$F^{C'C''}_{jk} E^{C'C}_{j_1} E^{C''C}_{k_2} = \sum_{i=1}^{3} (T_{i_1 j_1 k_1} T_{i_2 j_2 k_2} - T_{i_2 j_1 k_1} T_{i_1 j_2 k_2}) T_{i j_1 k_2}. \qquad (17.77)$$

Interchanging j_1, j_2 in $Texp_5$, we get

$$-F^{C'C''}_{jk} E^{C'C}_{j_2} E^{C''C}_{k_2} = \sum_{i=1}^{3} (T_{i_1 j_2 k_1} T_{i_2 j_1 k_2} - T_{i_2 j_2 k_1} T_{i_1 j_1 k_2}) T_{i j_2 k_2}. \qquad (17.78)$$

Interchanging k_1, k_2 in $Texp_5$, we get

$$-F^{C'C''}_{jk} E^{C'C}_{j_1} E^{C''C}_{k_1} = \sum_{i=1}^{3} (T_{i_1 j_1 k_2} T_{i_2 j_2 k_1} - T_{i_2 j_1 k_2} T_{i_1 j_2 k_1}) T_{i j_1 k_1}. \qquad (17.79)$$

Interchanging (j_1, k_1) and (j_2, k_2) in $Texp_5$, we get

$$F^{C'C''}_{jk} E^{C'C}_{j_2} E^{C''C}_{k_1} = \sum_{i=1}^{3} (T_{i_1 j_2 k_2} T_{i_2 j_1 k_1} - T_{i_2 j_2 k_2} T_{i_1 j_1 k_1}) T_{i j_2 k_1}. \qquad (17.80)$$

When $j_1 = j_2$ or $k_1 = k_2$, $Texp_5 = 0$ by expanding from left to right.
Define

$$v^C_{j_1 j_2 k_1 k_2} = -\sum_{i=1}^{3} (T_{i_1 j_1 k_1} T_{i_2 j_2 k_2} - T_{i_2 j_1 k_1} T_{i_1 j_2 k_2}) T_{i j_1 k_2}. \qquad (17.81)$$

Then

$$v^C_{j_1 j_2 k_1 k_2} = \begin{cases} 0, & \text{if } j_1 = j_2 \text{ or } k_1 = k_2; \\ -F^{C'C''}_{jk} E^{C'C}_{j_1} E^{C''C}_{k_2}, & \text{if } j \prec j_1 \prec j_2 \text{ and } k \prec k_1 \prec k_2, \\ & \text{or } j \prec j_2 \prec j_1 \text{ and } k \prec k_2 \prec k_1; \\ F^{C'C''}_{jk} E^{C'C}_{j_1} E^{C''C}_{k_2}, & \text{if } j \prec j_1 \prec j_2 \text{ and } k \prec k_2 \prec k_1, \\ & \text{or } j \prec j_2 \prec j_1 \text{ and } k \prec k_1 \prec k_2. \end{cases} \qquad (17.82)$$

Proposition 17.5.7. For any $1 \leq j_1, j_2, k_1, k_2 \leq 3$,

$$\frac{v^C_{j_1 j_2 k_1 k_2}}{v^C_{j_2 j_1 k_1 k_2}} = -\frac{E^{C'C}_{j_1}}{E^{C'C}_{j_2}}; \quad \frac{v^C_{j_1 j_2 k_1 k_2}}{v^C_{j_1 j_2 k_2 k_1}} = -\frac{E^{C''C}_{k_2}}{E^{C''C}_{k_1}}. \qquad (17.83)$$

Corollary 17.5.7. 1. For any $1 \leq j_l, k_l \leq 3$, where $1 \leq l \leq 4$,

$$\frac{v^C_{j_1j_2k_1k_2}}{v^C_{j_2j_1k_1k_2}} = \frac{v^C_{j_1j_2k_3k_4}}{v^C_{j_2j_1k_3k_4}}; \quad \frac{v^C_{j_1j_2k_1k_2}}{v^C_{j_1j_2k_2k_1}} = \frac{v^C_{j_3j_4k_1k_2}}{v^C_{j_3j_4k_2k_1}}. \tag{17.84}$$

2. For any $1 \leq i_l, j, j_l, k_l \leq 3$ where $1 \leq l \leq 2$,

$$\frac{v^C_{j_1j_2k_1k_2}}{v^C_{j_2j_1k_1k_2}} = -\frac{u^{C''C}_{i_1i_2jj_1}}{u^{C''C}_{i_1i_2jj_2}}. \tag{17.85}$$

Notice that (17.84) and (17.85) are groups of degree 6 constraints on T. (17.84) is closely related to Faugeras and Mourrain's third group of degree 6 constraints:

$$\begin{aligned} &|\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}||\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.l_1l_2}| \\ &\quad = |\mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}||\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.k_1l_2}|, \end{aligned} \tag{17.86}$$

where $\mathbf{T}_{.k_1k_2} = (T_{kk_1k_2})_{k=1..3}$.

