Analysis and Computation of Projective Invariants from Multiple Views in the Geometric Algebra Framework

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Abstract

A central task of computer vision is to automatically recognize objects in real-world scenes. The parameters defining image and object spaces can vary due to lighting conditions, camera calibration and viewing position. It is therefore desirable to look for geometric properties of the object which remain invariant under such changes. In this paper we present *geometric algebra* as a complete framework for the theory and computation of projective invariants formed from points and lines in computer vision. We will look at the formation of 3D projective invariants from multiple images, show how they can be formed from image coordinates and estimated tensors (F, fundamental matrix and T, trilinear tensor) and give results on simulated and real data.

Categories: Computer vision; invariants; Clifford/Geometric algebra; Grassmann-Cayley algebra; projective geometry; 3D projective invariants.

1 Introduction

The scope of *geometric invariance* was captured in the volume [16] and over the past decade or so invariance has been widely used for object recognition, matching and reconstruction [17]. Indeed, the currently fashionable topic of camera self-calibration can be cast in terms of looking for entities which are invariant under the class of similitudes. Thus, the study of invariants remains one of fundamental interest in computer vision. In this paper we will outline the use of *geometric algebra* (GA) in establishing a framework in which invariants can be derived and calculated. An important point to note here is that the *same* framework and approach can be used for extensions such as differential invariants and Lie algebra approaches.

Geometric algebra is a coordinate-free approach to geometry based on the algebras of Grassmann [5] and Clifford [3]. The algebra is defined on a space whose elements are called *multivectors*; a multivector is a linear combination of objects of different type, e.g. scalars and vectors. It has an associative and fully invertible product called the **geometric** or **Clifford** product. The existence of such a product and the calculus associated with the geometric algebra give the system tremendous power. Some preliminary applications of GA in the field of computer vision have already been given [13, 15], and here we will extend the discussion of geometric invariance given in [1, 12]. GA provides a very natural language for projective geometry as does the currently popular Grassmann-Cayley (GC) algebra, [2] (a system for computations with subspaces of finite-dimensional vector spaces). While the GC algebra expresses some ideas of projective geometry, such as the meet and join, very elegantly, it lacks an inner (regressive) product – the consequences of this are discussed more fully in [14]. The next section will give a brief introduction to GA. For a more complete introduction see [10] and for other brief summaries see [6, 15]. In this paper vectors will be bold quantities (except for basis vectors) and multivectors will not be bold. Lower case is used to denote vectors in 3D Euclidean space and upper case to denote vectors in 4D projective space.

2 Geometric Algebra: an outline

The algebras of Clifford and Grassmann are well known to pure mathematicians, but were long ago abandoned by physicists in favour of the vector algebra of Gibbs – which is still most commonly used today. The approach to Clifford algebra we adopt here was pioneered in the 1960's by David Hestenes who has, since then, worked on developing his version of Clifford algebra – which will be referred to as *geometric algebra* (GA) – into a unifying language for mathematics and physics [10].

2.1 The Geometric product and multivectors

Let $\mathcal{G}_n$ denote the geometric algebra of n-dimensions – this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford** product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ and can be expressed as a sum of its symmetric and antisymmetric parts

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}, \tag{1}$$

where the inner product $\boldsymbol{a}\cdot\boldsymbol{b}$ and the outer product $\boldsymbol{a}\wedge\boldsymbol{b}$ are defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \qquad\qquad \boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}). \tag{2}$$

The inner product of two vectors is the standard *scalar* or *dot* product and produces a scalar. The outer or wedge product of two vectors is a new quantity we call a **bivector**. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ – see figure 1. Thus, $\boldsymbol{b}\wedge\boldsymbol{a}$ will have the opposite orientation making the wedge product anticommutative as given in equation (2). The outer product is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a **trivector**, is interpreted as the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$ – see figure 1. The outer product of k vectors is a k-vector or k-blade, and such a quantity is said to have *grade* k. A **multivector** is made up of a linear combination of objects of different grade, i.e. scalar plus bivector etc. GA provides a means of manipulating multivectors which allows us to keep track of different grade objects simultaneously – much as one does with complex number operations. For a general multivector X, the notation $\langle X\rangle$ will mean

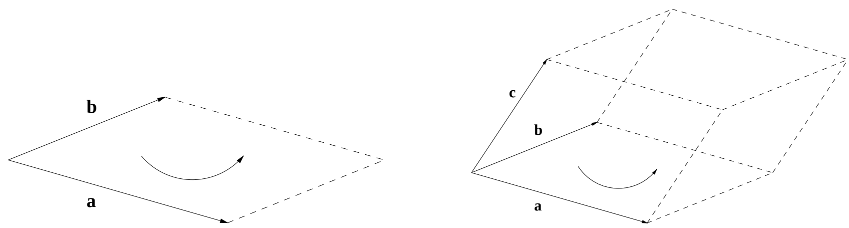

Figure 1: *Left:* The directed area, or bivector, $\boldsymbol{a}\wedge\boldsymbol{b}$. *Right:* The oriented volume, or trivector, $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$.

take the scalar part of X. In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist since there is no possibility of sweeping the volume element $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ over a 4th dimension. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I and is crucial when discussing duality.

We now end this introductory section by giving a very brief review of the geometric algebra approach to linear algebra. A more detailed review is found in [9].

