A NEW FRAMEWORK FOR THE FORMATION OF INVARIANTS AND MULTIPLE-VIEW CONSTRAINTS IN COMPUTER VISION

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ABSTRACT

In this paper we present *geometric algebra* as a new framework for the theory and computation of invariants and multiple-view constraints in computer vision. We discuss the formation of 3D projective invariants and a wholly geometric formulation of constraints from a number of images.

1. INTRODUCTION

Geometric algebra (GA) is a coordinate-free approach to geometry based on the algebras of Grassmann and Clifford; the system we adopt here was pioneered by David Hestenes [5]. We will outline the use of GA in the formulation of projective geometry and discuss the algebra of incidence. Using this we present a new methodology for the study of geometric invariance and multiple-view constraints (more detail can be found in [7, 1]). Throughout the paper the convention of summing over repeated indices is assumed.

2. GEOMETRIC ALGEBRA: AN OUTLINE

Let $\mathcal{G}_n$ denote the geometric algebra of n-dimensions. This is a graded linear space with vector addition, scalar multiplication and a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford** product. Any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ where

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}. \tag{1}$$

The inner product of two vectors, $\boldsymbol{a}\cdot\boldsymbol{b}$, is the standard *scalar* product and produces a scalar. The outer or wedge product of two vectors, $\boldsymbol{a}\wedge\boldsymbol{b}$, is a new quantity we call a **bivector**. This is a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ – see Figure 1.

Thus, $\boldsymbol{b}\wedge\boldsymbol{a}$ will have the opposite orientation making the outer product anticommutative. This is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a **trivector**, is the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$. The outer product of k vectors is a k-vector or k-blade, and is said to have *grade* k. A **multivector** (linear combination of objects of different type) is *homogeneous* if it contains terms of only a single grade. Geometric algebra allows us to manipulate multivectors in a way which keeps track of different grade objects.

In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I and is crucial to ideas of duality.

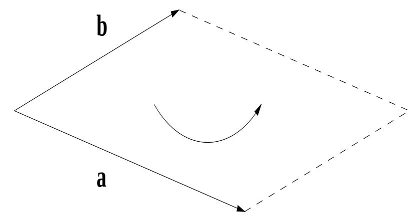

Figure 1: The directed area, or bivector, $\boldsymbol{a}\wedge\boldsymbol{b}$

Consider a linear function f mapping vectors to vectors in the same space. We can extend f to act linearly on multivectors via the **outermorphism**, $\underline{f}$, such that

$$\underline{f}(\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\ldots\wedge\boldsymbol{a}_r) = \underline{f}(\boldsymbol{a}_1)\wedge\underline{f}(\boldsymbol{a}_2)\wedge\ldots\wedge\underline{f}(\boldsymbol{a}_r). \tag{2}$$

$\underline{f}$ therefore preserves the grade of any r-vector it acts on. The action of $\underline{f}$ on general multivectors is then defined through linearity. Since the outermorphism preserves grade, the pseudoscalar of the space must be mapped onto some multiple of itself. The scale factor in this mapping is the **determinant** of $\underline{f}$;

$$\underline{f}(I) = \det(\underline{f})I. \tag{3}$$

3. PROJECTIVE SPACE

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$ (3D Euclidean space). As is usual, we associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. In these two spaces we define basis vectors: $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ in R^4 and $\{\sigma_1, \sigma_2, \sigma_3\}$ in $\mathcal{E}^3$ – noting that ${\gamma_j}^2 = -1$ for j=1,2,3 and $\gamma_4^2 = +1$ (so that $I^2 = -1$, where $I = \gamma_1\gamma_2\gamma_3\gamma_4$). We identify R^4 and $\mathcal{E}^3$ with the geometric algebras of 4 and 3 dimensions. Choosing γ_4 as a selected direction in R^4, we can define a mapping which associates the bivectors $\gamma_i\gamma_4$, $i = 1, 2, 3$, in R^4 with the vectors $\sigma_i = \gamma_i\gamma_4$, $i = 1, 2, 3$, in $\mathcal{E}^3$. This process is an application of what Hestenes calls the **projective split**.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is achieved by taking the geometric product of $\mathbf{X}$ and γ_4;

