Geometric Algebra: a Framework for Computing Point and Line Correspondences and Projective Structure using n Uncalibrated Cameras

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Abstract

In this paper we present geometric algebra as a system for analysing the geometry of multiple-view images. The power of this approach is illustrated by giving purely geometric derivations of the constraints for point and line correspondences in n-views and via a discussion of projective structure.

1. Introduction

Geometric algebra is a coordinate-free approach to geometry based on the algebras of Grassmann [5] and Clifford [3]. A basic introduction to the algebra is given in the accompanying paper in these proceedings [10] and in [1, 9], while a more complete treatment can be found in [8]. [10] also outlines the formulation of projective geometry using geometric algebra, the associated linear algebra framework and the interpretation of projective transformations. Using these basic results, this paper will discuss the algebra of incidence and then use this to formulate multiple view constraints and address projective reconstruction.

1.1. Algebra in projective space

This section will use the notation established in [10]. Consider three non-collinear points, P_1, P_2, P_3, represented by vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in $\mathcal{E}^3$ and by vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in R^4. The line L_{12} joining points P_1 and P_2 can be expressed in R^4 by the following bivector,

$$L_{12} = \mathbf{X}_1 \wedge \mathbf{X}_2. \tag{1}$$

Any point P, represented in R^4 by $\mathbf{X}$, on the line through P_1 and P_2, will satisfy

$$\mathbf{X} \wedge L_{12} = \mathbf{X} \wedge \mathbf{X}_1 \wedge \mathbf{X}_2 = 0. \tag{2}$$

This is therefore the constrained equation of the line in R^4. In general such an equation is telling us that $\mathbf{X}$ belongs to the subspace spanned by $\mathbf{X}_1$ and $\mathbf{X}_2$. Similarly, the plane Φ_{123} passing through points P_1, P_2, P_3 is expressed by the following trivector in R^4

$$\Phi_{123} = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3. \tag{3}$$

In $\mathcal{E}^3$ there are generally three types of intersections we wish to consider. We will look at two of these cases as an illustration and for these we will require the following general result, which gives the inner product of an r-blade, $A_r = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots\ldots \wedge \boldsymbol{a}_r$, and an s-blade, $B_s = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots\ldots \wedge \boldsymbol{b}_s$ (for $s \leq r$)

$$B_s \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots \wedge \boldsymbol{a}_r) =$$

$$\sum_{\boldsymbol{j}} \epsilon(j_1 j_2 \ldots. j_r) B_s \cdot (\boldsymbol{a}_{j_1} \wedge \boldsymbol{a}_{j_2} \wedge \ldots. \wedge \boldsymbol{a}_{j_s}) \boldsymbol{a}_{j_s+1} \wedge \ldots \wedge \boldsymbol{a}_{j_r} \tag{4}$$

where we sum over all combinations $\boldsymbol{j} = (j_1, j_2, \ldots., j_r)$ such that $j_1 < j_2 < \ldots. < j_r$ and $\epsilon(j_1 j_2 \ldots j_r) = +1$ or -1 according as $\boldsymbol{j}$ is an even or odd permutation of $(1, 2, 3, \ldots, r)$. See [8] for further discussion of this result.

1.1.1 Intersection of a line and a plane

Consider a line $A = \mathbf{X}_1 \wedge \mathbf{X}_2$ intersecting a plane $\Phi = \mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3$ – all vectors are in R^4. The intersection point is expressible using the *meet* operation

$$A \vee \Phi = (\mathbf{X}_1 \wedge \mathbf{X}_2) \vee (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3). \tag{5}$$