Consider a linear function f which maps vectors to vectors in the same space. We can extend f to act linearly on multivectors via the **outermorphism**, $\underline{f}$, defining the action of $\underline{f}$ on blades by

$$\underline{f}(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\ldots\wedge\boldsymbol{a}_r) = \underline{f}(\boldsymbol{a}_1)\wedge\underline{f}(\boldsymbol{a}_2)\wedge\ldots\wedge\underline{f}(\boldsymbol{a}_r). \tag{3}$$

We use the term outermorphism because $\underline{f}$ preserves the grade of any r-vector it acts on. We therefore know that the pseudoscalar of the space must be mapped onto some multiple of itself. The scale factor in this mapping is the **determinant** of $\underline{f}$;

$$\underline{f}(I) = \det(\underline{f})I. \tag{4}$$

This is much simpler than many definitions of the determinant enabling one to establish most properties of determinants with little effort.

3 Projective Geometry and the Projective Split

Since about the mid 1980's most of the computer vision literature discussing geometry and invariants has used the language of projective geometry (see appendix of [16]). As any point on a ray from the optical centre of a camera will map to the same point in the camera image plane it is easy to see why a 2D view of a 3D world might well be best expressed in projective space. In classical projective geometry one defines a 3D space, $\mathcal{P}^3$, whose points are in $1-1$ correspondence with lines through the origin in a 4D space, R^4. Similarly, k-dimensional subspaces of $\mathcal{P}^3$ are identified with $(k+1)$-dimensional subspaces of R^4. Such projective views can provide very elegant descriptions of the geometry of incidence (intersections, unions etc.). The projective space, $\mathcal{P}^3$, has no metric, the basis and metric are introduced in the associated 4D space. In this 4D space a coordinate description of a projective point is conventionally brought about by using *homogeneous coordinates*. Here we will briefly outline how projective geometry looks in the GA framework.

The basic projective geometry operations of meet and join are easily expressible in terms of standard operations within the geometric algebra. Firstly, to introduce the concepts of duality which are so important in projective geometry, we define the dual A^* of an r-vector A as

$$A^* = AI^{-1}. \tag{5}$$

In an n-dimensional geometric algebra one can define the **join** $J = A \bigwedge B$ of an r-vector, A, and an s-vector, B, by

$$J = A \wedge B \qquad \text{if } A \text{ and } B \text{ are linearly independent.} \tag{6}$$

If A and B are not linearly independent the join is not given simply by the wedge but by the subspace that they span. J can be interpreted as a *common dividend of lowest grade* and is defined up to a scale factor. It is easy to see that if $(r+s) \geq n$ then J will be the pseudoscalar for the space. In what follows we will use $\bigwedge$ for the join only when the blades A and B are not linearly independent, otherwise we will use the ordinary exterior product, $\wedge$.

If A and B have a common factor (i.e. there exists a k-vector C such that $A = A'C$ and $B = B'C$ for some A', B') then we can define the 'intersection' or **meet** of A and B as $A \vee B$ where [11]

$$(A \vee B)^* = A^* \wedge B^*. \tag{7}$$

That is, the dual of the meet is given by the join of the duals (a familiar result from classical projective geometry). The dual of $(A \vee B)$ is understood to be taken with respect to the *join* of A and B. In most cases of practical interest this join will be the whole space and the meet is therefore easily computed. A more useful expression for the meet (see [14]) is as follows

$$A \vee B = (A^* \cdot B). \tag{8}$$

We therefore have the very simple and readily computed relation of $A \vee B = (A^* \cdot B)$. The above concepts are discussed further in [11].

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$, a 3D space with a Euclidean metric. As mentioned earlier, we find it useful to associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. In these two distinct but related spaces we define basis vectors: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in R^4 and $(\sigma_1, \sigma_2, \sigma_3)$ in $\mathcal{E}^3$. We identify R^4 and $\mathcal{E}^3$ with the GAs of 4 and 3 dimensions, $\mathcal{G}_{(1,3,0)}$ and $\mathcal{G}_{(3,0,0)}$ (here $\mathcal{G}_{(p,q,r)}$ is a $p+q+r$-dimensional GA in which p, q and r basis vectors square to $+1$, -1 and 0 respectively). We require that vectors, bivectors and trivectors in R^4 will represent points, lines and planes in $\mathcal{E}^3$. Suppose we choose γ_4 as a selected direction in R^4, we can then define a mapping which associates the bivectors $\gamma_i\gamma_4$, $i = 1, 2, 3$, in R^4 with the vectors σ_i, $i = 1, 2, 3$, in $\mathcal{E}^3$;

$$\sigma_1 \equiv \gamma_1\gamma_4, \quad \sigma_2 \equiv \gamma_2\gamma_4, \quad \sigma_3 \equiv \gamma_3\gamma_4. \tag{9}$$

To preserve the Euclidean structure of the spatial vectors $\{\sigma_i\}$ (i.e. $\sigma_i^2 = +1$) we are forced to assume a non-Euclidean metric for the basis vectors in R^4. We choose to use $\gamma_4^2 = +1$, $\gamma_i = -1$, $i = 1, 2, 3$. This process of associating the higher and lower dimensional spaces is an application of what Hestenes calls the **projective split**.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is obtained by taking the geometric product of $\mathbf{X}$ and γ_4;

$$\mathbf{X}\gamma_4 = \mathbf{X}\cdot\gamma_4 + \mathbf{X}\wedge\gamma_4 = X_4\left(1 + \frac{\mathbf{X}\wedge\gamma_4}{X_4}\right) \equiv X_4(1 + \boldsymbol{x}). \tag{10}$$

Note that $\boldsymbol{x}$ contains terms of the form $\gamma_1\gamma_4$, $\gamma_2\gamma_4$, $\gamma_3\gamma_4$ or, via equation (9), terms in $\sigma_1, \sigma_2, \sigma_3$. We therefore associate the vector $\boldsymbol{x}$ in $\mathcal{E}^3$ with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4.

