Geometric Algebra as a Framework for the Perception–Action Cycle

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Abstract The design of technical systems with the capability of realizing perception–action cycles will be motivated an the base of fusion of robotics, computer vision, and signal processing or neural computation in a unified frame. This process of integration and the overcoming of well known limitations within the contributing disciplines at least partly may be supported by embedding the task in the frame of a powerful geometric interpretable algebra. As such geometric algebra will be presented with respect to its key ideas and some facets will be sketched of first realized applications in the frame of the mentioned disciplines.

1 Introduction

In this paper we will present a mathematical framework for embedding the realization of technical systems which are designed on principles of the perception–action cycle (PAC). The use of PAC as a design principle of systems which should have both capabilities of perception and action is motivated by ethology and has its theoretical roots in the theory of non–linear dynamic systems. PAC is the frame of autonomous behavior. It relates perception and action in a purposive manner. The global competence of such systems results from cooperation and competition of a set of behaviors, each as an observable manifestation of a certain kind of competence. If both acquired skill and experience are the sources to yield competence, there is hope also to gain such attractive system properties like robustness and adaptivity. The essence behind this extension of the active vision paradigm is a certain kind of equivalence between visual perception and action. That means both perceptual categories and those of actions are mutually supported and have to be mutually verified. Perception and action constitute the afferent and efferent interfaces of the agent to its environment. Using them in a mature stage the active agent stabilizes its relation to the environment by equalizing categories of perception with those of action. The first ones are defined by the experience that similar patterns cause similar actions (or reactions) and the second ones correspond to the skill that similar actions cause similar patterns.
Following that line it should be possible to design both technical visual systems with support of active components of movement and seeing robots. This necessitates the fusion of computer vision (as active vision), robotics, signal processing, and neural computation. It becomes obvious that representations will

take on central importance. They have to relate the agent with the environment in Euclidean space–time. Evaluating the actual situation with respect to the representation problem we have to state both serious shortcomings within the disciplines and gaps between them.

In order to overcome these problems the time is ripe to identify the deep reasons for this situation. Our hypothesis is the following: Linear algebra of vector spaces on real or complex numbers is a too poor language for representing phenomena of the world as complete and effective as necessary. This results in limited capabilities of reconstruction of complex objects from projected patterns in the approach of explicit representation of structure and slow learning rates of higher order correlations which are responsible for the structure of complex objects within an implicit representation approach of neural nets. Even if the ultimate goal of an agent is not visual reconstruction of the world, it has to find out the mapping rules between visual percepts and the world they stand for, constrained by its relations to that world.

What we need is a language which expresses a lot of group theoretical constrained geometric isomorphisms within the agent's mind. This has to be a rich algebraically structured geometry or a geometrically interpretable rich structured algebra. We may only refer to some of the serious limitations of linear algebra. In vector spaces of real numbers intrinsic representations are limited to zero order geometric entities (points) without any symmetry. Geometric transforms are restricted to translation. Lines and planes as geometric entities are no intrinsic conceptions of linear vector spaces but result from a process of construction using points and translation operations. Endowing vector spaces with complex numbers results in representations of first order entities (lines) with even/odd symmetry. Geometric transformations are enriched by rotation in the complex plane. In both cases the scalar product enables only to model bilinear relations between two vectors with an outcome as scalar. This poverty of operations hinders recognition of intrinsic dimensionality or reconstruction of higher order entities which constitute the perceivable world. Only by using richer numbers than real or complex ones, respectively by enabling richer symmetry conceptions, vector spaces can be modeled from subspaces which stand for irreducible invariants as members of a basis system of structure of higher order then zero or one.

