Geometric Algebra: a Framework for Computing Invariants in Computer Vision

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Abstract

In this paper we present *geometric algebra* as a new and complete framework for the theory and computation of invariants in computer vision and compare it with the currently popular Grassmann-Cayley (or Double) algebra. We will show that geometric algebra is a very elegant language for expressing all the ideas of projective geometry and is a system in which real computer implementations are straightforward. Using these techniques we will try to resolve some recent confusion in the literature over the formation of 3D projective invariants in terms of image coordinates.

1 Introduction

Geometric algebra is a coordinate-free approach to geometry based on the algebras of Grassmann [?] and Clifford [?]. The algebra is defined on a space whose elements are called *multivectors* and it has an associative and fully invertible product called the **geometric** or **Clifford** product. Some preliminary applications of geometric algebra in the field of computer vision have already been given [?, ?, ?], and here we extend these applications to include the study of geometric invariance. Geometric algebra provides a very natural language for projective geometry and has all the necessary equipment for the tasks which the Grassmann-Cayley algebra is currently used for. The Grassmann-Cayley algebra expresses the ideas of projective geometry, such as the meet and join, very elegantly, but it lacks an inner (regressive) product and some other key concepts. The next section will give a brief introduction to geometric algebra. For a more complete introduction see [?] and for other brief summaries see [?, ?, ?]. Given this background we can look at the familiar concepts of projective space and homogeneous coordinates, outline the formulation of projective geometry in the geometric algebra and introduce the concept of the *projective split*. We then deal with projective transformations and illustrate the formation of 1D, 2D and 3D projective invariants in this framework. We will illustrate the comparisons between our methods and those of the Grassmann-Cayley algebra by considering 3D projective invariants and discussing what we believe to be confusion which has occurred over this issue in the recent literature.

2 Geometric Algebra: an outline

The algebras of Clifford and Grassmann are well known to pure mathematicians, but were long ago abandoned by physicists in favour of the vector algebra of Gibbs, which is indeed what is most commonly used today in most areas of physics. The approach to Clifford algebra we adopt here was pioneered in the 1960's by David Hestenes [?] who has, since then, worked on developing his version of Clifford algebra – which will be referred to as *geometric algebra* – into a unifying language for mathematics and physics.

2.1 Basic Definitions

Let $\mathcal{G}_n$ denote the geometric algebra of n-dimensions – this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford** product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ and can be expressed as a sum of its symmetric and antisymmetric parts

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}, \tag{1}$$

where the inner product $\boldsymbol{a}\cdot\boldsymbol{b}$ and the outer product $\boldsymbol{a}\wedge\boldsymbol{b}$ are defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab}+\boldsymbol{ba}) \tag{2}$$

$$\boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab}-\boldsymbol{ba}). \tag{3}$$

The inner product of two vectors is the standard *scalar* or *dot* product and produces a scalar. The outer or wedge product of two vectors is a new quantity we call a **bivector**. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ – see Figure **??**.

Thus, $\boldsymbol{b}\wedge\boldsymbol{a}$ will have the opposite orientation making the wedge product anticommutative as given in equation **??**. The outer product is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a **trivector**, is interpreted as the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$. The outer product of k vectors is a k-vector or k-blade, and such a quantity is said to have *grade* k. A **multivector** (linear combination of objects of different type) is *homogeneous* if it contains terms of only a single grade. The geometric algebra provides a means of manipulating multivectors which allows us to keep track of different grade objects simultaneously – much as one does with complex number operations.

In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist since there is no possibility of sweeping the volume element $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ over a 4th dimension. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I and is crucial when discussing duality.

2.2 The Geometric Algebra of 3-D Space

In an n-dimensional space we can introduce an orthonormal basis of vectors $\{\sigma_i\}$ $i = 1, ..., n$, such that $\sigma_i \cdot \sigma_j = \delta_{ij}$. This leads to a basis for the entire algebra:

$$1, \quad \{\sigma_i\}, \quad \{\sigma_i \wedge \sigma_j\}, \quad \{\sigma_i \wedge \sigma_j \wedge \sigma_k\}, \quad \ldots, \quad \sigma_1 \wedge \sigma_2 \wedge \ldots \wedge \sigma_n. \tag{4}$$

Note that we shall not use bold symbols for these basis vectors. Any multivector can be expressed in terms of this basis. The basis for the 3-D space has $2^3 = 8$ elements given by:

$$\underbrace{1}_{scalar}, \underbrace{\{\sigma_1, \sigma_2, \sigma_3\}}_{vectors}, \underbrace{\{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\}}_{bivectors}, \underbrace{\{\sigma_1\sigma_2\sigma_3\} \equiv i}_{trivector}. \tag{5}$$

It can easily be verified that the trivector or pseudoscalar $\sigma_1\sigma_2\sigma_3$ squares to -1 and commutes with all multivectors in the 3-D space. We therefore give it the symbol i; noting that this is not the uninterpreted commutative scalar imaginary j used in quantum mechanics and engineering. Multiplication of the three basis vectors $\{\sigma_i\}$ by i results in the three basis bivectors;

$$\sigma_1\sigma_2 = i\sigma_3 \quad \sigma_2\sigma_3 = i\sigma_1 \quad \sigma_3\sigma_1 = i\sigma_2 \tag{6}$$

These simple bivectors rotate vectors in their own plane by 90° and it can easily be shown that the quaternion algebra is simply a subset of the geometric algebra of 3-space.

2.3 Formulation of Projective Geometry

Here we will outline the approach pioneered by Hestenes for using geometric algebra to discuss the algebra of incidence. For a more extended discussion we refer the reader to [?].

