What can Grassmann, Hamilton and Clifford tell us about Computer Vision and Robotics

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Abstract. Geometric algebra is a universal mathematical language which provides very comprehensive techniques for analyzing the complex geometric situations occurring in robotics and computer vision. The application of the 4D motor algebra for the linearization of the the hand-eye calibration problem is presented. Geometric algebra and its associated linear algebra framework is a very elegant language to express all the ideas of projective geometry. Using purely geometric derivations, the constraints for point and line correspondences in n-views and projective invariants are computed.

1 Introduction

W. K. Clifford (1876) defined the so called geometric product in an attempt to merge the algebra of the extension of H. Grassman (1877) with its quantitative interpretation of numbers and the algebra of the quaternions of W.R. Hamilton (1843) with its operational interpretation of numbers. The Clifford algebra is well known to pure mathematicians, but were long ago abandoned by physicists in favour of the vector algebra of Gibbs, which is indeed what is most commonly used today in most areas of physics. The approach to Clifford algebra we adopt here was pioneered in the 1960's by David Hestenes [4] who has, since then, worked on developing his version of Clifford algebra – which will be referred to as *geometric algebra* – into a unifying language for mathematics and physics. In this paper we will specify a geometric algebra $\mathcal{G}_n$ of the n-dimensional space by $\mathcal{G}_{p,q,r}$, where p, q and r stand for the number of basis vectors which squares to 1, -1 and 0 respectively and fulfill n=p+q+r. Its even subalgebra will be characterized by $\mathcal{G}^+_{p,q,r}$. In robotics, the emphasis of this paper is that of the hand-eye calibration using motion of lines. For computer vision geometric algebra provides a very natural language for non-metrical projective geometry and algebra of incidence. Using n uncalibrated cameras the constraints for point and line correspondences and projective invariants are computed.

2 Modelling the 3D Motion of Points, Lines and Planes

In this section we present the modelling of the 3D motion of basic geometric entities using the algebra of the motors $\mathcal{G}^+_{3,0,1}$. Firstly we summarize the dual expressions derived in [1] for points $\boldsymbol{q} = 1 + i\boldsymbol{x}$, planes $\phi = \boldsymbol{n} + id$ and lines $l_d = \boldsymbol{n} + i\boldsymbol{m}$, where the real part can be seen as the line direction and the dual part as the moment of the the line.

The motion of a point can be represented as

$$\boldsymbol{M}(1 + i\boldsymbol{x})\overline{\tilde{\boldsymbol{M}}} = \boldsymbol{T}\boldsymbol{R}(1 + i\boldsymbol{x})\tilde{\boldsymbol{R}}\boldsymbol{T} = 1 + i(\boldsymbol{R}\boldsymbol{x}\tilde{\boldsymbol{R}} + \boldsymbol{t}) \tag{1}$$

and of the line

$$\boldsymbol{l}_a = \boldsymbol{M}\boldsymbol{l}_b\tilde{\boldsymbol{M}} = \boldsymbol{R}\boldsymbol{n}_b\tilde{\boldsymbol{R}} + i(\boldsymbol{R}\boldsymbol{n}_b\tilde{\boldsymbol{R}}' + \boldsymbol{R}'\boldsymbol{n}_b\tilde{\boldsymbol{R}} + \boldsymbol{R}\boldsymbol{m}_b\tilde{\boldsymbol{R}}). \tag{2}$$

The motion of a plane can be seen as the motion of the dual of the point

$$\boldsymbol{M}(\boldsymbol{n} + id)\overline{\tilde{\boldsymbol{M}}} = \boldsymbol{R}\boldsymbol{n}\tilde{\boldsymbol{R}} + i(d + (\boldsymbol{R}\boldsymbol{n}\tilde{\boldsymbol{R}}) \cdot \boldsymbol{t}). \tag{3}$$

3 Motors for hand-eye calibration as a case of motion of lines

The well known hand-eye equation firstly formulated by Shiu and Ahmad [5] reads

$$\mathbf{AX} = \mathbf{XB} \tag{4}$$

where $\mathbf{A} = \mathbf{A}_1\mathbf{A}_2^{-1}$ and $\mathbf{B} = \mathbf{B}_1\mathbf{B}_2^{-1}$ express the elimination of the transformation hand-base to world. Equation (4) can be reformulated as $\mathbf{A}_2^{-1}\mathbf{A}_1\mathbf{Y} = \mathbf{YB}$. Now if $\mathbf{A}_2^{-1}\mathbf{A}_1$ is written as a function of the projection parameters it is possible to get an expression fully independent of the intrinsic parameters $\mathbf{C}$, i.e.