If we start with a vector $\boldsymbol{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ in $\mathcal{E}^3$, we can represent this in R^4 by the vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ such that

$$\boldsymbol{x} = \frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\gamma_1\gamma_4 + \frac{X_2}{X_4}\gamma_2\gamma_4 + \frac{X_3}{X_4}\gamma_3\gamma_4 = \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3, \tag{11}$$

$\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1, 2, 3$. This process can therefore be seen to be equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$. Thus, in this GA formulation we postulate distinct spaces in which we represent ordinary 3D quantities and their 4D projective counterparts, together with a well-defined way of moving between these spaces.

3.1 Projective transformations

Two of the main advantages of working in homogeneous coordinates arise from the facts that general displacements can be expressed in terms of a single matrix and some non-linear transformations in $\mathcal{E}^3$ become linear transformations in R^4. If a general point (x, y, z) in 3-D space is projected onto an image plane, the coordinates (x', y') in the image plane will be related to (x, y, z) via a transformation of the form:

$$x' = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{12}$$

Although clearly non-linear, this is expressible as the ratio of two linear transformations. To make this non-linear transformation in $\mathcal{E}^3$ into a linear transformation in R^4 we define a linear function $\underline{f}_p$ mapping vectors onto vectors in R^4 such that the action of $\underline{f}_p$ on the basis vectors $\{\gamma_i\}$ is given by

$$\underline{f}_p(\gamma_1) = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \tilde{\alpha}\gamma_4 \qquad \underline{f}_p(\gamma_2) = \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \tilde{\beta}\gamma_4$$

$$\underline{f}_p(\gamma_3) = \delta_1\gamma_1 + \delta_2\gamma_2 + \delta_3\gamma_3 + \tilde{\delta}\gamma_4 \qquad \underline{f}_p(\gamma_4) = \epsilon_1\gamma_1 + \epsilon_2\gamma_2 + \epsilon_3\gamma_3 + \tilde{\epsilon}\gamma_4 \tag{13}$$

A general point P in $\mathcal{E}^3$ given by $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ becomes the point $\mathbf{X} = (X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4)$ in R^4, where $x = X/W$, $y = Y/W$, $z = Z/W$. We can then see that $\underline{f}_p$ maps $\mathbf{X}$ onto $\mathbf{X}'$ where

$$\mathbf{X}' = \sum_{i=1}^{3}\{(\alpha_i X + \beta_i Y + \delta_i Z + \epsilon_i W)\gamma_i\} + (\tilde{\alpha}X + \tilde{\beta}Y + \tilde{\delta}Z + \tilde{\epsilon}W)\gamma_4 \tag{14}$$

The vector $\boldsymbol{x}' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3$ in $\mathcal{E}^3$ corresponds to $\mathbf{X}'$, where x' is given by

$$x' = \frac{\alpha_1 X + \beta_1 Y + \delta_1 Z + \epsilon_1 W}{\tilde{\alpha}X + \tilde{\beta}Y + \tilde{\delta}Z + \tilde{\epsilon}W} = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{15}$$

Similarly we have

$$y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha} x + \tilde{\beta} y + \tilde{\delta} z + \tilde{\epsilon}}, \qquad z' = \frac{\alpha_3 x + \beta_3 y + \delta_3 z + \epsilon_3}{\tilde{\alpha} x + \tilde{\beta} y + \tilde{\delta} z + \tilde{\epsilon}}. \tag{16}$$

Note that in general we would take $\alpha_3 = f\tilde{\alpha}$, $\beta_3 = f\tilde{\beta}$ etc. so that $z' = f$ (focal length), independent of the point chosen. Via this means the non-linear transformation in $\mathcal{E}^3$ becomes a linear transformation, $\underline{f}_p$, in R^4. We will see later that use of the linear function $\underline{f}_p$ makes the invariant nature of various quantities very easy to establish.

4 1-D and 2-D Projective Invariants from a Single View

In this section we will use the framework established so far to look at standard projective geometric invariants. We begin by looking at algebraic quantities which are invariant under projective transformations, arriving at these invariants in a way which can be naturally generalized from 1D to 2D to 3D.

The 1-D Cross-Ratio

The '*fundamental projective invariant*' of points on a line is the so-called **cross-ratio**, ρ,

defined as

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)},$$

where $t_1 = |PA|$, $t_2 = |PB|$, $t_3 = |PC|$, $t_4 = |PD|$ – see Figure 2.

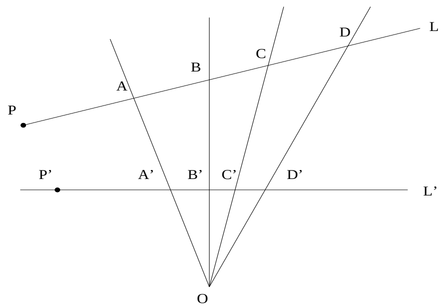

Figure 2: Formation of the 1D cross-ratio.