Using the definition of the meet given in [10] we have

$$A \vee \Phi = -A^* \cdot \Phi \tag{6}$$

since the pseudoscalar of R^4 (which we call I or sometimes I_4) squares to -1. This leads to

$$A^* \cdot \Phi = (AI^{-1}) \cdot \Phi = -(AI) \cdot \Phi. \tag{7}$$

Using equation 4 we can then expand the meet $A \vee \Phi$ as

$$\{(AI)\cdot(\mathbf{Y}_2\wedge\mathbf{Y}_3)\}\mathbf{Y}_1 + \{(AI)\cdot(\mathbf{Y}_3\wedge\mathbf{Y}_1)\}\mathbf{Y}_2 + \{(AI)\cdot(\mathbf{Y}_1\wedge\mathbf{Y}_2)\}\mathbf{Y}_3. \quad (8)$$

If $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4]$ is taken to be the magnitude of the pseudoscalar formed from the four vectors, then with some manipulation the meet reduces to (neglecting an overall minus sign)

$$A \vee \Phi = [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_2\mathbf{Y}_3]\mathbf{Y}_1 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3\mathbf{Y}_1]\mathbf{Y}_2 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_1\mathbf{Y}_2]\mathbf{Y}_3 \quad (9)$$

giving the intersection point (vector in R^4). Note that this is precisely the expansion of the meet given by the Grassmann-Cayley algebra [2]. We must identify the r-extensors of the Grassmann-Cayley algebra with r-blades in our geometric algebra.

The equivalent bracket in $\mathcal{E}^3$ is formed by evaluating the following volume

$$(\boldsymbol{x}_2 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_3 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_4 - \boldsymbol{x}_1)I_3^{-1}, \quad (10)$$

where we use the idea of the *projective split* discussed in [10], $\boldsymbol{x}_i = \frac{\mathbf{X}_i\wedge\gamma_4}{\mathbf{X}_i\cdot\gamma_4}$. We can summarize the above relationships between the brackets of 4 points in R^4 and $\mathcal{E}^3$ as follows

$$[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1} \equiv \{(\boldsymbol{x}_2 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_3 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_4 - \boldsymbol{x}_1)\}I_3^{-1}$$

1.1.2 Intersection of two planes

The intersection of two planes $\Phi_1 = \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3$ and $\Phi_2 = \mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3$ is given by the meet of Φ_1 and Φ_2;

$$\Phi_1 \vee \Phi_2 = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3) \vee (\mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3). \quad (11)$$

As before, this can be expanded using the definition of the meet and the fact that $(\Phi_1 I)\cdot\mathbf{Y}_i \equiv -[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_i]$, to give

$$\Phi_1\vee\Phi_2 = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_1](\mathbf{Y}_2\wedge\mathbf{Y}_3) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_2](\mathbf{Y}_3\wedge\mathbf{Y}_1) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_3](\mathbf{Y}_1\wedge\mathbf{Y}_2), \quad (12)$$

producing a line of intersection (bivector in R^4). Identifying 2-extensors with bivectors , the above expansion is seen to be the same as the expressions given in [2].

The intersection of two lines can be similarly discussed. The 4D algebra described above has been implemented using the computer algebra package MAPLE.

2. Point and line correspondences

We will now look at point and line correspondences between two, three and n cameras. For the analysis, let $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$, $(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ .... $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ define the image planes in views 1, 2, 3,..,n and let $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{c}_0$, ... $\boldsymbol{n}_0$ be the corresponding optical centres. We start with the well-understood case of two cameras.

2.1. Two cameras: the bilinear constraint

The projections of a world point $\mathbf{P}_i$ (represented by $\boldsymbol{x}_i$ and $\mathbf{X}_i$ in $\mathcal{E}^3$ and R^4) will be $\boldsymbol{a}'_i$ and $\boldsymbol{b}'_i$ in the two image planes and the R^4 representations of these quantities will be denoted by uppercase vectors, e.g. $\mathbf{A}'_i$ and $\mathbf{B}'_i$. $\mathbf{A}'_i$ can be expressed as the intersection of a line and image plane 1 (see figure 2, [10]):

$$\mathbf{A}'_i = (\mathbf{A}_0 \wedge \mathbf{X}_i) \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3) \quad (13)$$

$$= [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1 + [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_3\mathbf{A}_1]\mathbf{A}_2 + [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_1\mathbf{A}_2]\mathbf{A}_3$$