Our conception of the theory for PAC consists of two parts. First part is the so–called geometric or Clifford algebra as the global frame for representations of actions or patterns [1], [2], [3], [5], [7], [14]. To sketch out the basic conceptions and to demonstrate some first results is the topic of this article. The second part concerns the use of Lie algebra as local frame for both differential generation or recognition of patterns (see e.g. [15]). In such frame a unified architecture of a PAC system for the task of perception projects local patterns to the set of irreducible invariants in the frame of geometric algebra and successively from these projections complex patterns are constructed. Vice versa also local patterns of actions are generated from such set to construct from them complex patterns of motion and action. This algebraic constrained local approach also could be helpful to overcome the contemporary gap between geometric entities in Computer Vision and structural primitives in signal theory and thus to realize Faugeras' stratification approach [9] of vision on the signal level.

2 Geometric Algebra of Vector Spaces

What we need is a more general mathematics for easier modeling of both structures (perceived and generated) and operations. This mathematics has been formulated in the last century by W. K. Clifford (1876) as an algebra of directed numbers with both quantitative and operational interpretations. Clifford algebra (see [16] for a modern survey) can be seen as result of the unification of H. Grassmann's algebra of extensions (1844), which concerns the quantitative interpretation of numbers, with the algebra of quaternions, introduced by W. R. Hamilton (1844) as an operational interpretation of numbers. With this interpretation of the relations between the mentioned algebras we follow the ideas of David Hestenes [13] who presented Clifford algebra as a "unified language for mathematics and physics" [10] and named it geometric algebra (GA) as Clifford did. It is his great merit to reformulate and to work out Clifford algebra from a geometrical point of view as a framework for describing physical processes in the world. Thus, he also decided not to care about the long lasting debate of pure mathematicians on the priority of Grassmann's or Clifford's algebra in establishing a universal geometric algebra [8]. Indeed, Clifford algebra contains Grassmann algebra as a subalgebra.

Also the perception–action cycle concerns physical phenomena of the real world and some problems in modern physics are indeed comparable to those of computer vision, robotics or neural computation.

A geometric algebra G_n results from providing an n–dimensional vector space V_n in addition to vector addition and scalar multiplication with a non–commutative product. The geometric product is associative and distributive with respect to addition. The geometric product of two vectors $\mathbf{a}$, $\mathbf{b}$ is written $\mathbf{ab}$ and can be understood according

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

as the sum of a symmetric inner product $\alpha = \mathbf{a} \cdot \mathbf{b}$ and an antisymmetric outer product $\mathbf{B} = \mathbf{a} \wedge \mathbf{b}$. In case that $\mathbf{a}$, $\mathbf{b}$ are vectors — we say of grade one — the inner product corresponds to the scalar product — its result is a scalar (of grade zero), but the result of the outer product is a new entity of grade two. This is called a bivector. That means, in contrast to the scalar product of vector algebra, the geometric product of geometric algebra results in both contraction (inner product) and expansion (outer product) of the grade of entities. As a consequence we get from the product of n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ a multivector

$$\mathbf{A} = \langle \mathbf{A} \rangle_0 + \langle \mathbf{A} \rangle_1 + \ldots + \langle \mathbf{A} \rangle_n \quad ,$$

which is a mixture of multivector parts $\langle \mathbf{A} \rangle_r$ of grade r. Any multivector $\mathbf{A}_r = \langle \mathbf{A} \rangle_r$ is called homogeneous of grade r or r–vector. Only if such r–vector can be factored according

$$\mathbf{A}_r = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_r$$

it is called an r–blade. The geometric product of any two homogeneous multivectors $\mathbf{A}_r, \mathbf{B}_s$ results in a spectrum of multivectors of different grade

$$\mathbf{A}_r \mathbf{B}_s = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|} + \langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|+2} + \ldots + \langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s} \quad ,$$