A geometric algebra $\mathcal{G}_n$ can be written as $\mathcal{G}(p, q)$ where p and q are the dimensions of the maximal subspaces with positive and negative signatures respectively (the signature of a vector $\boldsymbol{a}$ is positive, negative or zero according as $\boldsymbol{a}^2 > 0, < 0, = 0$) – for real applications we find it useful to specify the signature to facilitate actual computations. We will see later that we adopt the standard Euclidean signature $\mathcal{G}(3, 0)$ for ordinary space, $\mathcal{E}^3$, but that we are forced to adopt a signature of $\mathcal{G}(1, 3)$ for the 4-dimensional space we associate with the projective space.

In Euclidean spaces of 2 and 3 dimensions the unit pseudoscalar squares to -1. In $\mathcal{G}(1, 3)$ it is easy to see that this is also the case. If γ_i, $i = 1, 2, 3, 4$ are our basis vectors in the 4D space, and ${\gamma_j}^2 = -1$ for j=1,2,3 and $\gamma_4^2 = +1$, then

$$(\gamma_1\gamma_2\gamma_3\gamma_4)(\gamma_1\gamma_2\gamma_3\gamma_4) = (\gamma_2\gamma_3\gamma_4)(\gamma_2\gamma_3\gamma_4) = -(\gamma_3\gamma_4)(\gamma_3\gamma_4) = -1. \tag{7}$$

The sign of I^2 depends on the signature of the space. In a given space any pseudoscalar P can be written as $P = \alpha I$ where α is a scalar. If I^{-1} is the inverse of I, so that $II^{-1} = 1$, then,

$$PI^{-1} = \alpha II^{-1} = \alpha \equiv [P] \tag{8}$$

where we have defined the **bracket** of the pseudoscalar P, $[P]$, as its magnitude, arrived at by multiplication on the right by I^{-1}. This bracket is precisely the same as the bracket of the Grassmann-Cayley algebra. The sign of the bracket does not depend on the signature of the space.

To introduce the concepts of duality we define the dual A^* of an r-vector A as

$$A^* = AI^{-1}. \tag{9}$$

¿From this definition we see that the dual of an r-vector is an $(n-r)$-vector (e.g. duality of lines $(r=1)$ and planes $(n-r=3-1)$ in 3 space). In an n-dimensional space, if A is an r-vector and B is an s-vector (such that $r+s=n$), it can be shown that the following identity for the dual of the outer product holds

$$[A\wedge B] = (A\wedge B)I^{-1} = A\cdot B^*. \tag{10}$$

We note that duality is simply multiplication by an element of the algebra, and that there is therefore no need to introduce a special operator or any concept of a different space.

In an n-dimensional geometric algebra one can define the **join** $J = A \bigwedge B$ of an r-vector A and an s-vector B by

$$J = A\wedge B \qquad \text{if } A \text{ and } B \text{ are linearly independent.} \tag{11}$$

If A and B are not linearly independent the join is not given simply by the wedge but by the subspace that they span. J can be interpreted as a 'common dividend of lowest grade' and is defined up to a scale factor. It is easy to see that if $(r+s) \geq n$ then J will be the pseudoscalar for the space. In what follows we will use $\bigwedge$ for the join only when the blades A and B are not linearly independent, otherwise we will use the ordinary outer product, $\wedge$.

If A and B have a common factor (i.e. there exists a k-vector C such that $A = A'C$ and $B = B'C$ for some A', B') then we can define the 'intersection' or **meet** of A and B as $A \vee B$ where

$$(A \vee B)^* = A^* \wedge B^*. \tag{12}$$

That is, the dual of the meet is given by the join of the duals. In equation **??** the dual of $(A \vee B)$ is understood to be taken with respect to the *join* of A and B. In most cases of practical interest this join will be the whole space and the meet is therefore easily computed so that a more useful expression for the meet is obtained as follows

$$A \vee B = (A^* \wedge B^*)I = (A^* \wedge B^*)(I^{-1}I)I = \pm(A^* \cdot B) \tag{13}$$

according as $I^2 = \pm 1$. We therefore have the very simple and readily computed relation of $A \vee B = \pm(A^* \cdot B)$. The above concepts are discussed further in [**?**].

2.4 Linear Algebra

In this section we will give a brief review of the geometric algebra approach to linear algebra. More detailed reviews can be found in [**?**, **?**].

Consider a linear function f which maps vectors to vectors in the same space. We can extend f to act linearly on multivectors via the **outermorphism**, $\underline{f}$, defining the action of $\underline{f}$ on blades by

$$\underline{f}(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\ldots\wedge\boldsymbol{a}_r) = \underline{f}(\boldsymbol{a}_1)\wedge\underline{f}(\boldsymbol{a}_2)\wedge\ldots\wedge\underline{f}(\boldsymbol{a}_r). \tag{14}$$

We use the term outermorphism because $\underline{f}$ preserves the grade of any r-vector it acts on. The action of $\underline{f}$ on general multivectors is then defined through linearity. $\underline{f}$ must therefore satisfy:

$$
\begin{aligned}
\underline{f}(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2) &= \underline{f}(\boldsymbol{a}_1) \wedge \underline{f}(\boldsymbol{a}_2) \\
\underline{f}(A_r) &= \langle \underline{f}(A_r) \rangle_r \\
\underline{f}(\alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2) &= \alpha_1 \underline{f}(\boldsymbol{a}_1) + \alpha_2 \underline{f}(\boldsymbol{a}_2)
\end{aligned} \tag{15}
$$

The outermorphism of a product of two linear functions is therefore the product of the outermorphisms, i.e. if $f(\boldsymbol{a}) = f_2(f_1(\boldsymbol{a}))$, then we can write $\underline{f} = \underline{f}_2 \underline{f}_1$.
Since the outermorphism preserves grade, we know that the pseudoscalar of the space must be mapped onto some multiple of itself. The scale factor in this mapping is the **determinant** of $\underline{f}$;

$$
\underline{f}(I) = \det(\underline{f}) I. \tag{16}
$$

This is much simpler than many definitions of the determinant. Using this definition, most properties of determinants can be established with little effort.