Let us express (17.86) in terms of Grassmann-Cayley algebra. Using the fact that $-\mathbf{T}_{.k_1k_2}$ is the coordinates of $C \wedge ((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{k_2}))$, we get

$$\begin{aligned} &|\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}|\, C \\ &\quad = \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{k_2})\Big)\Big) \\ &\qquad\qquad \vee_C \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{l_2})\Big)\Big) \\ &\qquad\qquad \vee_C \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{l_1}) \vee (C'' \wedge \hat{e}^{C''}_{l_2})\Big)\Big) \\ &\quad = C\ \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{k_2})\Big)\Big) \\ &\qquad\qquad \vee \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{l_2})\Big)\Big) \\ &\qquad\qquad \vee (C' \wedge \hat{e}^{C'}_{l_1}) \vee (C'' \wedge \hat{e}^{C''}_{l_2}). \end{aligned}$$

By (17.31), we have

$$\begin{aligned} &|\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}| \\ &\quad = \Big(C \wedge \Big((C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{k_2})\Big)\Big) \\ &\qquad\quad \vee (C' \wedge \hat{e}^{C'}_{k_1}) \vee (C'' \wedge \hat{e}^{C''}_{l_2}) \vee (C' \wedge \hat{e}^{C'}_{l_1})\ \ (C'' \wedge \hat{e}^{C''}_{l_2}) \vee C \\ &\quad = -(C' \wedge \hat{e}^{C'}_{k_1}) \vee C\ \ \ (C'' \wedge \hat{e}^{C''}_{l_2}) \vee C \\ &\qquad\quad (C' \wedge \hat{e}^{C'}_{k_1}) \vee (C' \wedge \hat{e}^{C'}_{l_1}) \vee (C'' \wedge \hat{e}^{C''}_{k_2}) \vee (C'' \wedge \hat{e}^{C''}_{l_2}). \end{aligned} \tag{17.87}$$

Define a mapping $V^C : (C' \wedge \mathcal{G}^2_4) \times (C' \wedge \mathcal{G}^2_4) \times (C'' \wedge \mathcal{G}^2_4) \times (C'' \wedge \mathcal{G}^2_4) \longrightarrow \mathbb{R}$ as follows:

$$V^C(L_1^{C'}, L_2^{C'}, L_1^{C''}, L_2^{C''}) = -(L_1^{C'} \vee C)(L_2^{C''} \vee C)(L_1^{C'} \vee L_2^{C'} \vee L_1^{C''} \vee L_2^{C''}). \tag{17.88}$$

Then

$$V^C(C' \wedge \hat{e}_{k_1}^{C'},\, C' \wedge \hat{e}_{l_1}^{C'},\, C'' \wedge \hat{e}_{k_2}^{C''},\, C'' \wedge \hat{e}_{l_2}^{C''}) = v_{k_1 l_1 k_2 l_2}^C, \tag{17.89}$$

According to (17.87),

$$|\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}| = v_{k_1l_1k_2l_2}^C. \tag{17.90}$$

Similarly, we have

$$\begin{aligned} |\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.l_1l_2}| &= v_{l_1k_1l_2k_2}^C, \\ |\mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.k_1l_2}\ \mathbf{T}_{.l_1l_2}| &= v_{l_1k_1k_2l_2}^C, \\ |\mathbf{T}_{.k_1k_2}\ \mathbf{T}_{.l_1k_2}\ \mathbf{T}_{.k_1l_2}| &= v_{k_1l_1l_2k_2}^C. \end{aligned} \tag{17.91}$$

Now (17.86) is equivalent to

$$\frac{v_{k_1l_1k_2l_2}^C}{v_{k_1l_1l_2k_2}^C} = \frac{v_{l_1k_1k_2l_2}^C}{v_{l_1k_1l_2k_2}^C}, \tag{17.92}$$

or more explicitly, the following identity:

$$\begin{aligned} &\frac{L_1^{C'} \vee C \quad L_2^{C''} \vee C \quad L_1^{C'} \vee L_2^{C'} \vee L_1^{C''} \vee L_2^{C''}}{L_1^{C'} \vee C \quad L_1^{C''} \vee C \quad L_1^{C'} \vee L_2^{C'} \vee L_2^{C''} \vee L_1^{C''}} \\ &= \frac{L_2^{C'} \vee C \quad L_2^{C''} \vee C \quad L_2^{C'} \vee L_1^{C'} \vee L_1^{C''} \vee L_2^{C''}}{L_2^{C'} \vee C \quad L_1^{C''} \vee C \quad L_2^{C'} \vee L_1^{C'} \vee L_2^{C''} \vee L_1^{C''}}. \end{aligned} \tag{17.93}$$

(17.84) is a straightforward generalization of it.