ranging from pure inner product $\langle \mathbf{A} \rangle_r \cdot \langle \mathbf{B} \rangle_s = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|}$ to pure outer product $\langle \mathbf{A} \rangle_r \wedge \langle \mathbf{B} \rangle_s = \langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s}$.
Thus, an n–dimensional vector space V_n uniquely determines a geometric algebra $G(\mathbf{A}) = G_n$ which itself spans a linear space of dimension 2^n. From a given set of n linearly independent vectors spanning V_n we get $\begin{pmatrix} n \\ r \end{pmatrix}$ linear independent r–blades. These r–blades themselves constitute a basis of the linear subspaces $G_r(\mathbf{A})$ of dimension $\begin{pmatrix} n \\ r \end{pmatrix}$ of all r–vectors in G_n. Each such r–blade $\mathbf{A}_r \in G_r(\mathbf{A})$ has a geometric interpretation as an uniquely oriented r–dimensional vector space $V_r = G_1(\mathbf{A}_r)$, consisting of all vectors $\mathbf{a}$ which satisfy $\mathbf{a} \wedge \mathbf{A}_r = 0$, as subspace of $V_n = G_1(\mathbf{A})$. Thus, any r–vector part $\langle \mathbf{A} \rangle_r$ can be understood as a projection of $\mathbf{A}$ into the space $G_r(V_n)$ and any r–vector $\mathbf{A}_r$ can be formulated as a sum of r–blades. Since V_n is the vector space of $\mathbf{A}_n$, it follows $\mathbf{a} \wedge \mathbf{A}_n = 0$ and $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n = \lambda I$ with I as unit pseudoscalar or direction of V_n. From the existence of such pseudoscalar follows the important intrinsic duality principle of geometric algebra which results from any unit r–blade and unit $(n-r)$–blade in $I_r I_{n-r} = I$. As a result of this the relation

$$\mathbf{A}^* = \mathbf{A} I^{-1}$$

defines a dual $\mathbf{A}^*$ of an r–vector $\mathbf{A} = \langle \mathbf{A} \rangle_r$ with respect to the unit pseudoscalar I. The grade of $\mathbf{A}^*$ is $(n-r)$ due to $\mathbf{A} I^{-1} = \mathbf{A} \cdot I^{-1}$. In case that $\mathbf{A}$ is an r–vector and $\mathbf{B}$ is an s–vector, it results the duality of inner and outer product corresponding to

$$\mathbf{A} \cdot \mathbf{B}^* = (\mathbf{A} \wedge \mathbf{B})^*$$

If $r + s = n$, then $\mathbf{A} \wedge \mathbf{B}$ equals a pseudoscalar $P = \lambda I$, and because $I I^{-1} = 1$, it follows

$$[P] = [\mathbf{A} \wedge \mathbf{B}] = (\mathbf{A} \wedge \mathbf{B}) I^{-1} = \mathbf{A} \cdot \mathbf{B}^* \quad .$$

Here $[P] \equiv \lambda$ is the bracket of the pseudoscalar P as used in Grassmann–Cayley algebra (see chapter 4). Because the sign of the bracket is independent of the signature or metric of G_n it is possible to define for that algebra also two qualitative operations. For any linear independent r–vector $\mathbf{A}$ and s–vector $\mathbf{B}$

$$\mathbf{C} = \mathbf{A} \wedge \mathbf{B}$$

is the join (or union) of both. Thus $\mathbf{C}$ is of grade $r + s$. On the other hand, if $\mathbf{A}$ and $\mathbf{B}$ are not linearly independent, the join represents the subspace which they span. As another qualitative operation the meet (or intersection) of $\mathbf{C} = \mathbf{A} \vee \mathbf{B}$ is indirectly defined with respect to the join as

$$\mathbf{C}^* = (\mathbf{A} \vee \mathbf{B})^* = \mathbf{A}^* \wedge \mathbf{B}^* \quad .$$

For $r + s = n$ the join will span the whole space and the meet as a multivector of grade $|r - s|$ will be easily computed as

$$\mathbf{C} = \pm(\mathbf{A}^* \cdot \mathbf{B}) \quad .$$

Both operations are of fundamental importance due to their constructive properties for instance in projective geometry (see chapter 4).