3 Projective Space and Projective Transformations

In classical projective geometry one defines a 3D space, $\mathcal{P}^3$, whose points are in $1-1$ correspondence with lines through the origin in a 4D space, R^4. Similarly, k-dimensional subspaces of $\mathcal{P}^3$ are identified with $(k+1)$-dimensional subspaces of R^4. Such projective views can provide very elegant descriptions of the geometry of incidence (intersections, unions etc.), but in order to carry out any real computations one is forced to introduce some sort of basis and associated metric. From a mathematical viewpoint the projective space, $\mathcal{P}^3$, would have no metric, the basis and metric are introduced in the associated 4D space. In this 4D space a coordinate description of a projective point is conventionally brought about by using *homogeneous coordinates*. The usefulness of the projective description of space is often only realised via the introduction of such homogeneous coordinates.

3.1 The Projective Split

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$, a 3D space with a Euclidean metric. Since any point on a line through some origin O will be mapped to a single point in the image plane, we will find it useful to associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. In these two distinct but related spaces we define basis vectors: $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ in R^4 and $\{\sigma_1, \sigma_2, \sigma_3\}$ in $\mathcal{E}^3$. We identify R^4 and $\mathcal{E}^3$ with the geometric algebras of 4 and 3 dimensions, $\mathcal{G}_4$ and $\mathcal{G}_3$. We require that vectors, bivectors and trivectors in R^4 will represent points, lines and planes in $\mathcal{E}^3$. Suppose we choose γ_4 as a selected direction in R^4, we can then define a mapping which associates the bivectors $\gamma_i \gamma_4$, $i = 1, 2, 3$, in R^4 with the vectors σ_i, $i = 1, 2, 3$, in $\mathcal{E}^3$;

$$
\sigma_1 \equiv \gamma_1 \gamma_4 \quad \sigma_2 \equiv \gamma_2 \gamma_4 \quad \sigma_3 \equiv \gamma_3 \gamma_4. \tag{17}
$$

To preserve the Euclidean structure of the spatial vectors $\{\sigma_i\}$ (i.e. $\sigma_i^2 = +1$) it is easy to see that we are forced to assume a non-Euclidean metric for the basis vectors in R^4. We choose to use

$\gamma_4^2 = +1$, $\gamma_i = -1$, $i = 1,2,3$. This process of associating the quantities in the higher dimensional space with quantities in the lower dimensional space is an application of what Hestenes calls the **projective split**.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is obtained by taking the geometric product of $\mathbf{X}$ and γ_4;

$$\mathbf{X}\gamma_4 = \mathbf{X}\cdot\gamma_4 + \mathbf{X}\wedge\gamma_4 = X_4\left(1 + \frac{\mathbf{X}\wedge\gamma_4}{X_4}\right) \equiv X_4(1+\boldsymbol{x}). \tag{18}$$

Note that $\boldsymbol{x}$ contains terms of the form $\gamma_1\gamma_4$, $\gamma_2\gamma_4$, $\gamma_3\gamma_4$ or, via the associations in equation **??**, terms in $\sigma_1, \sigma_2, \sigma_3$. We can therefore think of the vector $\boldsymbol{x}$ as a vector in $\mathcal{E}^3$ which is associated with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4.

If we start with a vector $\boldsymbol{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ in $\mathcal{E}^3$, we can represent this in R^4 by the vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ such that

$$\begin{aligned}\boldsymbol{x} &= \frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\gamma_1\gamma_4 + \frac{X_2}{X_4}\gamma_2\gamma_4 + \frac{X_3}{X_4}\gamma_3\gamma_4 \\ &= \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3,\end{aligned} \tag{19}$$

$\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1,2,3$. The process of representing $\boldsymbol{x}$ in a higher dimensional space can therefore be seen to be equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$. Thus, in this geometric algebra formulation we postulate distinct spaces in which we represent ordinary 3D quantities and their 4D projective counterparts, together with a well-defined way of moving between these spaces.

3.2 Projective transformations

It is well known that there are various advantages to working in homogeneous coordinates.

If a general point (x, y, z) in 3-D space is projected onto an image plane, the coordinates (x', y') in the image plane will be related to (x, y, z) via a transformation of the form:

$$x' = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{20}$$

To make this non-linear transformation in $\mathcal{E}^3$ into a linear transformation in R^4 we define a linear function $\underline{f}_p$ mapping vectors onto vectors in R^4 such that the action of $\underline{f}_p$ on the basis vectors $\{\gamma_i\}$ is given by

$$\begin{aligned}\underline{f}_p(\gamma_1) &= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \tilde{\alpha}\gamma_4 \\ \underline{f}_p(\gamma_2) &= \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \tilde{\beta}\gamma_4 \\ \underline{f}_p(\gamma_3) &= \delta_1\gamma_1 + \delta_2\gamma_2 + \delta_3\gamma_3 + \tilde{\delta}\gamma_4 \\ \underline{f}_p(\gamma_4) &= \epsilon_1\gamma_1 + \epsilon_2\gamma_2 + \epsilon_3\gamma_3 + \tilde{\epsilon}\gamma_4\end{aligned} \tag{21}$$