17.5.8 A Unified Treatment of Degree-six Constraints

In this section we make a comprehensive investigation of Faugeras and Mourrain's three groups of degree-six constraints. We have defined $u_{i_1i_2j_1j_2}^{C''C}$ in (17.47) to derive and generalize the first group of constraints. We are going to follow the same line to derive and generalize the other two groups of constraints.

The trifocal tensor T induces 6 kinds of linear mappings as shown in table 17.1. We have defined two linear mappings $t_i^{C''C}$ and $t_j^{CC'}$ in (17.19) and (17.30) respectively, which are generated by the T's. There are 6 such linear mappings as shown in table 17.2. Let

$$m^{C'} = L_1^{C'} \vee L_2^{C'}, \quad m^{C''} = L_1^{C''} \vee L_2^{C''}, \quad L^C = m_1^C \wedge_C m_2^C.$$

Here

$$t_{ijk}^{C''} = T_{i_1j_1k}T_{i_2j_2k} - T_{i_1j_2k}T_{i_2j_1k}, \tag{17.94}$$

where $i \prec i_1 \prec i_2$ and $j \prec j_1 \prec j_2$. t_{ijk}^C and $t_{ijk}^{C'}$ have been defined in (17.13) and (17.25) respectively.

Table 17.1. Linear mappings induced by T

Mapping	Definition	Matrix
$T_i^{C'C}$	$C'' \wedge \mathcal{G}_4^2 \longrightarrow C' \wedge \mathbb{R}^4$ $L^{C''} \mapsto C' \wedge ((C \wedge e_i^C) \vee L^{C''})$	$(-T_{ijk})_{j,k=1..3}$
$T_i^{C''C}$	$C' \wedge \mathcal{G}_4^2 \longrightarrow C'' \wedge \mathbb{R}^4$ $L^{C'} \mapsto C'' \wedge ((C \wedge e_i^C) \vee L^{C'})$	$(-T_{ijk})_{j,k=1..3}^T$
$T_j^{CC'}$	$C'' \wedge \mathcal{G}_4^2 \longrightarrow C \wedge \mathcal{G}_4^2$ $L^{C''} \mapsto C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee L^{C''})$	$(-T_{ijk})_{i,k=1..3}$
$T_j^{C''C'}$	$C \wedge \mathbb{R}^4 \longrightarrow C'' \wedge \mathbb{R}^4$ $m^C \mapsto C'' \wedge (m^C \vee (C' \wedge \hat{e}_j^{C'}))$	$(-T_{ijk})_{i,k=1..3}^T$
$T_k^{CC''}$	$C' \wedge \mathcal{G}_4^2 \longrightarrow C \wedge \mathcal{G}_4^2$ $L^{C'} \mapsto C \wedge (L^{C'} \vee (C'' \wedge \hat{e}_k^{C''}))$	$(-T_{ijk})_{i,j=1..3}$
$T_k^{C'C''}$	$C \wedge \mathbb{R}^4 \longrightarrow C' \wedge \mathbb{R}^4$ $m^C \mapsto C' \wedge (m^C \vee (C'' \wedge \hat{e}_k^{C''}))$	$(-T_{ijk})_{i,j=1..3}^T$

The mappings t's are well-defined because

$$\begin{aligned}
t_i^{C'C}(m^{C''}) &= -(C \wedge e_i^C) \vee m^{C''} \quad C' \wedge C \wedge e_i^C, \\
t_i^{C''C}(m^{C'}) &= -(C \wedge e_i^C) \vee m^{C'} \quad C'' \wedge C \wedge e_i^C, \\
t_j^{CC'}(m^{C''}) &= (C' \wedge \hat{e}_j^{C'}) \vee C \quad C \wedge ((C' \wedge \hat{e}_j^{C'}) \vee m^{C''}), \\
t_j^{C''C'}(L^C) &= -(C' \wedge \hat{e}_j^{C'}) \vee C \quad C'' \wedge (L^C \vee (C' \wedge \hat{e}_j^{C'})), \\
t_k^{CC''}(m^{C'}) &= (C'' \wedge \hat{e}_k^{C''}) \vee C \quad C \wedge (m^{C'} \vee (C'' \wedge \hat{e}_k^{C''})), \\
t_k^{C'C''}(L^C) &= -(C'' \wedge \hat{e}_k^{C''}) \vee C \quad C' \wedge (L^C \vee (C'' \wedge \hat{e}_k^{C''})).
\end{aligned} \tag{17.95}$$