$$\mathbf{X}\gamma_4 = \mathbf{X}\cdot\gamma_4 + \mathbf{X}\wedge\gamma_4 = X_4\left(1 + \frac{\mathbf{X}\wedge\gamma_4}{X_4}\right) \equiv X_4(1 + \boldsymbol{x}). \tag{4}$$

We think of the vector $\boldsymbol{x}$ as a vector in $\mathcal{E}^3$ which is associated with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4, i.e.

$$\boldsymbol{x} = \frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3, \tag{5}$$

$\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1, 2, 3$. This is equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$. We therefore have distinct spaces with a well-defined way of moving between these spaces.

3.1. Formulation of Projective Geometry

In a given space any pseudoscalar P can be written as $P = \alpha I$ where α is a scalar. If I^{-1} is the inverse of I, then

$$PI^{-1} = \alpha II^{-1} = \alpha \equiv [P] \tag{6}$$

where we have defined the **bracket**, $[P]$, of the pseudoscalar P. This bracket corresponds to the bracket of the Grassmann-Cayley algebra.

We define the **dual** A^* of an r-vector A as

$$A^* = AI^{-1}. \tag{7}$$

In an n-dimensional space, if A is an r-vector and B is an s-vector (such that $r + s = n$), we have

$$[A\wedge B] = (A\wedge B)I^{-1} = A\cdot B^*. \tag{8}$$

Here duality is simply multiplication by an element of the algebra. One can define the **join** $J = A \bigwedge B$ of an r-vector A and an s-vector B by

$$J = A\wedge B \quad \text{if } A \text{ and } B \text{ are linearly independent.} \tag{9}$$

If A and B have a common factor we can define the 'intersection' or **meet** of A and B as $A \vee B$ given by

$$(A \vee B)^* = A^*\wedge B^*, \tag{10}$$

where the dual is taken with respect to the *join* of A and B. If the join is the whole space the meet is given by

$$A \vee B = (A^*\wedge B^*)I = (A^*\wedge B^*)(I^{-1}I)I = \pm(A^*\cdot B) \tag{11}$$

according as $I^2 = \pm 1$. We therefore have the very simple relation of $A \vee B = \pm(A^*\cdot B)$. For more details see [7, 6].

3.2. Projective transformations

If a general point (x, y, z) in 3-D space is projected onto an image plane point, (x', y'), the coordinates are related by;

$$x' = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{12}$$

To make this non-linear transformation in $\mathcal{E}^3$ into a linear transformation in R^4 we define a linear function $\underline{f}$ (in R^4) where $\underline{f}$ is given by

$$\begin{aligned} \underline{f}(\gamma_1) &= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \tilde{\alpha}\gamma_4 \\ \underline{f}(\gamma_2) &= \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \tilde{\beta}\gamma_4 \text{ etc..} \end{aligned} \tag{13}$$

$\underline{f}$ maps $\mathbf{X}$ onto $\mathbf{X}'$ such that the vector $\boldsymbol{x}' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3$ in $\mathcal{E}^3$ is formed from $\mathbf{X}'$ via the projective split. Similarly for y' and z'.