and similarly for $\mathbf{B}'_i$ and $\mathbf{C}'_i$. We can define three planes through the optical centre of each camera, for example, Φ_{1j}, $j = 1, 2, 3$ are planes through $\mathbf{A}_0$ defined by

$$\Phi_{11} = \mathbf{A}_0\wedge\mathbf{A}_2\wedge\mathbf{A}_3, \quad \Phi_{12} = \mathbf{A}_0\wedge\mathbf{A}_3\wedge\mathbf{A}_1, \quad \Phi_{13} = \mathbf{A}_0\wedge\mathbf{A}_1\wedge\mathbf{A}_2. \quad (14)$$

Taking two views, say 1 and 2, the epipoles are defined as the intersections $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of the line joining $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ with the image planes. In R^4, $\mathbf{E}_1$ and $\mathbf{E}_2$ are found easily as pointed out by Carlsson [2], for example;

$$\mathbf{E}_1 = (\mathbf{A}_0\wedge\mathbf{B}_0) \vee (\mathbf{A}_1\wedge\mathbf{A}_2\wedge\mathbf{A}_3) \quad (15)$$

$$= [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1 + [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_3\mathbf{A}_1]\mathbf{A}_2 + [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_1\mathbf{A}_2]\mathbf{A}_3.$$

Since $\mathbf{E}_1$ lies in the plane defined by $\mathbf{A}_0\wedge\mathbf{B}_0\wedge\mathbf{B}_i$ for any i, we have

$$\mathbf{E}_1\wedge(\mathbf{A}_0\wedge\mathbf{B}_0\wedge\mathbf{B}_i) = 0. \quad (16)$$

Expanding this equation leads to $\mathcal{F}^T\boldsymbol{\varepsilon} = 0$, where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ if $\mathbf{E}_1 = \varepsilon_1\mathbf{A}_1 + \varepsilon_2\mathbf{A}_2 + \varepsilon_3\mathbf{A}_3$ and $\mathcal{F}$ is the well known fundamental matrix ($(\mathcal{F})_{ij} = [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_i\mathbf{B}_j]$). The coordinate vector of the epipolar point of view 1 therefore corresponds to the null-space of the transpose of the fundamental matrix $\mathcal{F}$. Now, the epipolar constraint is simply that $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{a}'_i$, $\boldsymbol{b}'_i$ are coplanar if $\boldsymbol{a}'_i$ and $\boldsymbol{b}'_i$ are projections of the same world point. This can be concisely written as $L_A \wedge L_B = 0$ where $L_A = \mathbf{A}_0 \wedge \mathbf{A}'_i$ and $L_B = \mathbf{B}_0 \wedge \mathbf{B}'_i$ or $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_i\mathbf{B}'_i] = 0$. Expressed in terms of the $\mathbf{A}'_i$, $\mathbf{B}'_i$ this gives

$$[\mathbf{A}_0\mathbf{B}_0(\alpha_{i1}\mathbf{A}_1 + \alpha_{i2}\mathbf{A}_2 + \alpha_{i3}\mathbf{A}_3)(\beta_{i1}\mathbf{B}_1 + \beta_{i2}\mathbf{B}_2 + \beta_{i3}\mathbf{B}_3)] = \boldsymbol{\alpha_i}^T\mathcal{F}\boldsymbol{\beta_i} = 0. \quad (17)$$