3 Geometric Algebra of Euclidean 3D–Space

The perception–action cycle takes place in Euclidean space–time where the agent is interested in recognizing and organizing processes in such world, even if for some tasks only projective or affine constraints of Euclidean space are used. In chapter 4 problems of projective geometry and kinematics are presented briefly. Following the facts of chapter 2, the geometric algebra of Euclidean 3D–space is 8–dimensional. Much higher–dimensional spaces result if geometric algebra is applied to manifolds, as in chapter 5.

An n–dimensional vector space endowed with an orthogonal basis $\{\sigma_l\}, l = 1, \ldots, n$, and a bilinear form such that $\sigma_l \cdot \sigma_k = \delta_{lk}$ results in a basis of the geometric algebra G_n:

$$1, \{\sigma_l\}, \{\sigma_l \sigma_k\}, \{\sigma_l \sigma_k \sigma_m\}, \ldots, \sigma_1 \sigma_2 \ldots \sigma_n \quad .$$

Thus, the basis of $G_3 = G(E_3)$ is composed of the following components

$$1, \{\sigma_1, \sigma_2, \sigma_3\}, \{\sigma_1\sigma_2 = \mathbf{i}_1, \sigma_2\sigma_3 = \mathbf{i}_2, \sigma_3\sigma_1 = \mathbf{i}_3\}, \sigma_1\sigma_2\sigma_3 = i \quad ,$$

which themselves constitute the basis vectors of the subspaces $G_r \subseteq G_3$. In this way i is the unit trivector or unit pseudoscalar of E_3 with $i^2 = -1$, and $\mathbf{i}_l, l = 1, 2, 3$ are the unit bivectors which are the basis vectors of the quaternion algebra. Any multivector $\mathbf{A} \in G_3$, $\mathbf{A} = \alpha + \mathbf{a} + \mathbf{B} + \mathbf{T}$ is component wise composed of multiples of these unit vectors of geometric algebra. However, due to the duality principle, we can change the basis of any r–vector part by its dual basis, e.g.

$$\mathbf{B} = B_1\mathbf{i}_1 + B_2\mathbf{i}_2 + B_3\mathbf{i}_3, \quad \mathbf{i}_1\mathbf{i}_2\mathbf{i}_3 = 1$$

or

$$\mathbf{B} = i\mathbf{b} \quad \text{with} \quad \mathbf{b} = B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3 \quad .$$

This will be useful if consideration of different interpretations of any multivector is of interest.

Any r–vector of G_3 can get an interpretation as geometric entity. Both points and lines are represented by vectors and planes are represented by bivectors. The relation of a line $\mathbf{a}$ with respect to a plane $\mathbf{B}$ is given by the geometric product $\mathbf{aB} = \mathbf{a} \cdot \mathbf{B} + \mathbf{a} \wedge \mathbf{B}$. If the line is on the plane we get $\mathbf{a} \wedge \mathbf{B} = 0$ whereas $\mathbf{a} \cdot \mathbf{B} = 0$ corresponds to a line perpendicular to the plane.

An operational interpretation of multivectors results by considering the 4–dimensional even subalgebra G_3^+. Any multivector $\mathbf{A} \in G_3^+$, $\mathbf{A} = \langle\mathbf{A}\rangle_0 + \langle\mathbf{A}\rangle_2 \equiv \alpha + \mathbf{B}$

is representing a spinor if $\mathbf{B} = i\mathbf{b}$ is used. A spinor stands for a rotation–dilation (not only in E_3). Indeed, the rotation component, represented by the rotor $\mathbf{R} = \pm e^{\mathbf{B}/2}$ with the rotation plane represented by the bivector $\mathbf{B}$, is a much more general way of expressing rotations then using matrix operations or using the frame of quaternions. The rotation $\mathbf{B}_r = \mathbf{R}\mathbf{A}_r\tilde{\mathbf{R}}$, where $\tilde{\mathbf{R}}$ stands for the conjugate of $\mathbf{R}$ and thus $\mathbf{R}\tilde{\mathbf{R}} = \tilde{\mathbf{R}}\mathbf{R} = 1$, of any r–vector $\mathbf{A}_r$ works for all spaces of any dimension on any type of objects, whatever grade. Moreover, it works without the use of external coordinates.