3.3. Algebra in projective space

Consider three non-collinear points, P_1, P_2, P_3, represented by vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in $\mathcal{E}^3$ and by vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in R^4. The line L_{12} joining points P_1 and P_2 can be expressed in R^4 by the following bivector,

$$L_{12} = \mathbf{X}_1\wedge\mathbf{X}_2. \tag{14}$$

Similarly, the plane Φ_{123} passing through points P_1, P_2, P_3 is expressed by the following trivector in R^4

$$\Phi_{123} = \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3. \tag{15}$$

Consider now a line $A = \mathbf{X}_1\wedge\mathbf{X}_2$ intersecting a plane $\Phi = \mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3$ – all vectors are in R^4. Using the *meet* operation we have ,[7, 1],

$$A\vee\Phi = [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_2\mathbf{Y}_3]\mathbf{Y}_1 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3\mathbf{Y}_1]\mathbf{Y}_2 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_1\mathbf{Y}_2]\mathbf{Y}_3 \tag{16}$$

where $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4]$ is the magnitude of the pseudoscalar formed from the four vectors – which agrees with the result in [2] if the r-extensors of the Grassmann-Cayley algebra are identified with r-blades.

The intersection of two planes $\Phi_1 = \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3$ and $\Phi_2 = \mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3$ is given by the meet of Φ_1 and Φ_2, which can be expanded [1] as

$$\begin{aligned} \Phi_1 \vee \Phi_2 = {}& [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_1](\mathbf{Y}_2\wedge\mathbf{Y}_3) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_2](\mathbf{Y}_3\wedge\mathbf{Y}_1) \\ & + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_3](\mathbf{Y}_1\wedge\mathbf{Y}_2), \end{aligned} \tag{17}$$

producing a line of intersection (bivector in R^4). This again agrees with the expressions given in [2].

4. INVARIANTS

The '*fundamental projective invariant*' of points on a line is the **cross-ratio**. When a point on line L is projected onto another line L', distances t and t' are related by a projective transformation of the form $t' = \frac{\alpha t + \beta}{\tilde{\alpha}t + \tilde{\beta}}$. This non-linear transformation in $\mathcal{E}^1$ can be made linear in R^2 by defining the linear function $\underline{f}_1$ mapping vectors onto vectors in R^2;

$$\underline{f}_1(\gamma_1) = \alpha\gamma_1 + \tilde{\alpha}\gamma_2, \quad \underline{f}_1(\gamma_2) = \beta\gamma_1 + \tilde{\beta}\gamma_2. \tag{18}$$

Consider 2 vectors $\mathbf{X}_1, \mathbf{X}_2$ in R^2. Form the bivector $\mathcal{S}_1 = \mathbf{X}_1\wedge\mathbf{X}_2 = \lambda_1 I_2$, where $I_2 = \gamma_1\gamma_2$ is the pseudoscalar for R^2. We now look at how $\mathcal{S}_1$ transforms under $\underline{f}_1$:

$$\mathcal{S}_1' = \mathbf{X}_1'\wedge\mathbf{X}_2' = \underline{f}_1(\mathbf{X}_1\wedge\mathbf{X}_2) = (\det\underline{f}_1)(\mathbf{X}_1\wedge\mathbf{X}_2). \tag{19}$$

Take 4 points on the line L whose corresponding vectors in R^2 are $\{\mathbf{X}_i\}$, $i = 1, .., 4$, and consider the ratio $\mathcal{R}_1$ of 2 wedge products, which will transform as follows under $\underline{f}_1$, $(\mathcal{R}_1 \rightarrow \mathcal{R}_1')$:

$$\mathcal{R}_1' = \frac{\mathbf{X}_1'\wedge\mathbf{X}_2'}{\mathbf{X}_3'\wedge\mathbf{X}_4'} = \frac{(\det\underline{f}_1)\mathbf{X}_1\wedge\mathbf{X}_2}{(\det\underline{f}_1)\mathbf{X}_3\wedge\mathbf{X}_4}. \tag{20}$$

$\mathcal{R}_1$ is therefore invariant under $\underline{f}_1$. In order to convert to 1D distances we must consider how the bivector $\mathcal{S}_1$ in R^2 projects down to $\mathcal{E}^1$.