$$\mathbf{A}_2^{-1}\mathbf{A}_1 = \begin{bmatrix} \mathbf{N}_2^{-1}\mathbf{N}_1 & \mathbf{N}_2^{-1}(\mathbf{n}_1 - \mathbf{n}_2) \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0^T & 1 \end{bmatrix}. \tag{5}$$

Taking into consideration the selected matrices and relations, this result allows anyway to consider the formulation of the hand-eye problem again with the standard equation (4) which can be solved in terms of motors as

$$\boldsymbol{M}_A\boldsymbol{M}_X = \boldsymbol{M}_X\boldsymbol{M}_B \tag{6}$$

where $\boldsymbol{M}_A = \boldsymbol{A} + i\boldsymbol{A}'$, $\boldsymbol{M}_B = \boldsymbol{B} + i\boldsymbol{B}'$ and $\boldsymbol{M}_X = \boldsymbol{R} + i\boldsymbol{R}'$. According to the congruence theorem of Chen [3] in this kind of problem the rotation angle and pitch of $\boldsymbol{M}_A$ and $\boldsymbol{M}_B$ remain invariant through out all the hand movements. Thus the consideration of this information can be neglected. It suffices to regard the rotation axis of the involved motors, i.e. the previous equation is reduced as the motion of the line axis of the hand towards the line axis of the camera. For that we can use the equation (2) for the computation of the real and dual components of $\boldsymbol{l}_A$, i.e.

$$\boldsymbol{l}_A = \boldsymbol{a} + i\boldsymbol{a}' = \boldsymbol{R}\boldsymbol{b}\tilde{\boldsymbol{R}} + i(\boldsymbol{R}\boldsymbol{b}\tilde{\boldsymbol{R}}' + \boldsymbol{R}\boldsymbol{b}'\tilde{\boldsymbol{R}} + \boldsymbol{R}'\boldsymbol{b}\tilde{\boldsymbol{R}}). \tag{7}$$

After some simple manipulations according the relation $\tilde{\boldsymbol{R}}\boldsymbol{R}' + \tilde{\boldsymbol{R}}'\boldsymbol{R}$ =0 we get the following matrix

$$\begin{bmatrix} \mathbf{a}-\mathbf{b} & [\mathbf{a}+\mathbf{b}]_\times & \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 3} \\ \mathbf{a}'-\mathbf{b}' & [\mathbf{a}'+\mathbf{b}']_\times & \mathbf{a}-\mathbf{b} & [\mathbf{a}+\mathbf{b}]_\times \end{bmatrix} \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{R}' \end{bmatrix} = 0 \tag{8}$$

where the matrix - we will call $\boldsymbol{S}$ - is a 6×8 matrix and the vector of unknowns $(\boldsymbol{R}^T, \boldsymbol{R}'^T)$ is 8-dimensional. Recall that we have two constraints on the unknowns so that the result is a unit dual rotor with

$$\boldsymbol{R}\tilde{\boldsymbol{R}} = 1 \qquad \text{and} \qquad \boldsymbol{R}\boldsymbol{R}' = 0. \tag{9}$$

Suppose now that $n \geq 2$ motions are given. We construct the $6n \times 8$ matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{S}_1^T \ \boldsymbol{S}_2^T \ \ldots \ \boldsymbol{S}_n^T \end{bmatrix}^T \tag{10}$$

which in the noise-free case has rank 6 and compute the singular value decomposition (SVD) of $\boldsymbol{T} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, where $\boldsymbol{\Sigma}$ is a diagonal matrix with the singular values, the columns of $\boldsymbol{U}$ are the left singular vectors, and the columns of $\boldsymbol{V}$ are the right singular vectors. If the rank is 6 than the last two right singular vectors $\mathbf{v_7}$ and $\mathbf{v_8}$ - corresponding to the two vanishing singular values - span the null space of $\boldsymbol{T}$. We write them as composed of two 4×1 vectors $\mathbf{v_7}^T = (\mathbf{u}_1^T, \mathbf{v}_1^T)$ and $\mathbf{v_8}^T = (\mathbf{u}_2^T, \mathbf{v}_2^T)$. A vector $(\boldsymbol{R}^T, \boldsymbol{R}'^T)$ satisfying $\boldsymbol{T}(\boldsymbol{R}^T, \boldsymbol{R}'^T)^T = 0$ must be a linear combination of $\mathbf{v_7}$ and $\mathbf{v_8}$, hence