4 Geometric Algebra of 4D–Space

The contemporary knowledge does not allow to work out the complete theory of perception of spatio–temporal equivalence classes in the frame of stratified space–time. This theory would permit to model projective, affine, or metric perception of structure from differential motion in space. Instead, the visual interpretation of the world from image sequences is treated either as stationary n–views problem of structure from motion problem with limited information capacity. By abandoning conceptions of time, kinematics often is considered as spatial transformation or rigid displacement in Euclidean space.
In this chapter we will show the use of geometric algebra for either problems of projective geometry or kinematics by embedding both tasks in different algebraic frames.
For the sake of generality, in the characterization of the geometric algebra the signature of G_n has to be considered. Writing $G_{p,q,r}$ instead, where $n = p+q+r$, refers to the number of basis elements which square to 1 for p, -1 for q, and zero for r. For example, $G_{3,0,0} = G(E_3)$ stands for $\sigma_l^2 = 1, \quad l = 1, 2, 3$.
As has been shown by Hestenes [11], geometric algebra is well suited to deal with problems of projective geometry. Because the projective space P_3 is a non–metric one, it has to be extended to an associated 4–dimensional vector space R_4 with the basis vectors γ_l, $l = 1, 2, 3, 4$. To become consistent with the signature of G_3 for the geometric algebra of the Euclidean space, the geometric algebra of R_4 has to be $G_{1,3,0}$ [14]. The correspondence between both spaces is given by $\sigma_i \equiv \gamma_i\gamma_4$, $i = 1, 2, 3$. This 16–dimensional space is spanned by the basis

$$1, \{\gamma_l\}, \{\gamma_4\gamma_k, i\gamma_4\gamma_k\}, \{i\gamma_l\}, i \quad ,$$

with $l = 1, 2, 3, 4$, $k = 1, 2, 3$, $\gamma_4^2 = 1$, $\gamma_k^2 = -1$, $i = \gamma_1\gamma_2\gamma_3\gamma_4$, and $i^2 = -1$. Here γ_4 plays the role of a selected direction. By computing the projective split of a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$

$$\mathbf{X}\gamma_4 = \mathbf{X}\cdot\gamma_4 + \mathbf{X}\wedge\gamma_4 = X_4\left(1 + \frac{\mathbf{X}\wedge\gamma_4}{X_4}\right) \equiv X_4(1+\mathbf{x})$$

any vector $\mathbf{X} \in R_4$ may be related to a vector $\mathbf{x} \in E_3$ and vice versa by

$$\frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\gamma_1\gamma_4 + \frac{X_2}{X_4}\gamma_2\gamma_4 + \frac{X_3}{X_4}\gamma_3\gamma_4 \equiv x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \mathbf{x}$$

it may be recognized that X_i represent the homogeneous coordinates of $\mathbf{x}$. The projective split [12] is a very powerful tool for any mapping of entities between spaces of different dimension [2], [4], [14]. In this way vectors, bivectors, and trivectors in $G_{1,3,0}$ correspond to points, lines, and planes in E_3. Using the join between any non–collinear points $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3} \in E_3$, respectively $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3} \in R_4$, a plane $\Pi \in G_{1,3,0}$ passing these points is represented by

$$\Pi = \mathbf{X_1} \wedge \mathbf{X_2} \wedge \mathbf{X_3} = L \wedge \mathbf{X_3}$$

with $L = \mathbf{X_1} \wedge \mathbf{X_2}$. As an example for the meet of entities in geometric algebra we consider the intersection $L \vee \Phi$ of the above defined line L with the plane $\Phi = \mathbf{Y_1} \wedge \mathbf{Y_2} \wedge \mathbf{Y_3}$. After using some algebra [2] the intersection point $\mathbf{Z} \in R_4$ is given by

$$\mathbf{Z} = L \vee \Phi = [\mathbf{X_1 X_2 Y_2 Y_3}]\mathbf{Y_1} + [\mathbf{X_1 X_2 Y_3 Y_1}]\mathbf{Y_2} + [\mathbf{X_1 X_2 Y_1 Y_2}]\mathbf{Y_3} \quad .$$