The epipolar or bilinear constraint has also been expressed [4, 7] in terms of intersections of the planes Φ_{ij}. For example, lines $L_A = \mathbf{A}_0 \wedge \mathbf{A}'_i$ and $L_B = \mathbf{B}_0 \wedge \mathbf{B}'_i$ intersect if $L_A \wedge L_B = 0$. Since $L_A = \alpha_{i1}(\mathbf{A}_0 \wedge \mathbf{A}_1) + \alpha_{i2}(\mathbf{A}_0 \wedge \mathbf{A}_2) + \alpha_{i3}(\mathbf{A}_0 \wedge \mathbf{A}_3)$ and $\mathbf{A}_0 \wedge \mathbf{A}_1 = \Phi_{12} \vee \Phi_{13}$ etc., we can write $L_A \wedge L_B = 0$ as

$$(\alpha_{i1}\Phi_{12} \vee \Phi_{13} + \alpha_{i2}\Phi_{13} \vee \Phi_{11} + \alpha_{i3}\Phi_{11} \vee \Phi_{12})\wedge \quad (18)$$

$$(\beta_{i1}\Phi_{22} \vee \Phi_{23} + \beta_{i2}\Phi_{23} \vee \Phi_{21} + \beta_{i3}\Phi_{21} \vee \Phi_{22}) = 0.$$

This is equivalent to equation 4 of [4]. If we had a 3rd camera we would have two further constraints from $L_A \wedge L_C = 0$, ($[\mathbf{A}_0\mathbf{C}_0\mathbf{A}'_i\mathbf{C}'_i] = 0$) and $L_B \wedge L_C = 0$, ($[\mathbf{B}_0\mathbf{C}_0\mathbf{B}'_i\mathbf{C}'_i] = 0$).

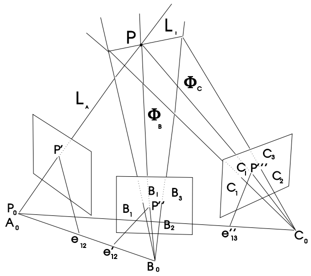

Figure 1. The trilinear constraint in terms of intersecting lines and planes.

2.2. Three cameras: the trilinear constraints

For point correspondences in three views we also have constraints of the following form;

$$L_A \wedge \{\Phi_{Bi} \vee \Phi_{Cj}\} = 0 \quad , L_B \wedge \{\Phi_{Ai} \vee \Phi_{Cj}\} = 0, \quad L_C \wedge \{\Phi_{Ai} \vee \Phi_{Bj}\} = 0 \quad (19)$$

where Φ_{Ak}, Φ_{Bk} and Φ_{Ck} are planes defined by $\Phi_{Ak} = \mathbf{A}_0 \wedge \mathbf{A}_k \wedge \mathbf{A}'_i$ etc. The first constraint in equation 19 is simply saying that line L_A and the line of intersection of planes Φ_{Bi} and Φ_{Cj} must intersect at a point – this point being P (drop subscript i on P, $\mathbf{A}'$ etc.), see Figure 1. Let us express this first constraint in terms of R^4 vectors

$$L_A \wedge \{\Phi_{Bi} \vee \Phi_{Cj}\} = \quad (20)$$

$$(\mathbf{A}_0 \wedge \mathbf{A}'_i) \wedge \{(\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}') \vee (\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}')\} = 0.$$

The points $\mathbf{A}', \mathbf{B}', \mathbf{C}'$ can also be expanded in terms of R^4 vectors; $\mathbf{A}' = \alpha_i \mathbf{A}_i$, $\mathbf{B}' = \beta_i \mathbf{B}_i$ and $\mathbf{C}' = \delta_i \mathbf{C}_i$. We can therefore write

$$\begin{aligned} \mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}' &= \beta_l (\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}_l) \equiv \beta_l \Phi^B_{il} \\ \mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}' &= \delta_m (\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}_m) \equiv \delta_m \Phi^C_{jm}, \end{aligned} \quad (21)$$

where we have now renamed the planes Φ_{11} etc. as given above. The constraint in equation 20 can now be written as

$$\alpha_k (\mathbf{A}_0 \wedge \mathbf{A}_k) \wedge \{\beta_l \delta_m (\Phi^B_{il} \vee \Phi^C_{jm})\} = 0 \quad (22)$$

which can be put into the form

$$\tilde{T}^{ij}_{klm} \alpha_k \beta_l \delta_m = 0 \quad (23)$$

where

$$\tilde{T}^{ij}_{klm} = [\mathbf{A}_0 \mathbf{A}_k (\Phi^B_{il} \vee \Phi^C_{jm})]. \quad (24)$$