A general point P in $\mathcal{E}^3$ given by $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ becomes the point $\mathbf{X} = (X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4)$ in R^4, where $x = X/W$, $y = Y/W$, $z = Z/W$. We can then see that $\underline{f}_p$ maps $\mathbf{X}$ onto $\mathbf{X}'$ where

$$\mathbf{X}' = \sum_{i=1}^{3}\{(\alpha_i X + \beta_i Y + \delta_i Z + \epsilon_i W)\gamma_i\} + (\tilde{\alpha} X + \tilde{\beta} Y + \tilde{\delta} Z + \tilde{\epsilon} W)\gamma_4 \tag{22}$$

The vector $\boldsymbol{x}' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3$ in $\mathcal{E}^3$ corresponds to $\mathbf{X}'$, where $\boldsymbol{x}'$ is as given in equation (**??**). Similar expressions are obtained for y' and z'.
Note that in general we would take $\alpha_3 = f\tilde{\alpha}$, $\beta_3 = f\tilde{\beta}$ etc. so that $z' = f$ (focal length), independent of the point chosen. Via this means the non-linear transformation in $\mathcal{E}^3$ becomes a linear transformation, $\underline{f}_p$, in R^4. Use of the linear function $\underline{f}_p$ makes the invariant nature of various quantities very easy to establish.

3.3 Algebra in projective space

There has been much recent interest in the use of the Grassmann-Cayley or double algebra as an elegant means of formulating the algebra of incidence [**?**, **?**, **?**, **?**]. Here we will show briefly how the main algebraic results of the Grassmann-Cayley algebra arise naturally when we express projective geometry in geometric algebra.
Consider three non-collinear points, P_1, P_2, P_3, represented by vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in $\mathcal{E}^3$ and by vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in R^4. The line L_{12} joining points P_1 and P_2 can be expressed in R^4 by the following bivector,

$$L_{12} = \mathbf{X}_1 \wedge \mathbf{X}_2. \tag{23}$$

Any point P, represented in R^4 by $\mathbf{X}$, on the line through P_1 and P_2, will satisfy

$$\mathbf{X} \wedge L_{12} = \mathbf{X} \wedge \mathbf{X}_1 \wedge \mathbf{X}_2 = 0. \tag{24}$$

This is therefore the equation of this line in R^4. In general such an equation is telling us that $\mathbf{X}$ belongs to the subspace spanned by $\mathbf{X}_1$ and $\mathbf{X}_2$.
Similarly, the plane Φ_{123} passing through points P_1, P_2, P_3 is expressed by the following trivector in R^4

$$\Phi_{123} = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3. \tag{25}$$

In $\mathcal{E}^3$ there are generally three types of intersections we wish to consider. We will look at two of these cases as an illustration and for these we will require the following general result, giving the inner product of an r-blade, $A_r = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots\ldots \wedge \boldsymbol{a}_r$, and an s-blade, $B_s = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots\ldots \wedge \boldsymbol{b}_s$ (for $s \leq r$)

$$B_s \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots\ldots \wedge \boldsymbol{a}_r) = \sum_{\boldsymbol{j}} \epsilon(j_1 j_2 \ldots. j_r) B_s \cdot (\boldsymbol{a}_{j_1} \wedge \boldsymbol{a}_{j_2} \wedge \ldots. \wedge \boldsymbol{a}_{j_s}) \boldsymbol{a}_{j_s+1} \wedge \ldots\ldots \wedge \boldsymbol{a}_{j_r} \tag{26}$$

where we sum over all combinations $\boldsymbol{j} = (j_1, j_2, \ldots., j_r)$ such that $j_1 < j_2 < \ldots. < j_r$. $\epsilon(j_1 j_2 \ldots j_r) = +1$ if $\boldsymbol{j}$ is an even permutation of $(1, 2, 3, \ldots, r)$ and -1 if it is an odd permutation. See [**?**] for further discussion of this result.

3.3.1 Intersection of a line and a plane

Consider a line $A = \mathbf{X}_1 \wedge \mathbf{X}_2$ intersecting a plane $\Phi = \mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3$ – all vectors are in R^4. The intersection point is expressible using the *meet* operation

$$A \vee \Phi = (\mathbf{X}_1 \wedge \mathbf{X}_2) \vee (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3). \tag{27}$$

Using the definition of the meet given in equation **??** we have

$$A \vee \Phi = -A^* \cdot \Phi \tag{28}$$

since the pseudoscalar for R^4, which we shall call I_4 if any ambiguity is possible, squares to -1. This leads to

$$A^* \cdot \Phi = (AI^{-1}) \cdot \Phi = -(AI) \cdot \Phi. \tag{29}$$

According to equation **??** we can then expand the meet as

$$\begin{aligned} A \vee \Phi &= (AI) \cdot (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3) \\ &= [\{(AI) \cdot (\mathbf{Y}_2 \wedge \mathbf{Y}_3)\}\mathbf{Y}_1 + \{(AI) \cdot (\mathbf{Y}_3 \wedge \mathbf{Y}_1)\}\mathbf{Y}_2 + \{(AI) \cdot (\mathbf{Y}_1 \wedge \mathbf{Y}_2)\}\mathbf{Y}_3]. \end{aligned} \tag{30}$$