It is fairly easy to show that for the projection through O of the collinear points A, B, C, D onto any line, ρ remains constant. For this 1D case, any point q on the line L can be written as $\boldsymbol{q} = t\sigma_1$ relative to P, where σ_1 is a unit vector in the direction of L. We then move up a dimension to a 2D space, with basis vectors (γ_1, γ_2) (we will call this R^2) in which $\boldsymbol{q}$ is represented by the vector $\mathbf{Q}$;

$$\mathbf{Q} = T\gamma_1 + S\gamma_2$$

where, as before, we associate $\boldsymbol{q}$ with the bivector

$$\frac{\mathbf{Q}\wedge\gamma_2}{\mathbf{Q}\cdot\gamma_2} = \frac{T}{S}\gamma_1\gamma_2 \equiv \frac{T}{S}\sigma_1$$

so that $t = T/S$. When a point on line L is projected onto another line L', the distances t and t' are related by a projective transformation of the form

$$t' = \frac{\alpha t + \beta}{\tilde{\alpha} t + \tilde{\beta}}. \tag{17}$$

This non-linear transformation in $\mathcal{E}^1$ can be made into a linear transformation in R^2 by defining the linear function $\underline{f}_1$ mapping vectors onto vectors in R^2;

$$\underline{f}_1(\gamma_1) = \alpha_1\gamma_1 + \tilde{\alpha}\gamma_2 \qquad \underline{f}_1(\gamma_2) = \beta_1\gamma_1 + \tilde{\beta}\gamma_2. \tag{18}$$

Consider 2 vectors $\mathbf{X}_1, \mathbf{X}_2$ in R^2. Form the bivector $\mathcal{S}_1 = \mathbf{X}_1 \wedge \mathbf{X}_2 = \lambda_1 I_2$ where $I_2 = \gamma_1\gamma_2$ is the pseudoscalar for R^2. We now look at how $\mathcal{S}_1$ transforms under $\underline{f}_1$:

$$\mathcal{S}_1' = \mathbf{X}_1' \wedge \mathbf{X}_2' = \underline{f}_1(\mathbf{X}_1 \wedge \mathbf{X}_2) = (\det \underline{f}_1)(\mathbf{X}_1 \wedge \mathbf{X}_2). \tag{19}$$

This last step follows since a linear function must map a pseudoscalar onto a multiple of itself, this multiple being the determinant of the function. Suppose that we now take 4 points on the line L whose corresponding vectors in R^2 are $\{\mathbf{X}_i\}$, $i = 1,..,4$, and consider the ratio $\mathcal{R}_1$ of the magnitudes of 2 wedge products;

$$\mathcal{R}_1 = \frac{(\mathbf{X}_1 \wedge \mathbf{X}_2)I_2^{-1}}{(\mathbf{X}_3 \wedge \mathbf{X}_4)I_2^{-1}} \tag{20}$$

where the inverse of the pseudoscalar, I_2^{-1}, has been used to produce a scalar. Then, under $\underline{f}_1$, $\mathcal{R}_1 \to \mathcal{R}_1'$, where

$$\mathcal{R}_1' = \frac{(\mathbf{X}_1' \wedge \mathbf{X}_2')I_2^{-1}}{(\mathbf{X}_3' \wedge \mathbf{X}_4')I_2^{-1}} = \frac{(\det \underline{f}_1)(\mathbf{X}_1 \wedge \mathbf{X}_2)I_2^{-1}}{(\det \underline{f}_1)(\mathbf{X}_3 \wedge \mathbf{X}_4)I_2^{-1}}. \tag{21}$$

$\mathcal{R}_1$ is therefore invariant under $\underline{f}_1$. However, we want to express our invariants in terms of *distances* on the 1D line; for this we must consider how the bivector $\mathcal{S}_1$ in R^2 projects down to $\mathcal{E}^1$.

$$\mathbf{X}_1 \wedge \mathbf{X}_2 = (T_1\gamma_1 + S_1\gamma_2)\wedge(T_2\gamma_1 + S_2\gamma_2) = (T_1S_2 - T_2S_1)\gamma_1\gamma_2 \equiv S_1S_2(T_1/S_1 - T_2/S_2)I_2 = S_1S_2(t_1 - t_2)I_2. \tag{22}$$

This expansion uses the fact that $\gamma_1 \wedge \gamma_1 = \gamma_2 \wedge \gamma_2 = 0$ and $\gamma_1 \cdot \gamma_2 = 0$. In order to form a projective invariant which is independent of the choice of the arbitrary scalars S_i, we must then take *ratios* of the bivectors $\mathbf{X}_i \wedge \mathbf{X}_j$ (so that $\det \underline{f}_1$ cancels) and *multiples* of such ratios so that the S_i's cancel. More precisely, consider the following expression

$$Inv_1 = \frac{(\mathbf{X}_3 \wedge \mathbf{X}_1)I_2^{-1}(\mathbf{X}_4 \wedge \mathbf{X}_2)I_2^{-1}}{(\mathbf{X}_4 \wedge \mathbf{X}_1)I_2^{-1}(\mathbf{X}_3 \wedge \mathbf{X}_2)I_2^{-1}}.$$

Then, in terms of distances along the lines, under the projective transformation $\underline{f}_1$, Inv_1 goes to Inv_1' where

$$Inv_1' = \frac{S_3S_1(t_3 - t_1)S_4S_2(t_4 - t_2)}{S_4S_1(t_4 - t_1)S_3S_2(t_3 - t_2)} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)}, \tag{23}$$

which is independent of the S_i's and is indeed the 1D classical projective invariant, the **cross-ratio**. Deriving the cross-ratio in this way enables us to easily generalize it to form invariants in higher dimensions.