$$\begin{aligned} \mathbf{X}_1\wedge\mathbf{X}_2 &= (T_1\gamma_1 + S_1\gamma_2)\wedge(T_2\gamma_1 + S_2\gamma_2) = \\ & (T_1 S_2 - T_2 S_1)\gamma_1\gamma_2 = S_1 S_2(t_1 - t_2)I_2. \end{aligned} \tag{21}$$

To form a projective invariant which is independent of the choice of the arbitrary scalars S_i, we then take *ratios* of the bivectors $\mathbf{X}_i \wedge \mathbf{X}_j$ (to cancel $\det \underline{f}_1$) and *multiples* of such ratios so that the S_i's cancel. Consider the following expression

$$Inv_1 = \frac{(\mathbf{X}_3 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_2) I_2^{-1}}{(\mathbf{X}_4 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_2) I_2^{-1}}.$$

In terms of 1D distances, under the projective transformation $\underline{f}_1$, Inv_1 goes to Inv_1' where

$$Inv_1' = \frac{S_3 S_1 (t_3 - t_1) S_4 S_2 (t_4 - t_2)}{S_4 S_1 (t_4 - t_1) S_3 S_2 (t_3 - t_2)} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)}, \tag{22}$$

which is independent of the S_i's and is indeed the 1D **cross-ratio**.

We can now extend these arguments to form invariant quantities in 2 and 3 dimensions by taking *multiples* of *ratios* of **trivectors** and **4-vectors**, e.g.

$$Inv_2 = \frac{(\mathbf{X}_5 \wedge \mathbf{X}_4 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_1) I_3^{-1}}{(\mathbf{X}_5 \wedge \mathbf{X}_1 \wedge \mathbf{X}_3) I_3^{-1} (\mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4) I_3^{-1}}.$$

$$Inv_3 = \frac{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5) I_4^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}. \tag{23}$$

4.1. 3D invariants in terms of image coordinates

From six general 3D points P_i, $i = 1, .., 6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4, we can form a number of 3D projective invariants. One such invariant is

$$Inv_3 = \frac{[\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \mathbf{X}_4][\mathbf{X}_4 \mathbf{X}_5 \mathbf{X}_2 \mathbf{X}_6]}{[\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_4 \mathbf{X}_5][\mathbf{X}_3 \mathbf{X}_4 \mathbf{X}_2 \mathbf{X}_6]}. \tag{24}$$

This is simply equation (23) written in terms of brackets. Recent work has used the Grassmann-Cayley algebra [2] to compute such invariants from a pair of images using image coordinates and the fundamental matrix, $\mathcal{F}$. Subsequent work by Csurka and Faugeras [3] attempts to correct some of Carlsson's expressions by including omitted scale factors. We will show that the resolution lies simply in reordering the bracket decomposition rather than finding large numbers of complicated scale factors.

Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \mathbf{X}_4] = (\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4) I_4{}^{-1}. \tag{25}$$

$(\mathbf{X}_i \wedge \mathbf{X}_j)$ represents the line joining points P_i and P_j. We let $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ be the centres of projection of the two cameras with the two camera image planes defined by $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ – see figure 2. The projection of points $\{P_i\}$ are given by the vectors $\{\boldsymbol{a}_i'\}$ and $\{\boldsymbol{b}_i'\}$. Note our vectors, $\boldsymbol{a}_i, \boldsymbol{b}_i, \ldots$ etc. are vectors in $\mathcal{E}^3$ with R^4 representations of $\mathbf{A}_i, \mathbf{B}_i, \ldots$, etc.