This is a *trilinear constraint* [6]. There are obviously 9 possible choices of the pair (ij). However, by expanding the bracket in (24) it can be shown that only 4 of these are independent – say $(1,1)$, $(2,2)$, $(1,2)$ and $(2,1)$. Since we had three original constraints, this leads to a total of 12 trilinearity constraints as noted by [4]. We note here that our tensor $\tilde{T}^{ij}_{klm}$ is related to Hartley's tensor [7] via;

$$\tilde{T}^{ij}_{klm} \longrightarrow T_{pqr} \quad (25)$$

where $p = 1$ if $(i,l) = (2,3)$, $p = 2$ if $(i,l) = (1,3)$ and $p = 3$ if $(i,l) = (1,2), (2,1)$. Similarly, $q = 1$ if $(j,m) = (2,3)$ etc.. We also note that for given (i,j) only certain values of (l,m) give non-zero expressions for $\tilde{T}$.

2.3. Line correspondences between three cameras

Here we outline the derivation of the trilinear constraints for lines. We will not need Plücker coordinates or indeed any constructions that we have not already discussed in the point case.

Given world points P_1 and P_2, whose R^4 representations are $\mathbf{P}_1$ and $\mathbf{P}_2$, the line L_{12} joining $\mathbf{P}_1$ and $\mathbf{P}_2$ can be expressed as $L_{12} = \mathbf{P}_1 \wedge \mathbf{P}_2$. L_{12} projects down to lines in the three image planes, these are

$$L^A_{12} = \mathbf{A}'_1 \wedge \mathbf{A}'_2 \quad L^B_{12} = \mathbf{B}'_1 \wedge \mathbf{B}'_2 \quad L^C_{12} = \mathbf{C}'_1 \wedge \mathbf{C}'_2. \quad (26)$$

As before, we can expand $\mathbf{A}'_i$ as $\alpha^i_j \mathbf{A}_j$. L^A_{12} can then be expanded in terms of the 'basis bivectors' L^A_k as follows

$$L^A_{12} = l_k L^A_k \quad (27)$$

where $L^A_1 = \mathbf{A}_2 \wedge \mathbf{A}_3$, $L^A_2 = \mathbf{A}_3 \wedge \mathbf{A}_1$ and $L^A_3 = \mathbf{A}_1 \wedge \mathbf{A}_2$ and $l_1 = \alpha^1_2 \alpha^2_3 - \alpha^1_3 \alpha^2_2$ etc. Similarly we have

$$L^B_{12} = l'_l L^B_l, \quad L^C_{12} = l''_m L^C_m. \quad (28)$$

Note here that the coefficients which describe the lines l'_k and l''_k have been denoted as such to coincide with Hartley's notation. To arrive at a constraint between the lines in the image planes we note that the line L_{12} can be expressed as the meet of the planes $(\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2)$ and $(\mathbf{C}_0 \wedge \mathbf{C}'_1 \wedge \mathbf{C}'_2)$. Also $L^A_{12} = \mathbf{A}'_1 \wedge \mathbf{A}'_2$ can be written as the meet of planes $(\mathbf{A}_0 \wedge \boldsymbol{P}_1 \wedge \boldsymbol{P}_2)$ and $(\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3)$. We therefore have the identity

$$\begin{aligned} \mathbf{A}'_1 \wedge \mathbf{A}'_2 = \quad & \{\mathbf{A}_0 \wedge \{(\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2) \vee \\ & (\mathbf{C}_0 \wedge \mathbf{C}'_1 \wedge \mathbf{C}'_2)\}\} \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3). \end{aligned} \quad (29)$$

Using the expansions in terms of the line coefficients this reduces to

$$l_k L^A_k = \{\mathbf{A}_0 \wedge \{l'_l \Phi^B_l \vee l''_m \Phi^C_m\}\} \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3). \quad (30)$$