4.1 The 2-D generalization of the Cross-Ratio

For points in a plane we again move up to a space with one higher dimension which we shall call R^3. Let a point P in the plane M be described by the vector $\boldsymbol{x}$ in $\mathcal{E}^2$ where $\boldsymbol{x} = x\sigma_1 + y\sigma_2$, and σ_1 and σ_2 are basis vectors in the plane M. In R^3 this point will be represented by $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3$ where $x = X/Z$ and $y = Y/Z$. As described earlier, we can define a general projective transformation via a linear function $\underline{f}_2$ mapping vectors to vectors in R^3 such that;

$$\underline{f}_2(\gamma_1) = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \tilde{\alpha}\gamma_3 \qquad \underline{f}_2(\gamma_2) = \beta_1\gamma_1 + \beta_2\gamma_2 + \tilde{\beta}\gamma_3 \qquad \underline{f}_2(\gamma_3) = \delta_1\gamma_1 + \delta_2\gamma_2 + \tilde{\delta}\gamma_3 \tag{24}$$

Consider 3 vectors (representing non-collinear points) $\mathbf{X}_i$, $i = 1, 2, 3$, in R^3 and form the trivector

$$\mathcal{S}_2 = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 = \lambda_2 I_3 \tag{25}$$

where $I_3 = \gamma_1\gamma_2\gamma_3$ is the pseudoscalar for R^3. As before, under the projective transformation given by $\underline{f}_2$, $\mathcal{S}_2$ transforms to $\mathcal{S}_2'$ where $\mathcal{S}_2' = \det \underline{f}_2 \mathcal{S}_2$.

Therefore, the ratio of the magnitudes of any trivectors is invariant under $\underline{f}_2$. To project down into $\mathcal{E}^2$ we use the fact that $\mathbf{X}_i\gamma_3 = Z_i(1 + \boldsymbol{x}_i)$ under the projective split to write

$$\begin{aligned} \mathcal{S}_2 {I_3}^{-1} &= \langle \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 {I_3}^{-1} \rangle = \langle \mathbf{X}_1\gamma_3\gamma_3\mathbf{X}_2\mathbf{X}_3\gamma_3\gamma_3 {I_3}^{-1} \rangle \\ &= Z_1 Z_2 Z_3 \langle (1 + \boldsymbol{x}_1)(1 - \boldsymbol{x}_2)(1 + \boldsymbol{x}_3)\gamma_3 {I_3}^{-1} \rangle. \end{aligned} \tag{26}$$

Where the $\boldsymbol{x}_i$ represent vectors in $\mathcal{E}^2$. We can only form a scalar part from the expression within the brackets by taking products of a vector, 2 spatial vectors (i.e. vectors made up of the σs) and ${I_3}^{-1}$, so that

$$\mathcal{S}_2 {I_3}^{-1} = Z_1 Z_2 Z_3 \langle (\boldsymbol{x}_1\boldsymbol{x}_3 - \boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{x}_2\boldsymbol{x}_3)\gamma_3 {I_3}^{-1} \rangle = Z_1 Z_2 Z_3 \{ (\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \} {I_2}^{-1}. \tag{27}$$

It is then clear that we should take multiples of such ratios so that the arbitrary scalars Z_i cancel. For 4 points in a plane, there are only 4 possible combinations of $Z_i Z_j Z_k$ and we cannot have a situation where we multiply two ratios of the form $\mathbf{X}_i \wedge \mathbf{X}_j \wedge \mathbf{X}_k$ together and have all the Z's cancelling. For 5 coplanar points $\{\mathbf{X}_i\}$, $i = 1, .., 5$, there are several ways of achieving the desired cancellation, for example

$$Inv_2 = \frac{(\mathbf{X}_5 \wedge \mathbf{X}_4 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_1) I_3^{-1}}{(\mathbf{X}_5 \wedge \mathbf{X}_1 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4) I_3^{-1}}.$$

According to equation (27) we can interpret this ratio in $\mathcal{E}^2$ as

$$Inv_2 = \frac{(\boldsymbol{x}_5 - \boldsymbol{x}_4) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_3) I_2^{-1} (\boldsymbol{x}_5 - \boldsymbol{x}_2) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_1) I_2^{-1}}{(\boldsymbol{x}_5 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_3) I_2^{-1} (\boldsymbol{x}_5 - \boldsymbol{x}_2) \wedge (\boldsymbol{x}_5 - \boldsymbol{x}_4) I_2^{-1}} = \frac{A_{543} A_{521}}{A_{513} A_{524}} \tag{28}$$

where $\frac{1}{2} A_{ijk}$ is the area of the triangle defined by the 3 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k$. This invariant is regarded as the 2D generalization of the 1D cross-ratio.

5 3-D Projective Invariants from Multiple Views

In this section we begin by looking at the generalization of the cross–ratio in 3D and then consider how we actually *compute* projective invariants from image coordinates in two and three cameras views.

5.1 The 3-D generalization of the Cross–Ratio

For general points in $\mathcal{E}^3$ we have seen that we move up one dimension to work in the 4D space R^4. The point $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ in $\mathcal{E}^3$ is written as $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4$, where $x = X/W$, $y = Y/W$, $z = Z/W$. As before, a non-linear projective transformation in $\mathcal{E}^3$ becomes a linear transformation, described by a linear function $\underline{f}_3$ in R^4.