Let the intersection of the lines joining points $\{\boldsymbol{a}_1'$ and $\boldsymbol{a}_2'\}$ and $\{\boldsymbol{a}_3'$ and $\boldsymbol{a}_4'\}$ be $\boldsymbol{a}_{1234}'$ ($\mathbf{A}_{1234}'$ in R^4). $\mathbf{B}_{1234}'$ is defined similarly in the second image plane. It can be shown [7] that by decomposing as in equation (25) it is possible to write the bracket S_{1234} as given in [2]

$$S_{1234} = [\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \mathbf{X}_4] \equiv [\mathbf{A}_0 \mathbf{B}_0 \mathbf{A}_{1234}' \mathbf{B}_{1234}']. \tag{26}$$

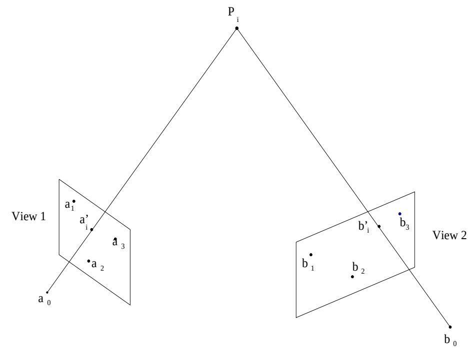

Figure 2: Defining points of two camera planes

Note that when we take ratios of brackets we must ensure that the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ occurs in both numerator and denominator so that the arbitrary factors cancel. For example, Inv_3 can be written as

$$\frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_4 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_3 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}} \tag{27}$$

where this decomposition rule has been obeyed. In [3] it is claimed that the following invariant of 6 points

$$\frac{[\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \mathbf{X}_4][\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_5 \mathbf{X}_6]}{[\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_3 \mathbf{X}_5][\mathbf{X}_1 \mathbf{X}_2 \mathbf{X}_4 \mathbf{X}_6]}, \tag{28}$$

is not invariant when expressed in Carlsson's terms. Their solution is to include a large number of correcting scale factors. One decomposition of this expression is

$$\frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_5 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_6)\} I_4{}^{-1}}. \tag{29}$$

Here the same bivectors do *not* appear in both the numerator and denominator so the scale factors will not cancel. However, we are free to rearrange equation (28) in the following way;

$$\frac{[\mathbf{X}_1 \mathbf{X}_4 \mathbf{X}_2 \mathbf{X}_3][\mathbf{X}_1 \mathbf{X}_5 \mathbf{X}_2 \mathbf{X}_6]}{[\mathbf{X}_1 \mathbf{X}_5 \mathbf{X}_2 \mathbf{X}_3][\mathbf{X}_1 \mathbf{X}_4 \mathbf{X}_2 \mathbf{X}_6]}. \tag{30}$$

The decomposition now looks like

$$\frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_3)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_3)\} I_4{}^{-1} \{(\mathbf{X}_1 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}} \tag{31}$$

and we see that the *same* bivectors appear in both numerator and denominator and therefore all scale factors cancel. Writing

$$Inv_3 = \frac{[\mathbf{A}_0 \mathbf{B}_0 \mathbf{A}_{1423}' \mathbf{B}_{1423}'][\mathbf{A}_0 \mathbf{B}_0 \mathbf{A}_{1526}' \mathbf{B}_{1526}']}{[\mathbf{A}_0 \mathbf{B}_0 \mathbf{A}_{1523}' \mathbf{B}_{1523}'][\mathbf{A}_0 \mathbf{B}_0 \mathbf{A}_{1426}' \mathbf{B}_{1426}']} \tag{32}$$

will indeed produce an invariant thus dispensing with the need for the scale factors proposed in [3].

5. POINT CORRESPONDENCES

For this analysis, let $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$, $(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$, $\ldots$, $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ define the image planes and let $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{c}_0, \ldots, \boldsymbol{n}_0$ be the corresponding optical centres.