This can then be simplified using the definition of the meet;

$$l_k L^A_k = l'_l l''_m \{[\mathbf{A}_0 \{\Phi^B_l \vee \Phi^C_m\} \mathbf{A}_n] L^A_n\}. \quad (31)$$

From this it is clear that the relationship between the l's is

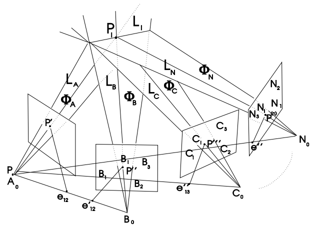

Figure 2. The quadrilinear constraint in terms of pairs of intersecting planes.

$$\begin{aligned} l_k &= l'_l l''_m \{[\mathbf{A}_0\{\Phi^B_l \vee \Phi^C_m\}\mathbf{A}_k] \\ &= l'_l l''_m \{[\mathbf{A}_0\mathbf{A}_k\{\Phi^B_l \vee \Phi^C_m\}] \\ &= l'_l l''_m T_{klm} \end{aligned} \quad (32)$$

from the definition of $\tilde{T}$ in terms of Hartley's tensor. This is precisely the constraint obtained by Hartley, but here it is arrived at via purely geometric reasoning.

2.4. Point correspondences between n cameras: n-linear constraints

In $\mathcal{E}^3$ there are just three important intersections; the intersection of a line and a plane, a plane and a plane, and a line and a line and it is therefore unlikely that taking more and more cameras will continue to give more constraint information. Using the condition that two lines intersect in space we can relate 4 views; we do this by joining two sets of intersecting planes, see Figure 2. If we have n views let us choose 4 of these views and denote them by A, B, C and N. $\Phi_{Aj} \vee \Phi_{Bk}$ gives a line passing through world point P as does $\Phi_{Cl} \vee \Phi_{Nm}$. We therefore have the condition

$$\{\Phi_{Aj} \vee \Phi_{Bk}\} \wedge \{\Phi_{Cl} \vee \Phi_{Nm}\} = 0. \quad (33)$$

If $\boldsymbol{N}' = \eta_1\boldsymbol{N}_1 + \eta_2\boldsymbol{N}_2 + \eta_3\boldsymbol{N}_3$ then this condition can be written as

$$\alpha_r\beta_s\delta_t\eta_u\{(\Phi^A_{jr} \vee \Phi^B_{ks})\wedge(\Phi^C_{lt} \vee \Phi^N_{mu})\} = 0. \quad (34)$$

The bracketed quantity above can be expanded in terms of the bilinear and trilinear constraints in a similar manner to that given by [4]. Therefore for a set up of n cameras or a moving sensor the general equation for computing bi- tri- and quadri-linear constraints is

$$\{\Phi_{Kk} \vee \Phi_{Ll}\} \wedge \{\Phi_{Mm} \vee \Phi_{Nn}\} = 0 \quad (35)$$

where K,L,M and N are any four of the n cameras or any four views from a moving observer. Note that this equation subsumes the two and three camera cases, i.e. for two cameras use L_K instead of $\{\Phi_{Kk} \vee \Phi_{Ll}\}$ and L_L instead of $\{\Phi_{Mm} \vee \Phi_{Nn}\}$ and for three cameras use L_K instead of $\{\Phi_{Kk} \vee \Phi_{Ll}\}$ and $\{\Phi_{Ll} \vee \Phi_{Mm}\}$ instead of $\{\Phi_{Mm} \vee \Phi_{Nn}\}$.

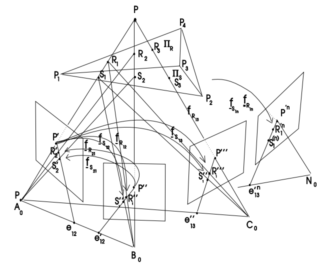

Figure 3. Invariant projective depth using n uncalibrated cameras.