Noting that $(AI) \cdot (\mathbf{Y}_i \wedge \mathbf{Y}_j)$ is a scalar, we can evaluate the above by taking scalar parts. If we write $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4]$ as a shorthand for the magnitude of the pseudoscalar formed from the four vectors, then with some manipulation we can see that the meet reduces to (neglecting an overall minus sign)

$$A \vee \Phi = [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_2\mathbf{Y}_3]\mathbf{Y}_1 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3\mathbf{Y}_1]\mathbf{Y}_2 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_1\mathbf{Y}_2]\mathbf{Y}_3 \tag{31}$$

giving the intersection point (vector in R^4). Note that this is precisely the expansion of the meet that would result from the analysis in the Grassmann-Cayley algebra [?, ?]. We can see that we must identify the r-extensors of the Grassmann-Cayley algebra with r-blades in our geometric algebra. Also, the definition of the bracket of four vectors in R^4 as the magnitude of the pseudoscalar formed from the outer product of the vectors is equivalent to its definition as the determinant of the four vectors in the Grassmann-Cayley algebra.
¿From the definition of the bracket given above it is easy to show that the equivalent bracket in $\mathcal{E}^3$ is formed by evaluating the following volume

$$(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1) I_3^{-1} \tag{32}$$

where, as before, $\boldsymbol{x}_i = \frac{\mathbf{X}_i \wedge \gamma_4}{\mathbf{X}_i \cdot \gamma_4}$. If the $W_i = 1$, we can summarize the above relationships between the brackets of 4 points in R^4 and $\mathcal{E}^3$ as follows

$$[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} \equiv \{(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1)\} I_3^{-1}. \tag{33}$$

3.3.2 Intersection of two planes

We now consider the intersection of two planes $\Phi_1 = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3$ and $\Phi_2 = \mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3$. The meet of Φ_1 and Φ_2 is given by

$$\Phi_1 \vee \Phi_2 = (\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3) \vee (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3). \tag{34}$$

As before, this can be expanded as

$$\Phi_1 \vee \Phi_2 = \{(\Phi_1 I) \cdot \mathbf{Y}_1\}(\mathbf{Y}_2 \wedge \mathbf{Y}_3) + \{(\Phi_1 I) \cdot \mathbf{Y}_2\}(\mathbf{Y}_3 \wedge \mathbf{Y}_1) + \{(\Phi_1 I) \cdot \mathbf{Y}_3\}(\mathbf{Y}_1 \wedge \mathbf{Y}_2). \tag{35}$$

Following the arguments of the previous section we can show that $(\Phi_1 I)\cdot\mathbf{Y}_i \equiv -[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_i]$, so that the meet is

$$\Phi_1 \vee \Phi_2 = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_1](\mathbf{Y}_2\wedge\mathbf{Y}_3) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_2](\mathbf{Y}_3\wedge\mathbf{Y}_1) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_3](\mathbf{Y}_1\wedge\mathbf{Y}_2), \qquad (36)$$

producing a line of intersection (bivector in R^4). If one identifies the 2-extensors of the Grassmann-Cayley algebra with bivectors in the geometric algebra, the above expansion is seen to be the same as the expressions given in [?].

The intersection of two lines can be similarly discussed.

The 4D algebra described above has been implemented using the computer algebra package MAPLE and all of the operations discussed here are easily evaluated.

4 Invariance using Geometric Algebra

In this section we will use the framework established so far to show how standard geometric invariants are derived in this approach.

4.1 1-D and 2-D Projective Invariants from a Single View

The 1-D Cross-Ratio

The '*fundamental projective invariant*' of points on a line is the so-called **cross-ratio**, ρ, defined as

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)},$$

where $t_1 = |PA|$, $t_2 = |PB|$, $t_3 = |PC|$, $t_4 = |PD|$, for points P, A, B, C, D on some line L. It is fairly easy to show that for the projection through some point O of the collinear points A, B, C, D onto any line L', ρ remains constant. For this 1D case, any point q on the line L can be written as $\boldsymbol{q} = t\sigma_1$ relative to P, where σ_1 is a unit vector in the direction of L. We then move up a dimension to a 2D space, with basis vectors (γ_1, γ_2), we will call R^2 in which $\boldsymbol{q}$ is represented by the vector $\mathbf{Q}$;

$$\mathbf{Q} = T\gamma_1 + S\gamma_2$$

where, as before, we associate $\boldsymbol{q}$ with the bivector $\frac{\mathbf{Q}\wedge\gamma_2}{\mathbf{Q}\cdot\gamma_2} = \frac{T}{S}\sigma_1$ so that $t = T/S$. When a point on line L is projected onto another line L', the distances t and t' are related by a projective transformation of the form

$$t' = \frac{\alpha t + \beta}{\tilde{\alpha} t + \tilde{\beta}}. \qquad (37)$$

This non-linear transformation in $\mathcal{E}^1$ can be made into a linear transformation in R^2 by defining the linear function $\underline{f}_1$ mapping vectors onto vectors in R^2;

$$\begin{aligned} \underline{f}_1(\gamma_1) &= \alpha_1\gamma_1 + \tilde{\alpha}\gamma_2 \\ \underline{f}_1(\gamma_2) &= \beta_1\gamma_1 + \tilde{\beta}\gamma_2. \end{aligned}$$

Consider 2 vectors $\mathbf{X}_1, \mathbf{X}_2$ in R^2. Form the bivector $\mathcal{S}_1 = \mathbf{X}_1 \wedge \mathbf{X}_2 = \lambda_1 I_2$, where $I_2 = \gamma_1\gamma_2$ is the pseudoscalar for R^2. We now look at how $\mathcal{S}_1$ transforms under $\underline{f}_1$:

$$\mathcal{S}_1' = \mathbf{X}_1' \wedge \mathbf{X}_2' = \underline{f}_1(\mathbf{X}_1 \wedge \mathbf{X}_2) = (\det \underline{f}_1)(\mathbf{X}_1 \wedge \mathbf{X}_2). \tag{38}$$