For any $1 \leq i_1, i_2, j_1, j_2, k_1, k_2 \leq 3$, let

Table 17.2. Linear mappings induced by t

Mapping	Definition	Matrix
$t_i^{C'C}$	$C'' \wedge \mathbb{R}^4 \longrightarrow C' \wedge \mathcal{G}_4^2$ $m^{C''} \mapsto T_i^{C'C}(L_1^{C''}) \wedge_{C'} T_i^{C'C}(L_2^{C''})$	$(t_{ijk}^C)_{j,k=1..3}$
$t_i^{C''C}$	$C' \wedge \mathbb{R}^4 \longrightarrow C'' \wedge \mathcal{G}_4^2$ $m^{C'} \mapsto T_i^{C''C}(L_1^{C'}) \wedge_{C''} T_i^{C''C}(L_2^{C'})$	$(t_{ijk}^C)_{j,k=1..3}^T$
$t_j^{CC'}$	$C'' \wedge \mathbb{R}^4 \longrightarrow C \wedge \mathbb{R}^4$ $m^{C''} \mapsto T_j^{CC'}(L_1^{C''}) \vee_C T_j^{CC'}(L_2^{C''})$	$(t_{ijk}^{C'})_{i,k=1..3}$
$t_j^{C''C'}$	$C \wedge \mathcal{G}_4^2 \longrightarrow C'' \wedge \mathcal{G}_4^2$ $L^C \mapsto T_j^{C''C'}(m_1^C) \wedge_{C''} T_j^{C''C'}(m_2^C)$	$(t_{ijk}^{C'})_{i,k=1..3}^T$
$t_k^{CC''}$	$C' \wedge \mathbb{R}^4 \longrightarrow C \wedge \mathbb{R}^4$ $m^{C'} \mapsto T_k^{CC''}(L_1^{C'}) \vee_C T_k^{CC''}(L_2^{C'})$	$(t_{ijk}^{C''})_{i,j=1..3}$
$t_k^{C'C''}$	$C \wedge \mathcal{G}_4^2 \longrightarrow C' \wedge \mathcal{G}_4^2$ $L^C \mapsto T_k^{C'C''}(m_1^C) \wedge_{C'} T_k^{C'C''}(m_2^C)$	$(t_{ijk}^{C''})_{i,j=1..3}^T$

$$
\begin{aligned}
u_{i_1i_2j_1j_2}^{C''C} &= \sum_{k=1}^{3} t_{i_1j_1k}^{C} T_{i_2j_2k},\\
u_{i_1i_2j_1j_2}^{C''C'} &= \sum_{k=1}^{3} t_{i_1j_1k}^{C'} T_{i_2j_2k},\\
u_{i_1i_2k_1k_2}^{C'C} &= \sum_{j=1}^{3} t_{i_1jk_1}^{C} T_{i_2jk_2},\\
u_{i_1i_2k_1k_2}^{C'C''} &= \sum_{j=1}^{3} t_{i_1jk_1}^{C''} T_{i_2jk_2},\\
u_{j_1j_2k_1k_2}^{CC'} &= \sum_{i=1}^{3} t_{ij_1k_1}^{C'} T_{ij_2k_2},\\
u_{j_1j_2k_1k_2}^{CC''} &= \sum_{i=1}^{3} t_{ij_1k_1}^{C''} T_{ij_2k_2}.
\end{aligned}
\tag{17.96}
$$

Then

$$
\begin{aligned}
u^{C''C}_{i_1i_2j_1j_2}C'' &= t^{C''C}_{i_1}(C' \wedge e^{C'}_{j_1}) \vee T^{C''C}_{i_2}(C' \wedge \hat{e}^{C'}_{j_2}),\\
u^{C''C'}_{i_1i_2j_1j_2}C'' &= t^{C''C'}_{j_1}(C \wedge \hat{e}^{C}_{i_1}) \vee T^{C''C'}_{j_2}(C \wedge e^{C}_{i_2}),\\
u^{C'C}_{i_1i_2k_1k_2}C' &= t^{C'C}_{i_1}(C'' \wedge e^{C''}_{k_1}) \vee T^{C'C}_{i_2}(C'' \wedge \hat{e}^{C''}_{k_2}),\\
u^{C'C''}_{i_1i_2k_1k_2}C' &= t^{C'C''}_{k_1}(C \wedge \hat{e}^{C}_{i_1}) \vee T^{C'C''}_{k_2}(C \wedge e^{C}_{i_2}),\\
u^{CC'}_{j_1j_2k_1k_2}C &= t^{CC'}_{j_1}(C'' \wedge e^{C''}_{k_1}) \vee T^{CC'}_{j_2}(C'' \wedge \hat{e}^{C''}_{k_2}),\\
u^{CC''}_{j_1j_2k_1k_2}C &= t^{CC''}_{k_1}(C' \wedge e^{C'}_{j_1}) \vee T^{CC''}_{k_2}(C' \wedge \hat{e}^{C'}_{j_2}).
\end{aligned} \tag{17.97}
$$

Expanding the right-hand side of the above equalities, we can get a factored form of the u's, from which we get the following constraints.