In [14] the framework of geometric algebra has been applied for computing $3D$ projective invariants. In [2] geometric algebra has been used to compute point correspondences between n cameras and invariant projective depth by taking into account n–linear constraints. It has been shown that geometric algebra is superior to the recently used Grassmann–Cayley or Double algebra with respect to both elegance of derivations and gain of geometric insight of operations.

Besides, geometric algebra is not special to problems of projective geometry. This has been shown in [7], [4]. There the hand–eye calibration in visual robotics could be solved as a linear problem. The simultaneous estimation of translation and rotation is a nonlinear problem by itself. However, choosen a dual quaternion or motor algebra this becomes a linear problem.

Dual quaternions belong to the general class of composed numbers $a = b + \omega c$, where the algebraic operator ω specifies complex numbers for $\omega^2 = -1$, double numbers for $\omega^2 = 1$ and dual numbers for $\omega^2 = 0$. In the last case the term b is called the real part and c corresponds to the dual part of a.

Clifford [6] recognized that dual quaternions are representations of a so–called screw motion, which can be understood as a rigid motion (that means coupled rotation and translation) of lines in E_3. He introduced the dual quaternions with the name motors as abbreviation of "moment and vector". Motors are the multivectors of the even 8–dimensional subalgebra $G^{+}_{0,3,1}$ of $G_{0,3,1}$, which is the geometric algebra of a 4–dimensional vector space R_4 with a pseudometric $\gamma_4^2 = 0$, $\gamma_l^2 = -1$, $l = 1, 2, 3$. In accordance to the requirement of the algebraic operator of dual numbers $G_{0,3,1}$ is endowed with a unit pseudoscalar $i = \gamma_1\gamma_2\gamma_3\gamma_4$ which squares $i^2 = 0$. The basis of the degenerated algebra $G^{+}_{0,3,1}$ is $(1, \{\gamma_4\gamma_l\}, \{i\gamma_4\gamma_l\}, i)$, where $\{\gamma_4\gamma_l\}$ define the basis of the real part and $\{i\gamma_4\gamma_l\}$ define the basis of the dual part of motors [4].

The basic geometric interpretation of a motor $\mathbf{M}$ corresponds to the sum of two non–coplanar lines, represented in the dual basis of $G^{+}_{0,3,1}$, i.e.

$$\begin{aligned} \mathbf{M} &= \mathbf{X_1 X_2} + \mathbf{X_3 X_4} \\ &= (a_0 + a_1\gamma_4\gamma_1 + a_2\gamma_4\gamma_2 + a_3\gamma_4\gamma_3) \end{aligned}$$

$$
\begin{aligned}
&+i(b_0 + b_1\gamma_4\gamma_1 + b_2\gamma_4\gamma_2 + b_3\gamma_4\gamma_3) \\
= \quad & \mathbf{R} + i\mathbf{R}' \quad .
\end{aligned}
$$

A motor therefore can be represented as a dual rotor. On the other hand a motor is a coupled translation–rotation, i.e.

$$\mathbf{M} = \mathbf{TR} = \left(1 + i\frac{\mathbf{t}}{2}\right)\mathbf{R} \quad .$$