This last step follows since a linear function must map a pseudoscalar onto a multiple of itself, this multiple being the determinant of the function. Suppose that we now take 4 points on the line L whose corresponding vectors in R^2 are $\{\mathbf{X}_i\}$, $i = 1,..,4$, and consider the ratio $\mathcal{R}_1$ of 2 wedge products;

$$\mathcal{R}_1 = \frac{\mathbf{X}_1 \wedge \mathbf{X}_2}{\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{39}$$

Then, under $\underline{f}_1$, $\mathcal{R}_1 \rightarrow \mathcal{R}_1'$, where

$$\mathcal{R}_1' = \frac{\mathbf{X}_1' \wedge \mathbf{X}_2'}{\mathbf{X}_3' \wedge \mathbf{X}_4'} = \frac{(\det \underline{f}_1)\mathbf{X}_1 \wedge \mathbf{X}_2}{(\det \underline{f}_1)\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{40}$$

$\mathcal{R}_1$ is therefore invariant under $\underline{f}_1$. However, we want to express our invariants in terms of distances on the 1D line; for this we must consider how the bivector $\mathcal{S}_1$ in R^2 projects down to $\mathcal{E}^1$.

$$\begin{aligned} \mathbf{X}_1 \wedge \mathbf{X}_2 &= (T_1\gamma_1 + S_1\gamma_2) \wedge (T_2\gamma_1 + S_2\gamma_2) \\ &= (T_1S_2 - T_2S_1)\gamma_1\gamma_2 \\ &= S_1S_2(t_1 - t_2)I_2. \end{aligned} \tag{41}$$

In order to form a projective invariant which is independent of the choice of the arbitrary scalars S_i, we must then take *ratios* of the bivectors $\mathbf{X}_i \wedge \mathbf{X}_j$ (so that $\det \underline{f}_1$ cancels) and *multiples* of such ratios so that the S_i's cancel. More precisely, consider the following expression

$$Inv_1 = \frac{(\mathbf{X}_3 \wedge \mathbf{X}_1)I_2^{-1}(\mathbf{X}_4 \wedge \mathbf{X}_2)I_2^{-1}}{(\mathbf{X}_4 \wedge \mathbf{X}_1)I_2^{-1}(\mathbf{X}_3 \wedge \mathbf{X}_2)I_2^{-1}}$$

Then, in terms of distances along the lines, under the projective transformation $\underline{f}_1$, Inv_1 goes to Inv_1' where

$$Inv_1' = \frac{S_3S_1(t_3 - t_1)S_4S_2(t_4 - t_2)}{S_4S_1(t_4 - t_1)S_3S_2(t_3 - t_2)} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)}, \tag{42}$$

which is independent of the S_i's and is indeed the 1D classical projective invariant, the **cross-ratio**. Deriving the cross-ratio in this way enables us to easily generalize it to form invariants in higher dimensions.

The 2-D generalization of the Cross-Ratio

For points in a plane we again move up to a space with one higher dimension which we shall call R^3. Let a point P in the plane M be described by the vector $\boldsymbol{x}$ in $\mathcal{E}^2$ where $\boldsymbol{x} = x\sigma_1 + y\sigma_2$. In R^3 this point will be represented by $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3$ where $x = X/Z$ and $y = Y/Z$. As before, we can define a general projective transformation via a linear function $\underline{f}_2$ mapping vectors to vectors in R^3. We can then follow through the arguments used in the 1D case to form invariant quantities by taking *multiples* of *ratios* of **trivectors**, e.g.

$$Inv_2 = \frac{(\mathbf{X}_5 \wedge \mathbf{X}_4 \wedge \mathbf{X}_3)I_3^{-1}(\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_1)I_3^{-1}}{(\mathbf{X}_5 \wedge \mathbf{X}_1 \wedge \mathbf{X}_3)I_3^{-1}(\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4)I_3^{-1}}$$

We can then interpret this ratio in $\mathcal{E}^2$ as

$$Inv_2 = \frac{(\boldsymbol{x}_5 - \boldsymbol{x}_4)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_3)I_2^{-1}(\boldsymbol{x}_5 - \boldsymbol{x}_2)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_1)I_2^{-1}}{(\boldsymbol{x}_5 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_3)I_2^{-1}(\boldsymbol{x}_5 - \boldsymbol{x}_2)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_4)I_2^{-1}} = \frac{A_{543}A_{521}}{A_{513}A_{524}} \tag{43}$$

where $\frac{1}{2}A_{ijk}$ is the area of the triangle defined by the 3 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k$. This invariant is regarded as a 2D generalization of the 1D cross-ratio.

4.2 3-D Projective Invariants from Multiple Views

4.2.1 The 3-D generalization of the Cross-Ratio

When considering general points in $\mathcal{E}^3$ we have seen that we move up one dimension to work in the 4D space R^4. The point $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ in $\mathcal{E}^3$ is written as $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4$, where $x = X/W$, $y = Y/W$, $z = Z/W$. As before, a non-linear projective transformation in $\mathcal{E}^3$ becomes a linear transformation, described by the linear function $\underline{f}_3$ in R^4.