Constraints from $u^{C''C}_{i_1i_2j_1j_2}$: (see also subsection 17.5.5)

$$
u^{C''C}_{i_1i_2j_1j_2} = \begin{cases} 0, \text{ if } i_1 = i_2;\\ -F^{CC'}_{i_1j_1}E^{C'C}_{j_2}E^{CC''}_{i}, \text{ if } i \prec i_1 \prec i_2;\\ F^{CC'}_{i_1j_1}E^{C'C}_{j_2}E^{CC''}_{i}, \text{ if } i \prec i_2 \prec i_1. \end{cases} \tag{17.98}
$$

Two constraints can be obtained from $u^{C''C}_{i_1i_2j_1j_2}$:

1. For any $1 \le i_l, j_l \le 3$, where $1 \le l \le 4$,

$$
\frac{u^{C''C}_{i_1i_2j_1j_2}}{u^{C''C}_{i_1i_2j_1j_3}} = \frac{u^{C''C}_{i_3i_4j_4j_2}}{u^{C''C}_{i_3i_4j_4j_3}}. \tag{17.99}
$$

2. Let $i \prec i_1 \prec i_2$. Then for any $1 \le j_l \le 3$ where $1 \le l \le 4$,

$$
\frac{u^{C''C}_{i_1i_2j_1j_2}}{u^{C''C}_{i_1ij_1j_2}} = \frac{u^{C''C}_{i_1i_2j_3j_4}}{u^{C''C}_{i_1ij_3j_4}}. \tag{17.100}
$$

Define $U^{C''C} : (C \wedge \mathbb{R}^4) \times (C \wedge \mathbb{R}^4) \times (C' \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}^2_4) \longrightarrow \mathbb{R}$ by

$$
\begin{aligned}
&U^{C''C}(m^C_1, m^C_2, m^{C'}, L^{C'}) =\\
&\qquad -(m^C_1 \vee m^{C'})(C \vee L^{C'})(C'' \vee (m^C_1 \wedge_C m^C_2)).
\end{aligned} \tag{17.101}
$$

Then

$$
U^{C''C}(C \wedge e^C_{i_1}, C \wedge e^C_{i_2}, C' \wedge e^{C'}_{j_1}, C' \wedge \hat{e}^{C'}_{j_2}) = u^{C''C}_{i_1i_2j_1j_2}. \tag{17.102}
$$

Constraints from $u^{C''C'}_{i_1i_2j_1j_2}$: If $i_1 \ne i_2$, then

$$u^{C''C'}_{i_1 i_2 j_1 j_2} = \begin{cases} 0, \text{ if } j_1 = j_2; \\ E^{C'C}_{j_1} E^{CC''}_{i_1} F^{CC'}_{i_2 j}, \text{ if } j \prec j_1 \prec j_2; \\ -E^{C'C}_{j_1} E^{CC''}_{i_1} F^{CC'}_{i_2 j}, \text{ if } j \prec j_2 \prec j_1. \end{cases} \tag{17.103}$$

Two constraints can be obtained from $u^{C''C'}_{i_1 i_2 j_1 j_2}$:

1. Let $i_1 \neq i_2$, $i_3 \neq i_4$. Then for any $1 \leq j_1, j_2 \leq 3$,

$$\frac{u^{C''C'}_{i_1 i_2 j_1 j_2}}{u^{C''C'}_{i_1 i_2 j_2 j_1}} = \frac{u^{C''C'}_{i_3 i_4 j_1 j_2}}{u^{C''C'}_{i_3 i_4 j_2 j_1}}. \tag{17.104}$$

2. Let $i \prec i_1 \prec i_2$. Then for any $1 \leq j_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{C''C'}_{i_1 i_2 j_1 j_2}}{u^{C''C'}_{i i_2 j_1 j_2}} = \frac{u^{C''C'}_{i_1 i_2 j_3 j_4}}{u^{C''C'}_{i i_2 j_3 j_4}}. \tag{17.105}$$