Here the term $\mathbf{T}$ defines a so–called translator as a rotation plane displaced from the origin of reference by vector $\mathbf{t}$ and with the same orientation of that vector. By augmenting the space E_3 by using R_4 instead, rigid displacements of lines is a very attractive alternative of that of points. In dual number representation a line $\mathbf{l}_d$,

$$\mathbf{l}_d = \mathbf{n} + i\mathbf{n} \wedge \mathbf{p} = \mathbf{n} + i\mathbf{m}, \quad i^2 = 0 \quad ,$$

represents by its real part the line direction $\mathbf{n}$ and by its dual part the moment $\mathbf{m}$ which results from vector $\mathbf{n}$ and any vector $\mathbf{p}$ touching the line. Its motion in terms of motors reads

$$\mathbf{l}'_d = \mathbf{M}\mathbf{l}_d\tilde{\mathbf{M}} = \mathbf{R}\mathbf{n}\tilde{\mathbf{R}} + i(\mathbf{R}\mathbf{n}\tilde{\mathbf{R}}' + \mathbf{R}'\mathbf{n}\tilde{\mathbf{R}} + \mathbf{R}\mathbf{m}\tilde{\mathbf{R}}) \quad .$$

Of course also the other entities exist in the augmented space R_4 but their use in tasks of rigid transformations is limited in comparison to those of lines.
It seems that unrestricted merging of projective and kinematic tasks in the frame of geometric algebra would be possible if the Euclidean space E_3 would be algebraically extended to R_5.

5 Geometric Algebra of Manifolds

Because geometric algebra plays the role of a very general scheme of embedding any task of the PAC, it will be important also for analysis of multi–dimensional signals and for (neural) mapping of perceived signals onto those of motor control. ¿From the last problem follows the necessity to reconsider the role of the linear associator in neural nets and to enrich it with the capability to process multivectors. This topic [3] should be passed over here. Instead, we will consider some problems of multi–dimensional signal processing, mentioned in the introduction, with strong relevance to early visual processing.
The well known Fourier transform hitherto is inable to provide us with possibilities of representing real multi–dimensional signals. That means it is only limited to the separable case. This is strongly related with its linear nature. On the other side real multi–dimensional structures are constituted in a non–linear manner from one–dimensional basis functions. To become adequate for multi–dimensional signals and simultaneously keeping its linear structure, a multi–dimensional Fourier transform has to be algebraically extended to the requested

dimension. In [5] a Clifford Fourier transform (CFT) $F^c(\mathbf{u})$ of an n–dimensional signal $f(\mathbf{x})$

$$F^c(\mathbf{u}) = \int \ldots \int f(\mathbf{x}) Q_{\mathbf{u}}(\mathbf{x}) d^n \mathbf{x}$$

has been introduced by defining its basis functions as

$$Q_{\mathbf{u}}(\mathbf{x}) = \prod_{k=1}^{n} e^{-j_k 2\pi u_k x_k} \quad .$$

In contrast to the classic approach each one–dimensional component transforms to its own complex domain, indicated by the n different imaginaries j_k. From this follows that the quaternionic Fourier transform (QFT), which is adequate to two–dimensional signals, splits such signal into four components in the quaternionic Fourier domain. Therefore not only more symmetry conceptions result but also only on this way a multi–dimensional phase can be defined.
Another topic of future importance will be the design of local operators which in the linear vector space of signals represent non–linear operators but in contrary to this in the multivector space of geometric algebra represent linear ones. This problem is related to the problem of estimation of higher order correlations of signals within the operator support. In principle each pixel of that domain contributes with one dimension to the signal space and consequently very high–order relations may be estimated. But this is hindered by the computational complexity of the nonlinear nature of operators. By using Volterra series approach for the local estimation of signal structure and by embedding the task in the frame of geometric algebra, the estimation problems become linear ones in the corresponding multivector subspace of geometric algebra.

6 Conclusion

We presented Clifford or geometric algebra as a powerful and general scheme of embedding any problem of perception–action cycle, ranging from kinematics via projective geometry to signal theory. The paper summarizes some key ideas of the algebra in relation to different applications in the mentioned fields. Although the development of methodology is in its infancy yet, its potential becomes visible. Both reformulations of well known approaches and extension to new approaches will result in overcoming of existing limitations in the design of technical systems which might be able to perform perception–action cycles in real–time.

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