Again, as in the 1D and 2D case we form invariant quantities by taking *multiples* of *ratios* of **4-vectors**, e.g.

$$Inv_3 = \frac{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4)I_4^{-1}(\mathbf{X}_4 \wedge \mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6)I_4^{-1}}{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5)I_4^{-1}(\mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6)I_4^{-1}}. \tag{44}$$

Using the arguments of the previous sections we can write

$$(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4)I_4^{-1} \equiv W_1W_2W_3W_4\{(\boldsymbol{x}_2 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_3 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_4 - \boldsymbol{x}_1)\}I_3^{-1}. \tag{45}$$

We can therefore see that the invariant Inv_3 is the 3D equivalent of the 1D cross-ratio and consists of ratios of volumes;

$$Inv_3 = \frac{V_{1234}V_{4526}}{V_{1245}V_{3426}}, \tag{46}$$

where V_{ijkl} is the volume of the solid formed by the 4 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k, \boldsymbol{x}_l$.

Conventionally all of these invariants are well known but above we have outlined a general process for generating projective invariants in any dimension which is straightforward and simple.

4.2.2 3D projective invariants in terms of image coordinates

Suppose we have six general 3D points P_i, $i = 1,..,6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4 respectively. We have seen in section 4.2.1 that 3D projective invariants can be formed from these points, and an example of such an invariant is

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{47}$$

This is simply equation **??** rewritten in terms of brackets. If it is possible to express the bracket $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ in terms of the image coordinates of points P_i, P_j, P_k, P_l, then we can easily compute this invariant. Some of the most recent work on this problem has utilized the Grassmann-Cayley algebra [?, ?, ?]. It has been shown [?] that it is not possible to compute general 3D invariants from a single image and in [?] Carlsson tried to show how to compute such invariants from a pair of images in terms of the image coordinates and the fundamental matrix, $\boldsymbol{F}$, using the Grassmann-Cayley algebra. Subsequent work by Csurka and Faugeras [?] claims to have corrected some of Carlsson's expressions by including omitted scale factors. Here we will translate the approach of Carlsson into the geometric algebra framework in an attempt to address this question. We will also show that the claim of Csurka *et al.* is indeed correct but that the resolution lies simply in reordering the bracket decomposition rather than finding large numbers of omitted scale factors. Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1} = (\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1}. \tag{48}$$

The quantities $(\mathbf{X}_1 \wedge \mathbf{X}_2)$ and $(\mathbf{X}_3 \wedge \mathbf{X}_4)$ represent the line joining points P_1 and P_2, and P_3 and P_4. We let $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ be the centres of projection of the two cameras and suppose that the two camera image planes can be defined by the two sets of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ – see figure **??**. Let the projection of points $\{P_i\}$ through the centres of projection onto the image planes be given by the vectors $\{\boldsymbol{a}'_i\}$ and $\{\boldsymbol{b}'_i\}$. Note that this notation follows that of Carlsson [?] but that our vectors, $\boldsymbol{a}_i, \boldsymbol{b}_i$, etc. are ordinary vectors in $\mathcal{E}^3$. We then let the representations of these vectors in R^4 be $\mathbf{A}_i, \mathbf{B}_i, \mathbf{A}'_i, \mathbf{B}'_i$..., etc.

Expressing $\mathbf{X}_1\wedge\mathbf{X}_2$ (resp. $\mathbf{X}_3\wedge\mathbf{X}_4$) as the meet of the planes U_1 and V_1 (resp. U_2 and V_2) through the image points and the centres of projection gives

$$\mathbf{X}_1\wedge\mathbf{X}_2 = (\mathbf{A}_0\wedge\mathbf{A}'_1\wedge\mathbf{A}'_2) \vee (\mathbf{B}_0\wedge\mathbf{B}'_1\wedge\mathbf{B}'_2) \equiv U_1\wedge V_1 \tag{49}$$

$$\mathbf{X}_3\wedge\mathbf{X}_4 = (\mathbf{A}_0\wedge\mathbf{A}'_3\wedge\mathbf{A}'_4) \vee (\mathbf{B}_0\wedge\mathbf{B}'_3\wedge\mathbf{B}'_4) \equiv U_2\wedge V_2. \tag{50}$$

S_{1234} can therefore be written as $S_{1234} = \{(U_1 \vee V_1) \wedge (U_2 \vee V_2)\}\, I_4{}^{-1}$. ¿From the definition of the meet this equation can be simplified to

$$S_{1234} = -\,\{(U_1 \vee U_2)\wedge(V_1 \vee V_2)\}\, I_4{}^{-1}. \tag{51}$$

Using the similar expression for $V_1 \vee V_2$, S_{1234} then reduces to

$$S_{1234} = -\,\{[\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3][\mathbf{B}_0\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_4][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_4\mathbf{B}'_3] + [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4][\mathbf{B}_0\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_3][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_3\mathbf{B}'_4]\}\,. \tag{52}$$

Now consider a point in the first image plane which is the intersection of the lines joining points $\{\boldsymbol{a}'_1$ and $\boldsymbol{a}'_2\}$ and $\{\boldsymbol{a}'_3$ and $\boldsymbol{a}'_4\}$ – call this point $\boldsymbol{a}'_{1234}$. Let the R^4 representation of this point be $\mathbf{A}'_{1234}$. We can expand the points $\mathbf{A}'_i$ and $\mathbf{A}'_{1234}$ in terms of the R^4 vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$:

$$\mathbf{A}'_i = \alpha_{i,1}\mathbf{A}_1 + \alpha_{i,2}\mathbf{A}_2 + \alpha_{i,3}\mathbf{A}_3 \quad i = 1, 2, 3 \tag{53}$$

$$\mathbf{A}'_{1234} = \alpha_{1234,1}\mathbf{A}_1 + \alpha_{1234,2}\mathbf{A}_2 + \alpha_{1234,3}\mathbf{A}_3. \tag{54}$$