5.1. Two cameras: the bilinear constraint

The projections of a world point P_i (represented by $\boldsymbol{x}_i$ and $\mathbf{X}_i$ in $\mathcal{E}^3$ and R^4) will be $\boldsymbol{a}'_i$ and $\boldsymbol{b}'_i$ in the two image planes ($\mathbf{A}'_i$ and $\mathbf{B}'_i$ in R^4). $\mathbf{A}'_i$ can be expressed as the intersection of a line and image plane 1, see figure 2:

$$\mathbf{A}'_i = (\mathbf{A}_0 \wedge \mathbf{X}_i) \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3) \tag{33}$$

$$= [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_2\mathbf{A}_3]\mathbf{A}_1 + [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_3\mathbf{A}_1]\mathbf{A}_2 + [\mathbf{A}_0\mathbf{X}_i\mathbf{A}_1\mathbf{A}_2]\mathbf{A}_3$$

and similarly for $\mathbf{B}'_i$ and $\mathbf{C}'_i$. We can define three planes through the optical centre of each camera, for example, $\Phi_{1j},\ j = 1, 2, 3$ are planes through $\mathbf{A}_0$ defined by

$$\Phi_{11} = \mathbf{A}_0\wedge\mathbf{A}_2\wedge\mathbf{A}_3, \quad \Phi_{12} = \mathbf{A}_0\wedge\mathbf{A}_3\wedge\mathbf{A}_1, \quad \Phi_{13} = \mathbf{A}_0\wedge\mathbf{A}_1\wedge\mathbf{A}_2. \tag{34}$$

The epipolar constraint is simply that $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{a}'_i$, $\boldsymbol{b}'_i$ are coplanar. This can be concisely written as $L_A \wedge L_B = 0$ where $L_A = \mathbf{A}_0\wedge\mathbf{A}'_i$ and $L_B = \mathbf{B}_0\wedge\mathbf{B}'_i$, giving $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_i\mathbf{B}'_i] = 0$. Expressed in terms of the $\mathbf{A}'_i$, $\mathbf{B}'_i$ this gives

$$[\mathbf{A}_0\mathbf{B}_0(\alpha_{i1}\mathbf{A}_1 + \alpha_{i2}\mathbf{A}_2 + \alpha_{i3}\mathbf{A}_3)(\beta_{i1}\mathbf{B}_1 + \beta_{i2}\mathbf{B}_2 + \beta_{i3}\mathbf{B}_3)]$$

$$= \boldsymbol{\alpha_i}^T \mathcal{F} \boldsymbol{\beta_i} = 0, \tag{35}$$

where $\mathcal{F}_{ij} = [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_i\mathbf{B}_i]$, is the well known fundamental matrix.

5.2. Three cameras: the trilinear constraints

For point correspondences in three views we have constraints of the following form;

$$L_A\wedge\{\Phi_{Bi} \vee \Phi_{Cj}\} = 0 \quad , L_B\wedge\{\Phi_{Ai} \vee \Phi_{Cj}\} = 0,$$
$$L_C\wedge\{\Phi_{Ai} \vee \Phi_{Bj}\} = 0 \tag{36}$$

where Φ_{Ak}, Φ_{Bk} and Φ_{Ck} are planes defined by $\Phi_{Ak} = \mathbf{A}_0 \wedge \mathbf{A}_k \wedge \mathbf{A}'_i$ etc. The first constraint in equation (36) is simply saying that line L_A and the line of intersection of planes Φ_{Bi} and Φ_{Cj} must intersect at a point – this point being P (drop subscript i on P, $\mathbf{A}'$ etc.). We have

$$L_A\wedge\{\Phi_{Bi} \vee \Phi_{Cj}\} = \tag{37}$$

$$(\mathbf{A}_0\wedge\mathbf{A}'_i)\wedge\{(\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}') \vee (\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}')\} = 0.$$

Using $\mathbf{A}' = \alpha_i\mathbf{A}_i$, $\mathbf{B}' = \beta_i\mathbf{B}_i$ and $\mathbf{C}' = \delta_i\mathbf{C}_i$ then enables us to write

$$\begin{aligned} \mathbf{B}_0\wedge\mathbf{B}_i\wedge\mathbf{B}' &= \beta_l(\mathbf{B}_0\wedge\mathbf{B}_i\wedge\mathbf{B}_l) \equiv \beta_l\Phi^B_{il} \\ \mathbf{C}_0\wedge\mathbf{C}_j\wedge\mathbf{C}' &= \delta_m(\mathbf{C}_0\wedge\mathbf{C}_j\wedge\mathbf{C}_m) \equiv \delta_m\Phi^C_{jm}, \end{aligned} \tag{38}$$