Consider 4 vectors in R^4, $\{\mathbf{X}_i\}$, $i = 1, .., 4$. Form the 4-vector $\mathcal{S}_3 = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 = \lambda_3 I_4$ where $I_4 = \gamma_1\gamma_2\gamma_3\gamma_4$ is the pseudoscalar for R^4. As before, $\mathcal{S}_3$ transforms to $\mathcal{S}_3'$ under $\underline{f}_3$;

$$\mathcal{S}_3' = \mathbf{X}_1' \wedge \mathbf{X}_2' \wedge \mathbf{X}_3' \wedge \mathbf{X}_4' = \det \underline{f}_3 \mathcal{S}_3. \tag{29}$$

The ratio of the magnitudes of any two 4-vectors is therefore invariant under $\underline{f}_3$ and we must take multiples of such ratios to ensure the arbitrary scale factors W_i cancel. With 5 general points there are 5 possible combinations of $W_iW_jW_kW_l$ and it is then simple to show that one cannot take multiples of ratios such that the W factors cancel. However, for 6 points one can do this, and an example of such an invariant is

$$Inv_3 = \frac{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5) I_4^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}. \tag{30}$$

Using the arguments of the previous sections we can write

$$(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} \equiv W_1W_2W_3W_4 \{(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1)\} I_3^{-1}. \tag{31}$$

The invariant Inv_3 is therefore the 3D equivalent of the 1D cross-ratio and consists of ratios of volumes;

$$Inv_3 = \frac{V_{1234} V_{4526}}{V_{1245} V_{3426}}, \tag{32}$$

where V_{ijkl} is the volume of the solid formed by the 4 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k, \boldsymbol{x}_l$.

Conventionally all of these invariants are well known but above we have outlined a general, straightforward process for generating projective invariants in any dimension.

5.2 3D point projective invariants from image coordinates in two views

Suppose we have six general 3D points P_i, $i = 1, .., 6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4 respectively. We have seen above that 3D projective invariants can be formed from these points, and an example of such an invariant is

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{33}$$

This is simply equation (30) rewritten in terms of *brackets*, where the bracket is defined by $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1}$. If it is possible to express the bracket $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ in terms of the **image coordinates** of points P_i, P_j, P_k, P_l, then this invariant will be readily computable in practice. Some of the most recent work which has addressed this problem has utilized the Grassmann-Cayley (GC) algebra [2, 4]. It has been shown that it is not possible to compute general 3D invariants from a single image and in [2] Carlsson discussed the computation of such invariants from a pair of images in terms of the image coordinates and the fundamental matrix, $\boldsymbol{F}$, using the GC algebra. Here we will show how the approach of Carlsson looks in the geometric algebra framework and in the following section, extend the technique to deal with three views.

Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1} = (\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1}. \qquad (34)$$

The quantities $L_{12} = (\mathbf{X}_1 \wedge \mathbf{X}_2)$ and $L_{34} = (\mathbf{X}_3 \wedge \mathbf{X}_4)$ are the R^4 representations of the lines joining points P_1 and P_2, and P_3 and P_4. Now, by expressing these lines in the world as intersections of planes through the optical centres of the cameras, it can be shown [14] that it is possible to write the above bracket as

$$(\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1} = (\mathbf{A}_0\wedge\mathbf{B}_0\wedge\mathbf{A}'_{1234}\wedge\mathbf{B}'_{1234})I_4^{-1}. \qquad (35)$$

where $\mathbf{A}_{1234}$ is the R^4 representation of the intersection in the first image plane (denoted A, having optical centre $\mathbf{A}_0$) of the projections of lines L_{12} and L_{34}, and $\mathbf{B}_{1234}$ is the same intersection in the second image plane (denoted B, having optical centre $\mathbf{B}_0$) – see figure 3.

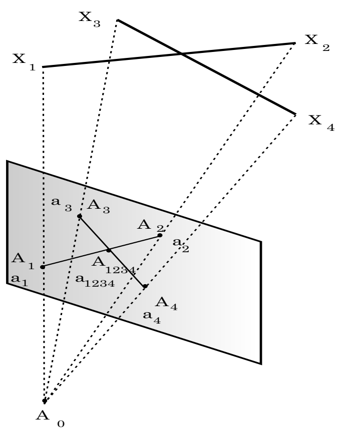

Figure 3: Two world lines projected into an image plane. Both R^4 and $\mathcal{E}^3$ representations are shown.

From this expression it is not difficult to go one stage further and write the bracket in terms of image coordinates and the fundamental matrix. It can be shown (see [14]) that, in terms of the *observed* quantities, we can write the invariant as

$$Inv_3 = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})\mu_{1245}(\mu_{3426}-1)\lambda_{1245}(\lambda_{3426}-1)}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})\mu_{4526}(\mu_{1234}-1)\lambda_{4526}(\lambda_{1234}-1)} \qquad (36)$$

where the $\boldsymbol{\delta}$s and the $\boldsymbol{\epsilon}$s are the observed intersection points in the two image planes and $\boldsymbol{F}$ is the estimated fundamental matrix (using sets of matching image points for example). The μs and λs are defined by expanding the image points as follows. Since $\boldsymbol{a}_r$, $\boldsymbol{a}_s$ and $\boldsymbol{a}_{pqrs}$ are collinear we can write

$$\boldsymbol{a}_{pqrs} = \mu_{pqrs}\boldsymbol{a}_s + (1-\mu_{pqrs})\boldsymbol{a}_r \quad \text{and} \quad \boldsymbol{b}_{pqrs} = \lambda_{pqrs}\boldsymbol{b}_s + (1-\lambda_{pqrs})\boldsymbol{b}_r. \tag{37}$$

To summarize; given the coordinates of a set of 6 corresponding points in the two image planes (from non-coplanar world points) we can form 3D projective invariants provided we have some estimate of $\boldsymbol{F}$.