Define $U^{C''C'} : (C \wedge \mathcal{G}_4^2) \times (C \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}_4^2) \times (C' \wedge \mathcal{G}_4^2) \longrightarrow \mathbb{R}$ by

$$U^{C''C'}(L^C, m^C, L_1^{C'}, L_2^{C'}) = (L^C \vee C'')(L_1^{C'} \vee C)(m^C \vee L_1^{C'} \vee L_2^{C'}). \tag{17.106}$$

When $i_1 \neq i_2$,

$$U^{C''C'}(C \wedge \hat{e}^C_{i_1}, C \wedge e^C_{i_2}, C' \wedge \hat{e}^{C'}_{j_1}, C' \wedge \hat{e}^{C'}_{j_2}) = u^{C''C'}_{i_1 i_2 j_1 j_2}. \tag{17.107}$$

Constraints from $u^{C'C}_{i_1 i_2 k_1 k_2}$:

$$u^{C'C}_{i_1 i_2 k_1 k_2} = \begin{cases} 0, \text{ if } i_1 = i_2; \\ -E^{C''C}_{k_2} E^{CC'}_{i} F^{CC''}_{i_1 k_1}, \text{ if } i \prec i_1 \prec i_2; \\ E^{C''C}_{k_2} E^{CC'}_{i} F^{CC''}_{i_1 k_1}, \text{ if } i \prec i_2 \prec i_1. \end{cases} \tag{17.108}$$

Two constraints can be obtained from $u^{C'C}_{i_1 i_2 k_1 k_2}$:

1. For any $1 \leq i_l, k_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{C'C}_{i_1 i_2 k_1 k_2}}{u^{C'C}_{i_1 i_2 k_1 k_3}} = \frac{u^{C'C}_{i_3 i_4 k_4 k_2}}{u^{C'C}_{i_3 i_4 k_4 k_3}}. \tag{17.109}$$

2. Let $i \prec i_1 \prec i_2$. Then for any $1 \leq k_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{C'C}_{i_1 i_2 k_1 k_2}}{u^{C'C}_{i_1 i k_1 k_2}} = \frac{u^{C'C}_{i_1 i_2 k_3 k_4}}{u^{C'C}_{i_1 i k_3 k_4}}. \tag{17.110}$$

Define $U^{C'C} : (C \wedge \mathbb{R}^4) \times (C \wedge \mathbb{R}^4) \times (C'' \wedge \mathbb{R}^4) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} &U^{C'C}(m_1^C, m_2^C, m^{C''}, L^{C''}) = \\ &\qquad -(m_1^C \vee m^{C''})(C \vee L^{C''})(C' \vee (m_1^C \wedge_C m_2^C)). \end{aligned} \tag{17.111}$$

Then

$$U^{C'C}(C \wedge e_{i_1}^C, C \wedge e_{i_2}^C, C'' \wedge e_{k_1}^{C''}, C'' \wedge \hat{e}_{k_2}^{C''}) = u_{i_1 i_2 k_1 k_2}^{C'C}. \tag{17.112}$$

Constraints from $u_{i_1 i_2 k_1 k_2}^{C'C''}$: If $i_1 \neq i_2$, then

$$u_{i_1 i_2 k_1 k_2}^{C'C''} = \begin{cases} 0, \text{ if } k_1 = k_2; \\ E_{k_1}^{C''C} E_{i_1}^{CC'} F_{i_2 k}^{CC''}, \text{ if } k \prec k_1 \prec k_2; \\ -E_{k_1}^{C''C} E_{i_1}^{CC'} F_{i_2 k}^{CC''}, \text{ if } k \prec k_2 \prec k_1. \end{cases} \tag{17.113}$$

Two constraints can be obtained from $u_{i_1 i_2 k_1 k_2}^{C'C''}$:

1. Let $i_1 \neq i_2$ and $i_3 \neq i_4$. Then for any $1 \leq k_1, k_2 \leq 3$,

$$\frac{u_{i_1 i_2 k_1 k_2}^{C'C''}}{u_{i_1 i_2 k_2 k_1}^{C'C''}} = \frac{u_{i_3 i_4 k_1 k_2}^{C'C''}}{u_{i_3 i_4 k_2 k_1}^{C'C''}}. \tag{17.114}$$

2. Let $i \prec i_1 \prec i_2$. Then for any $1 \leq k_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u_{i_1 i_2 k_1 k_2}^{C'C''}}{u_{i i_2 k_1 k_2}^{C'C''}} = \frac{u_{i_1 i_2 k_3 k_4}^{C'C''}}{u_{i i_2 k_3 k_4}^{C'C''}}. \tag{17.115}$$