where the planes Φ_{11} etc. have been remaned as given above. The constraint in equation (37) is now

$$\alpha_k(\mathbf{A}_0\wedge\mathbf{A}_k)\wedge\{\beta_l\delta_m(\Phi^B_{il} \vee \Phi^C_{jm})\} = 0 \tag{39}$$

which can be put into the form

$$\tilde{T}^{ij}_{klm}\alpha_k\beta_l\delta_m = 0 \tag{40}$$

where

$$\tilde{T}^{ij}_{klm} = [\mathbf{A}_0\mathbf{A}_k(\Phi^B_{il} \vee \Phi^C_{jm})]. \tag{41}$$

This is a *trilinear constraint*. There are obviously 9 possible choices of the pair (ij). However, by expanding the bracket in equation (41) it can be shown that only 4 of these are independent. Since we had three original constraints, this leads to a total of 12 trilinearity constraints as noted by [4].

Our tensor $\tilde{T}^{ij}_{klm}$ is related to Hartley's tensor, T_{pqr} [4], via;

$$\tilde{T}^{ij}_{klm} \longrightarrow T_{pqr} \tag{42}$$

where $p = 1$ if $(i, l) = (2, 3)$, $p = 2$ if $(i, l) = (1, 3)$ and $p = 3$ if $(i, l) = (1, 2), (2, 1)$. Similarly, $q = 1$ if $(j, m) = (2, 3)$ etc.. We also note that for given (i, j) only certain values of (l, m) give non-zero expressions for $\tilde{T}$.

The derivation of the trilinear constraints for lines is given in [1].

5.3. Unifying the point constraints for n-views

If we have n views let us choose 4 of these views and denote them by A, B, C and N. $\Phi_{Aj} \vee \Phi_{Bk}$ gives a line passing through world point P as does $\Phi_{Cl} \vee \Phi_{Nm}$. We therefore have the condition

$$\{\Phi_{Aj} \vee \Phi_{Bk}\}\wedge\{\Phi_{Cl} \vee \Phi_{Nm}\} = 0. \tag{43}$$

If $\boldsymbol{N}' = \eta_1\boldsymbol{N}_1 + \eta_2\boldsymbol{N}_2 + \eta_3\boldsymbol{N}_3$ then this condition can be written as

$$\alpha_r\beta_s\delta_t\eta_u\{(\Phi^A_{jr} \vee \Phi^B_{ks})\wedge(\Phi^C_{lt} \vee \Phi^N_{mu})\} = 0. \tag{44}$$

Therefore for n cameras or a moving sensor the general equation for computing bi-, tri- and quadri-linear constraints is

$$\{\Phi_{Kk} \vee \Phi_{Ll}\}\wedge\{\Phi_{Mm} \vee \Phi_{Nn}\} = 0 \tag{45}$$

where K,L,M and N are any four cameras or any four views from a moving observer. This equation subsumes the two and three camera cases, i.e. for two cameras use L_K instead of $\{\Phi_{Kk} \vee \Phi_{Ll}\}$ and L_L instead of $\{\Phi_{Mm} \vee \Phi_{Nn}\}$ and for three cameras use L_K instead of $\{\Phi_{Kk} \vee \Phi_{Ll}\}$ and $\{\Phi_{Ll} \vee \Phi_{Mm}\}$ instead of $\{\Phi_{Mm} \vee \Phi_{Nn}\}$.

6. CONCLUSIONS

We have shown how geometric algebra can be used in the formation and computation of invariants and in deriving a single constraint statement which holds for 1, 2, 3 or 4 views. The framework provides a single mathematical language to replace the multitude of distinct systems currently in use and can be used for most computer vision problems.

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