5.3 3D point projective invariants from image coordinates in three views

The technique used to form the 3D projective invariants for two views can be straightforwardly extended to give expressions for invariants of three views. Consider the scenario shown in figure 4, which shows four world points, $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ (or two lines $\mathbf{X}_1 \wedge \mathbf{X}_2$ and $\mathbf{X}_3 \wedge \mathbf{X}_4$) projected into three camera planes, where we use the same notation as in previous sections. As before, we can write these world lines as the intersection of planes. Using GA it is also possible to write the components of the *trilinear tensor*, T, [8], in terms of intersections of lines and planes [13], and hence write the bracket $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]$ in terms of T_{ijk} (components of T) and values in the image plane. If l^B_{ij} and l^C_{ij} are the vectors representing the standard homogeneous coordinates of the lines joining the projections of world points $\mathbf{X}_i$ and $\mathbf{X}_j$ in planes B and C respectively, the expression for Inv_3 in terms of observed or estimated quantities is (see [14])

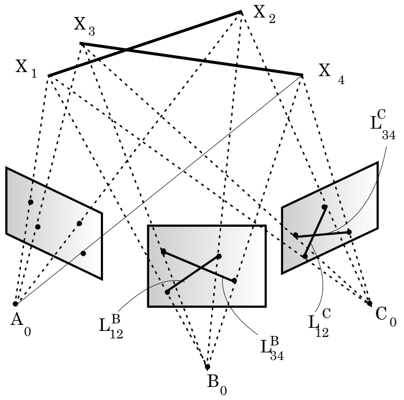

Figure 4: Two world lines projected down into three camera planes. The points and lines of interest are indicated in each frame. All quantities shown with their R^4 representations.

$$Inv_3 = \frac{[T_{ijk}\delta^A_{1234,i}l^B_{12,j}l^C_{34,k}][T_{ijk}\delta^A_{4526,i}l^B_{45,j}l^C_{26,k}]\mu_{1245}(\mu_{3426}-1)}{[T_{ijk}\delta^A_{1245,i}l^B_{12,j}l^C_{45,k}][T_{ijk}\delta^A_{3426,i}l^B_{34,j}l^C_{26,k}]\mu_{4526}(\mu_{1234}-1)}. \tag{38}$$

where the δs and μs are from image A and take the same meaning as previously. In equation (38) the quantities are all *observed* quantities or entities we form from *observed* quantities.

6 Experimental Results

In this section we present results for the formation of 3D invariants from two and three views on both simulated and real data. Throughout this section feature points (corners) were extracted automatically but matched by hand. The authors note that automatic matching is, itself, often a hard problem, but the work presented here addresses only the subsequent analysis, assuming such matching has been successfully performed.

The simulations (carried out in Maple) involve generating four different sets, S_i $i = 1, .., 4$, of 6 points;

$$S_i = \{\mathbf{X}_1^i, \mathbf{X}_2^i, \mathbf{X}_3^i, \mathbf{X}_4^i, \mathbf{X}_5^i, \mathbf{X}_6^i\}$$

within a spherical region whose dimensions were around a tenth of the distance of the centre of the volume from the camera's optical centre. These sets of points are then observed from four different viewpoints, so that the four sets of image coordinates for set S_i are given by s_{ij}, $j = 1, .., 4$;

$$s_{ij} = \{\boldsymbol{x}_{j1}^i, \boldsymbol{x}_{j2}^i, \boldsymbol{x}_{j3}^i, \boldsymbol{x}_{j4}^i, \boldsymbol{x}_{j5}^i, \boldsymbol{x}_{j6}^i\}$$

For each set of 6 points the three linearly independent invariants $\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^3$ are formed, where these are the standard invariants given as follows

$$\begin{aligned} \mathcal{I}^1 &= \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]} \quad \mathcal{I}^2 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]} \\ \mathcal{I}^3 &= \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_6][\mathbf{X}_6\mathbf{X}_5\mathbf{X}_2\mathbf{X}_4]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_6\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_6\mathbf{X}_2\mathbf{X}_4]}. \end{aligned} \tag{39}$$

These I's are formed using a) views 1 & 2, b) views 2 & 3, c) views 1,2 & 3 and d) views 2,3 & 4 – in a) and b) the fundamental matrix is calculated by standard linear means and in c) and d) the trifocal tensor is derived also from a simple linear algorithm ([8]). Although these simple linear methods do not enforce the necessary constraints on F and T, the resulting estimates were adequate for the purposes of the experiments shown here. In the experiments with real data below, F and T were also formed in this way.

These invariants of each set were represented as 3D vectors, $\mathbf{v}_i = [\mathcal{I}_i^1, \mathcal{I}_i^2, \mathcal{I}_i^3]^T$. The comparison of the invariants was done using 'Euclidean distances' of the vectors, $d(\boldsymbol{v}_i, \boldsymbol{v}_j)$, where

$$d(\boldsymbol{v}_i, \boldsymbol{v}_j) = \left[1 - \left|\frac{\boldsymbol{v}_i \cdot \boldsymbol{v}_j}{||\boldsymbol{v}_i||||\boldsymbol{v}_j||}\right|\right]^{\frac{1}{2}}. \tag{40}$$

For any $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ the distance $d(\mathbf{v}_i, \boldsymbol{v}_j)$ lies between 0 and 1 and it does not vary when $\boldsymbol{v}_i$ or $\boldsymbol{v}_j$ is multiplied by a nonzero constant – this follows Hartley's analysis given in [7].

Figure 5 shows two sets of tables. The (i, j)th entry in the top left-hand box shows the distance, $d(\boldsymbol{v}_i, \boldsymbol{v}_j)$, between invariants for set S_i formed from views 1 & 2 and invariants for set S_j formed from views 2 & 3, when gaussian noise of $\sigma = 0.005$ was added to the image points. The boxes to the right of this show the same thing for increasing σ. The bottom row shows the equivalent for invariants formed from three views using the expression given in the previous section; here the (i,j)th entry in the left-hand box shows the distance, $d(\boldsymbol{v}_i, \boldsymbol{v}_j)$, between invariants for set S_i formed from views 1,2 & 3, and invariants for set S_j formed from views 2, 3 & 4. Clearly, we would like the diagonal elements to be as close as possible to zero since the invariants should be the same in all views in the zero noise case. The off-diagonal elements give some indication of the usefulness of the invariants in distinguishing between sets of points (we would like these to be as close to 1 as possible – although there is, of course, no guarantee that this will be the case). We can see that, in terms of the diagonal elements being close to zero, the performance of the invariants based on trilinearities is better than those based on bilinearities. However, it appears that, for greater noise values, T has slightly poorer distinguishing ability (i.e. off-diagonal elements are, on average, higher for F).