Define $U^{C'C''} : (C \wedge \mathcal{G}_4^2) \times (C \wedge \mathbb{R}^4) \times (C'' \wedge \mathcal{G}_4^2) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow \mathbb{R}$ by

$$U^{C'C''}(L^C, m^C, L_1^{C''}, L_2^{C''}) = (L^C \vee C'')(L_1^{C''} \vee C)(m^C \vee L_1^{C''} \vee L_2^{C''}). \tag{17.116}$$

When $i_1 \neq i_2$,

$$U^{C'C''}(C \wedge \hat{e}_{i_1}^C, C \wedge e_{i_2}^C, C'' \wedge \hat{e}_{k_1}^{C''}, C'' \wedge \hat{e}_{k_2}^{C''}) = u_{i_1 i_2 k_1 k_2}^{C'C''}. \tag{17.117}$$

Constraints from $u_{j_1 j_2 k_1 k_2}^{CC'}$: If $k_1 \neq k_2$, then

$$u_{j_1 j_2 k_1 k_2}^{CC'} = \begin{cases} 0, \text{ if } j_1 = j_2; \\ -E_{j_1}^{C'C} E_{k_2}^{C''C} F_{j k_1}^{C'C''}, \text{ if } j \prec j_1 \prec j_2; \\ E_{j_1}^{C'C} E_{k_2}^{C''C} F_{j k_1}^{C'C''}, \text{ if } j \prec j_2 \prec j_1. \end{cases} \tag{17.118}$$

Two constraints can be obtained from $u_{j_1 j_2 k_1 k_2}^{CC'}$:

1. Let $k_1 \neq k_2$ and $k_3 \neq k_4$. Then for any $1 \leq j_1, j_2 \leq 3$,

$$\frac{u^{CC'}_{j_1 j_2 k_1 k_2}}{u^{CC'}_{j_2 j_1 k_1 k_2}} = \frac{u^{CC'}_{j_1 j_2 k_3 k_4}}{u^{CC'}_{j_2 j_1 k_3 k_4}}. \tag{17.119}$$

2. Let $k \prec k_1 \prec k_2$, then for any $1 \leq j_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{CC'}_{j_1 j_2 k_1 k_2}}{u^{CC'}_{j_1 j_2 k_1 k}} = \frac{u^{CC'}_{j_3 j_4 k_1 k_2}}{u^{CC'}_{j_3 j_4 k_1 k}}. \tag{17.120}$$

Define $U^{CC'} : (C' \wedge \mathcal{G}_4^2) \times (C' \wedge \mathcal{G}_4^2) \times (C'' \wedge \mathbb{R}^4) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} &U^{CC'}(L_1^{C'}, L_2^{C'}, m^{C''}, L^{C''}) = \\ &\quad -(L_1^{C'} \vee C)(L^{C''} \vee C)(L_1^{C'} \vee L_2^{C'} \vee m^{C''}). \end{aligned} \tag{17.121}$$

When $k_1 \neq k_2$,

$$U^{CC'}(C' \wedge \hat{e}^{C'}_{j_1}, C' \wedge \hat{e}^{C'}_{j_2}, C'' \wedge e^{C''}_{k_1}, C'' \wedge \hat{e}^{C''}_{k_2}) = u^{CC'}_{j_1 j_2 k_1 k_2}. \tag{17.122}$$

Constraints from $u^{CC''}_{j_1 j_2 k_1 k_2}$: If $j_1 \neq j_2$, then

$$u^{CC''}_{j_1 j_2 k_1 k_2} = \begin{cases} 0, \text{ if } k_1 = k_2; \\ E^{C'C}_{j_2} E^{C''C}_{k_1} F^{C'C''}_{j_1 k}, \text{ if } k \prec k_1 \prec k_2; \\ -E^{C'C}_{j_2} E^{C''C}_{k_1} F^{C'C''}_{j_1 k}, \text{ if } k \prec k_2 \prec k_1. \end{cases} \tag{17.123}$$

Two constraints can be obtained from $u^{CC''}_{j_1 j_2 k_1 k_2}$:

1. Let $j_1 \neq j_2$ and $j_3 \neq j_4$. Then for any $1 \leq k_1, k_2 \leq 3$,

$$\frac{u^{CC''}_{j_1 j_2 k_1 k_2}}{u^{CC''}_{j_1 j_2 k_2 k_1}} = \frac{u^{CC''}_{j_3 j_4 k_1 k_2}}{u^{CC''}_{j_3 j_4 k_2 k_1}}. \tag{17.124}$$

2. Let $j \prec j_1 \prec j_2$, then for any $1 \leq k_l \leq 3$ where $1 \leq l \leq 4$,

$$\frac{u^{CC''}_{j_1 j_2 k_1 k_2}}{u^{CC''}_{j_1 j k_1 k_2}} = \frac{u^{CC''}_{j_1 j_2 k_3 k_4}}{u^{CC''}_{j_1 j k_3 k_4}}. \tag{17.125}$$