3. Projective structure using n uncalibrated cameras

Here we will use the geometric algebra formulation of projective geometry to compute the *projective depth* discovered by Shashua [11]. Projective depth is simply the cross-ratio of projected points lying on an epipolar line of any of the n cameras.

3.1. Homomorphic transformations

Consider a world point P and 4 other distinct points $P_i, i = 1,2,3,4$ defining a tetrahedron. Let $\pi_R = \mathbf{P}_1\wedge\mathbf{P}_3\wedge\mathbf{P}_4$ and $\pi_S = \mathbf{P}_1\wedge\mathbf{P}_2\wedge\mathbf{P}_3$ and assume P does not lie on either of these two planes – see figure 3. Let $\mathbf{R}_i$ and $\mathbf{S}_i$ be the intersections of the line joining the optical centre of the ith camera with point P with the planes π_R and π_S, e.g. $\mathbf{R}_1 = \pi_R \vee (\mathbf{A}_0\wedge\mathbf{P})$. Let $\mathbf{R}^n_i$ and $\mathbf{S}^n_i$ be the projections of the points $\mathbf{R}_i$ and $\mathbf{S}_i$ onto the nth image planes – e.g. $\mathbf{R}^2_1 = (\mathbf{B}_0\wedge\mathbf{R}_1) \vee (\mathbf{B}_1\wedge\mathbf{B}_2\wedge\mathbf{B}_3)$ etc. Note that $\mathbf{R}^i_i$ and $\mathbf{S}^i_i$ are simply the projections of the world point P onto the ith image plane, e.g. $\mathbf{R}^1_1 = \mathbf{S}^1_1 = (\mathbf{A}_0\wedge\mathbf{P}) \vee (\mathbf{A}_1\wedge\mathbf{A}_2\wedge\mathbf{A}_3)$. Let us call the ith image plane ψ_i.

In order to compute a cross-ratio which will be defined later, we must be able to calculate the image coordinates of $\mathbf{R}^n_i, \mathbf{S}^n_i$. We can do this by finding the *homomorphic* transformations or homographies relating projected points in one image plane to the projected points in another. Consider the homography between image planes ψ_i and ψ_j due to the plane π_S. If the projections of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ onto ψ_i and ψ_j are $\{\mathbf{P}^i_k\}$ and $\{\mathbf{P}^j_k\}$, for $k = 1,2,3$, then the linear function $\underline{f}^S_{ij}$ representing this transformation must satisfy

$$\underline{f}^S_{ij}(\mathbf{P}^i_k) = \mathbf{P}^j_k \quad \text{for } k = 1,2,3. \quad (36)$$

Here we are working in R^3 so that the non-linear projective transformations in $\mathcal{E}^2$ (plane to plane) become linear – the above linear-function representation is outlined in [10]. Similarly, the corresponding homography due to the plane π_R is represented by the linear function $\underline{f}^R_{ij}$ given by

$$\underline{f}_{ij}^R(\mathbf{P}_k^i) = \mathbf{P}_k^j \quad \text{for } k = 1, 3, 4. \tag{37}$$

If four point correspondences from each plane are known then these linear functions can be recovered up to a scale factor by simple linear techniques. Since the homographies must map the epipole in one image plane onto the epipole in the other, we can choose the epipoles as the fourth point if these are known; $\underline{f}_{ij}^R(\mathbf{E}_{ji}) = \mathbf{E}_{ij}$ etc.

3.2. Computing the projective depth

The fundamental projective invariant in 1D is the cross-ratio. We can form a cross-ratio from the collinear points $\mathbf{P}, \mathbf{R}_1, \mathbf{S}_1, \mathbf{A}_0$, namely

$$\rho = \frac{(\mathbf{S}_1 \wedge \mathbf{A}_0) I_2^{-1}}{(\mathbf{R}_1 \wedge \mathbf{A}_0) I_2^{-1}} \frac{(\mathbf{R}_1 \wedge \mathbf{P}) I_2^{-1}}{(\mathbf{S}_1 \wedge \mathbf{P}) I_2^{-1}}. \tag{38}$$