Similarly, if $\mathbf{B}'_{1234}$ is the R^4 representation of the intersection point $\boldsymbol{b}'_{1234}$ in the second image plane (i.e. the intersection of the lines joining $\{\boldsymbol{b}'_1$ and $\boldsymbol{b}'_2\}$ and $\{\boldsymbol{b}'_3$ and $\boldsymbol{b}'_4\}$) then we can write the points $\mathbf{B}'_i$ and $\mathbf{B}'_{1234}$ as

$$\begin{aligned} \mathbf{B}'_i &= \beta_{i,1}\mathbf{B}_1 + \beta_{i,2}\mathbf{B}_2 + \beta_{i,3}\mathbf{B}_3 \quad i = 1, 2, 3 \\ \mathbf{B}'_{1234} &= \beta_{1234,1}\mathbf{B}_1 + \beta_{1234,2}\mathbf{B}_2 + \beta_{1234,3}\mathbf{B}_3. \end{aligned} \tag{55}$$

We can form $\mathbf{A}'_{1234}$ from the meet of the line $\mathbf{A}'_1 \wedge \mathbf{A}'_2$ and the plane $\mathbf{A}_0 \wedge \mathbf{A}'_3 \wedge \mathbf{A}'_4$;

$$\mathbf{A}'_{1234} = [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3\mathbf{A}'_4]\mathbf{A}_0 + [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4\mathbf{A}_0]\mathbf{A}'_3 + [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}_0\mathbf{A}'_3]\mathbf{A}'_4. \tag{56}$$

Similarly for $\mathbf{B}'_{1234}$;

$$\mathbf{B}'_{1234} = [\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_3\mathbf{B}'_4]\mathbf{B}_0 + [\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_4\mathbf{B}_0]\mathbf{B}'_3 + [\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}_0\mathbf{B}'_3]\mathbf{B}'_4. \tag{57}$$

Therefore we can evaluate the bracket $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]$ as follows

$$\begin{aligned} [\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}] &= -\{\mathbf{A}_0 \wedge \mathbf{A}'_{1234}\} \wedge \{\mathbf{B}_0 \wedge \mathbf{B}'_{1234}\} I_4{}^{-1} \\ &= -\{[\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4\mathbf{A}_0][\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}_0\mathbf{A}'_3][\mathbf{A}_0\mathbf{A}'_3\mathbf{B}_0\mathbf{B}'_4] + \\ &\quad [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}_0\mathbf{A}'_3][\mathbf{B}'_1\mathbf{B}'_2\mathbf{B}'_4\mathbf{B}_0][\mathbf{A}_0\mathbf{A}'_4\mathbf{B}'_3\mathbf{B}_0]\}. \end{aligned} \tag{58}$$

After some reordering of terms we note that this is precisely the expression for the bracket S_{1234} given in equation **??**. Thus, to summarize, we are able to write the bracket of the 4 points (in R^4) as

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] \equiv [\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]. \tag{59}$$

Note here that the bracket $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]$ has been equated to $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]$ by the process of splitting up the bracket into two parts, $\mathbf{X}_1 \wedge \mathbf{X}_2$ and $\mathbf{X}_3 \wedge \mathbf{X}_4$ and then expressing each of these lines (bivectors) as the meet of two planes (trivectors) as in equation **??**. Thus, when we take ratios of brackets to cancel out scale factors and form our invariants we must ensure that, if we want to express the brackets in the form of equation **??**, the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ must occur in the numerator and denominator so that these arbitrary factors cancel. In the case of Inv_3, we have

$$Inv_3 = \frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_4 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_3 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}} \tag{60}$$

so we see that this decomposition rule has been obeyed. Now in [**?**] it is claimed that the invariant of 6 points which can be thought of as arising from 4 points and a line, namely

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]}. \tag{61}$$

is not invariant when expressed in Carlsson's terms. Their solution is to include a large number of correcting scale factors. If we were to decompose the expression as given in equation ?? in the manner outlined previously, we would have

$$Inv_3 = \frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_5 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_6)\} I_4{}^{-1}}. \tag{62}$$

It is clear that the same bivectors do *not* appear in both the numerator and denominator and therefore there will not be the required cancelling of scale factors. However, suppose we simply rearrange equation ?? in the following way;

$$Inv_3' = \frac{[\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{63}$$

This can always be done since interchanging vectors in $\mathbf{X}_i \wedge \mathbf{X}_j \wedge \mathbf{X}_k \wedge \mathbf{X}_l$ simply changes the sign of the pseudoscalar. Now, the decomposition would look like

$$Inv_3' = \frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_3)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_3)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}. \tag{64}$$

We now see that the *same* bivectors appear in both numerator and denominator and therefore that all scale factors should cancel. Writing

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]} \equiv \frac{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1423}\mathbf{B}'_{1423}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1526}\mathbf{B}'_{1526}]}{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1523}\mathbf{B}'_{1523}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1426}\mathbf{B}'_{1426}]} \tag{65}$$

where $\mathbf{A}_{ijkl}$ is the point in R^4 corresponding to the intersection point $\boldsymbol{a}'_{ijkl}$ as defined previously, will indeed produce an invariant. There is thus no need for the introduction of the scale factors proposed in [?].

5 Conclusions

We have presented a brief introduction to the techniques of geometric algebra and shown how they can be used in the algebra of incidence and in the formation and computation of invariants. For intersections of planes, lines etc. and for the discussion of projective transformations it is useful to work in a 4D space we have called R^4. We find that we do not need to invoke the standard concepts or machinery of classical projective geometry, all that is needed is the idea of the *projective split* relating the quantities in R^4 to quantities in our 3D world. We believe that with this approach we can achieve everything that has currently been achieved with the standard approaches, but that we can do it in a more geometrically intuitive fashion.

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