Define $U^{CC''} : (C' \wedge \mathbb{R}^4) \times (C' \wedge \mathcal{G}_4^2) \times (C'' \wedge \mathcal{G}_4^2) \times (C'' \wedge \mathcal{G}_4^2) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} &U^{CC''}(m^{C'}, L^{C'}, L_1^{C''}, L_2^{C''}) = (L_1^{C''} \vee C)(L^{C'} \vee C) \\ &\quad (m^{C'} \vee L_1^{C''} \vee L_2^{C''}). \end{aligned} \tag{17.126}$$

When $j_1 \neq j_2$,

$$U^{CC''}(C' \wedge e^{C'}_{j_1}, C' \wedge \hat{e}^{C'}_{j_2}, C'' \wedge \hat{e}^{C''}_{k_1}, C'' \wedge \hat{e}^{C''}_{k_2}) = u^{CC''}_{j_1 j_2 k_1 k_2}. \tag{17.127}$$

We have

$$V^{C''}(m_1^C, m_2^C, L_1^{C'}, L_2^{C'}) = \begin{cases} U^{C''C}(m_1^C, m_2^C, L_1^{C'} \vee L_2^{C'}, L_2^{C'}), \text{ if } L_1^{C'} \vee L_2^{C'} \neq 0; \\ -U^{C''C'}(m_1^C \wedge_C m_2^C, m_1^C, L_2^{C'}, L_1^{C'}), \text{ if } m_1^C \vee m_2^C \neq 0. \end{cases} \tag{17.128}$$

Thus

$$v^{C''}_{i_1 i_2 j_1 j_2} = \begin{cases} u^{C''C}_{i_1 i_2 j j_2}, \text{ if } j \prec j_1 \prec j_2; \\ -u^{C''C}_{i_1 i_2 j j_2}, \text{ if } j \prec j_2 \prec j_1; \\ -u^{C''C'}_{i i_1 j_2 j_1}, \text{ if } i \prec i_1 \prec i_2; \\ u^{C''C'}_{i i_1 j_2 j_1}, \text{ if } i \prec i_2 \prec i_1. \end{cases} \tag{17.129}$$

Comparing these constraints, we find that the constraints (17.65), (17.66) from $V^{C''}$ are equivalent to the constraints (17.104), (17.105) from $U^{C''C'}$, and are included in the constraints (17.99), (17.100) from $U^{C''C}$. Faugeras and Mourrain's first group of constraints is a special case of any of (17.65), (17.104) and (17.99). Similarly, Faugeras and Mourrain's second group of constraints is a special case of any of (17.109), (17.114).

We also have

$$V^{C}(L_1^{C'}, L_2^{C'}, L_1^{C''}, L_2^{C''}) = \begin{cases} U^{CC'}(L_1^{C'}, L_2^{C'}, L_1^{C''} \vee L_2^{C''}, L_2^{C''}), \text{ if } L_1^{C''} \vee L_2^{C''} \neq 0; \\ -U^{CC''}(L_1^{C'} \wedge_C L_2^{C'}, L_1^{C'}, L_2^{C''}, L_1^{C''}), \text{ if } L_1^{C'} \vee L_2^{C'} \neq 0. \end{cases} \tag{17.130}$$

Thus

$$v^{C}_{j_1 j_2 k_1 k_2} = \begin{cases} u^{CC''}_{j j_1 k_2 k_1}, \text{ if } j \prec j_1 \prec j_2; \\ -u^{CC''}_{j j_1 k_2 k_1}, \text{ if } j \prec j_2 \prec j_1; \\ -u^{CC'}_{j_1 j_2 k k_2}, \text{ if } k \prec k_1 \prec k_2; \\ u^{CC'}_{j_1 j_2 k k_2}, \text{ if } k \prec k_2 \prec k_1. \end{cases} \tag{17.131}$$

The constraints (17.84), (17.85) from V^C are equivalent to the constraints (17.124), (17.125) from $U^{C''C'}$, and are also equivalent to the constraints (17.119), (17.120) from $U^{C''C}$. Faugeras and Mourrain's third group of constraints is a special case of any of (17.84), (17.124) and (17.119).

17.6 Conclusion

In this chapter we propose a new algebraic representation for image points obtained from a pinhole camera, based on Hestenes and Ziegler's idea of projective split. We reformulate camera modeling and calibration, epipolar and trifocal geometries with this new representation. We also propose a systematic approach to derive constraints on epipolar and trifocal tensors, by which we have not only derived all known constraints, but also made considerable generalizations.