In the case of real images we use a sequence of images taken by a moving robot equipped with a binocular head. Figure 6 shows an example of images taken with the left and right eyes – the experimental setup roughly matched the simulations in terms of ratios of object distance to object size. Image pairs, one from the left sequence and one from the right sequence were taken to form invariants using F. For the formation of invariants using T, two from the left and one from the right sequence were used. 38 points were semi-automatically taken and 6 sets of 6 general points were selected. The vector of invariants for each set was formed using both F and T and the set of distances found are shown in figure 7. Again, the diagonal elements are smaller for the invariants calculated using T, and now, differing from the simulations, we also see that the off-diagonal components are, on average, larger for T. We therefore see that computing the invariants from 3 views appears to be more robust and useful than computing them from 2 views – one would expect this from a theoretical viewpoint. Another reason for preferring the invariants formed from three views is that degenerate or almost degenerate configurations of points will be less likely.

Invariants using F

0	0.59	0.67	0.460
	0	0.515	0.68
		0.59	0
			0.69

0.148	0.600	0.920	0.724
	0.60	0.96	0.755
		0.71	0.97
			0.596

0.90	0.838	0.69	0.96
	0.276	0.693	0.527
		0.98	0.59
			0.663

Invariants using T

0	0.59	0.31	0.63
	0	0.63	0.338
		0.134	0.67
			0.29

0.031	0.1	0.352	0.66
	0.031	0.337	0.67
		0.31	0.67
			0.518

0.00	0.64	0.452	0.70
	0.063	0.77	0.545
		0.321	0.63
			0.643

Figure 5: The distance matrices show the performance of the invariants with increasing Gaussian noise σ: 0.005, 0.025 and 0.04 (left to right).

7 Conclusions

We have presented a brief introduction to the techniques of geometric algebra and shown how they can be used for projective geometry and in the formation and computation

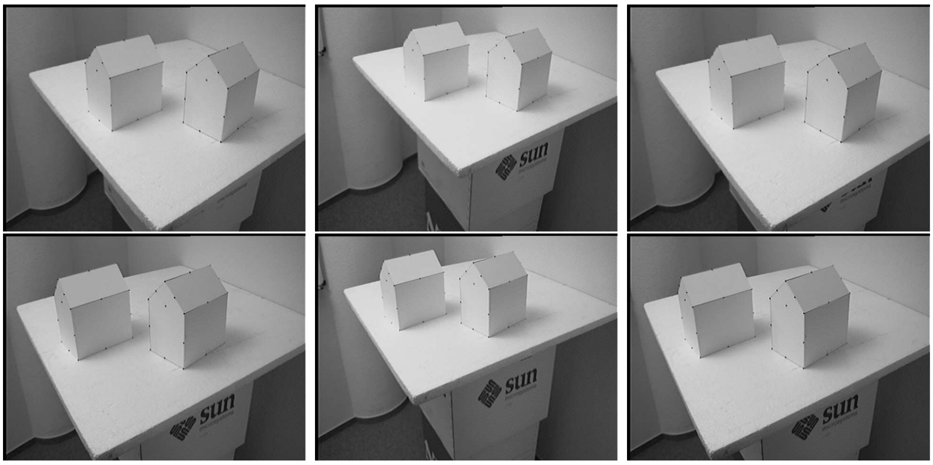

Figure 6: *I*mage sequence taken during navigation by the binocular head of a mobile robot. The upper and lower rows shows the left and right eye images respectively.

using F

0.04	0.79	0.646	0.130	0.679	0.89
	0.023	0.2535	0.278	0.268	0.89
		0.0167	0.723	0.606	0.862
			0.039	0.808	0.91
				0.039	0.808
					0.039

using T

0.021	0.779	0.346	0.930	0.759	0.81
	0.016	0.305	0.378	0.780	0.823
		0.003	0.83	0.678	0.97
			0.02	0.908	0.811
				0.008	0.791
					0.01

Figure 7: Distance matrices showing the performance of the computed invariants using bilinearities (left) and trilinearities (right) for the real image sequence.

of invariants. For intersections of planes, lines etc. and for the discussion of projective transformations it is useful to work in a 4D space we have called R^4. We find that we do not need to invoke the standard concepts or machinery of classical projective geometry, all that is needed is the idea of the *projective split* relating the quantities in R^4 to quantities in our 3D world. For real computations in the space R^4 we have a 4D geometric algebra with a Lorentzian metric. We can therefore use the extensive symbolic algebra packages (for use with MAPLE) which have been developed for work in relativity, quantum mechanics and cosmology using the *spacetime algebra*, also a 4D geometric algebra with a Lorentzian metric. Analysing problems using geometric algebra provides the enormous advantage of working in a system which can be used for most areas of computer vision and which has very powerful associated linear algebra and calculus frameworks. In addition, we have shown how the geometric insight provided by GA can be used to extend existing work on 3D projective invariants from two views to three views and have given explicit expressions for forming such invariants in terms of measurable quantities. Initial results indicate that such invariants are more robust than their 2-view counterparts.

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