See [10] for a discussion of the formation of invariants in the geometric algebra framework. ρ will be invariant when projected onto any other image plane. Consider this cross-ratio in the image plane of the second camera;

$$\rho = \frac{(\mathbf{S}_1^2 \wedge \mathbf{E}_{12}) I_2^{-1}}{(\mathbf{R}_1^2 \wedge \mathbf{E}_{12}) I_2^{-1}} \frac{(\mathbf{R}_1^2 \wedge \mathbf{P}^2) I_2^{-1}}{(\mathbf{S}_1^2 \wedge \mathbf{P}^2) I_2^{-1}}. \tag{39}$$

If we know the linear functions $\underline{f}_{12}^S, \underline{f}_{12}^R$, then we can write this ratio as

$$\rho = \frac{(\underline{f}_{12}^S(\mathbf{P}^1) \wedge \mathbf{E}_{12}) I_2^{-1}}{(\underline{f}_{12}^R(\mathbf{P}^1) \wedge \mathbf{E}_{12}) I_2^{-1}} \frac{(\underline{f}_{12}^S(\mathbf{P}^1) \wedge \mathbf{P}^2) I_2^{-1}}{(\underline{f}_{12}^R(\mathbf{P}^1) \wedge \mathbf{P}^2) I_2^{-1}}. \tag{40}$$

Recall that the homographies were determined only up to a scale factor, therefore we are free to choose the scale factors for $\underline{f}^S$ and $\underline{f}^R$ such that

$$\frac{(\underline{f}_{12}^S(\mathbf{P}^1) \wedge \mathbf{E}_{12}) I_2^{-1}}{(\underline{f}_{12}^R(\mathbf{P}^1) \wedge \mathbf{E}_{12}) I_2^{-1}} = 1. \tag{41}$$

Assuming that each homography found between planes i, j, is scaled in this way, we have a general form for the ratio, which we write as k, given by

$$k = \left((\underline{f}_{ij}^S(\mathbf{P}^i) \wedge \mathbf{P}^j) I_2^{-1} \right) / \left((\underline{f}_{ij}^R(\mathbf{P}^i) \wedge \mathbf{P}^j) I_2^{-1} \right). \tag{42}$$

This is the invariant termed *projective depth* in [11]. If we have a number of views available then, in this framework, a more robust estimate of k would be given by

$$k = \frac{1}{n} \sum_{(i \neq j)} \frac{(\underline{f}_{ij}^S(\mathbf{P}^i) \wedge \mathbf{P}^j) I_2^{-1}}{(\underline{f}_{ij}^R(\mathbf{P}^i) \wedge \mathbf{P}^j) I_2^{-1}}, \tag{43}$$

where n is the number of estimates used. We can write (42) as

$$(\underline{f}_{ij}^R(\mathbf{P}^i) \wedge \mathbf{P}^j) = k (\underline{f}_{ij}^S(\mathbf{P}^i) \wedge \mathbf{P}^j), \tag{44}$$

which can then be rearranged to give

$$(\underline{f}_{ij}^R(\mathbf{P}^i) + k \underline{f}_{ij}^S(\mathbf{P}^i)) \wedge \mathbf{P}^j = 0. \tag{45}$$

This tells us that $\mathbf{P}^j$ is parallel to $(\underline{f}_{ij}^R(\mathbf{P}^i) + k \underline{f}_{ij}^S(\mathbf{P}^i))$. Therefore, if the linear functions $\underline{f}_{ij}^S, \underline{f}_{ij}^R$ and the invariant k are known, we can projectively reconstruct $\mathbf{P}^j$.

4. Conclusions

This paper shows how the constraints relating point and line correspondences in multiple views are formed in a geometrically intuitive manner using the geometric algebra framework. For the case of three views, the derivations of the trilinear constraints are derived using nothing more than the intersections of planes and lines – there is no introduction of matrices or Plücker coordinates. Finally, we illustrate how to form a useful projective invariant [11] in this framework.

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