$$\begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{R}' \end{bmatrix} = \lambda_1 \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{bmatrix} + \lambda_2 \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{v}_2 \end{bmatrix}.$$

The two degrees of freedom are fixed by the constraints (9) which imply two quadratic equations in λ_1 and λ_2 which can be solved after simple algebraic manipulations. The computational algorithm and more technical details of this approach can be found in [1].

4 Point and line linear constraints

In this section we will outline the geometric algebra approach to the analysis of point and line correspondences between two and three cameras. For the projective space P^3 we choose the geometric algebra $\mathcal{G}_{1,3,0}$ [2]. Let $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$, $(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ define the image planes in views 1, 2, 3 and let $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{c}_0$ be the corresponding optical centres. We start with the well understood case of two cameras.

4.1 Two cameras: the bilinear or epipolar constraint

Now, the epipolar constraint is simply that $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{a}'_i$, $\boldsymbol{b}'_i$ are coplanar if $\boldsymbol{a}'_i$ and $\boldsymbol{b}'_i$ are projections of the same world point. This can be concisely written as $L_A \wedge L_B = 0$ where $L_A = \mathbf{A}_0 \wedge \mathbf{A}'_i$ and $L_B = \mathbf{B}_0 \wedge \mathbf{B}'_i$ or $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_i\mathbf{B}'_i] = 0$. The latter expressed in terms of the $\mathbf{A}'_i$, $\mathbf{B}'_i$

$$[\mathbf{A}_0\mathbf{B}_0(\alpha_{i1}\mathbf{A}_1 + \alpha_{i2}\mathbf{A}_2 + \alpha_{i3}\mathbf{A}_3)(\beta_{i1}\mathbf{B}_1 + \beta_{i2}\mathbf{B}_2 + \beta_{i3}\mathbf{B}_3)] = \boldsymbol{\alpha_i}^T \tilde{\boldsymbol{F}} \boldsymbol{\beta_i} = 0. \quad (11)$$

4.2 Three cameras: the trilinear constraints

For point correspondences in three views we also have constraints of the following form;

$$L_A \wedge \{\Phi_{Bi} \vee \Phi_{Cj}\} = 0, \quad L_B \wedge \{\Phi_{Ai} \vee \Phi_{Cj}\} = 0, \quad L_C \wedge \{\Phi_{Ai} \vee \Phi_{Bj}\} = 0 \quad (12)$$

where Φ_{Ak}, Φ_{Bk} and Φ_{Ck} are planes through the optical centres of views 1, 2 and 3 and through the world point P_i defined by $\Phi_{Ak} = \mathbf{A}_0 \wedge \mathbf{A}_k \wedge \mathbf{A}'_i$ etc. So, the first constraint in equation (12) is simply saying that line L_A and the line of intersection of planes Φ_{Bi} and Φ_{Cj} must intersect at a point – this point being P (drop subscript i on P, $\mathbf{A}'$ etc.), see Figure 1. Let us express this first constraint in terms of R^4 vectors

$$L_A \wedge \{\Phi_{Bi} \vee \Phi_{Cj}\} = (\mathbf{A}_0 \wedge \mathbf{A}') \wedge \{(\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}') \vee (\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}')\} = 0. \quad (13)$$

The points can be also expanded in terms of the R^4 vectors $\mathbf{A}' = \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_3$, $\mathbf{B}' = \beta_1\mathbf{B}_1 + \beta_2\mathbf{B}_2 + \beta_3\mathbf{B}_3$ and $\mathbf{C}' = \delta_1\mathbf{C}_1 + \delta_2\mathbf{C}_2 + \delta_3\mathbf{C}_3$.We can therefore write $\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}' \beta_l (\mathbf{B}_0 \wedge \mathbf{B}_i \wedge \mathbf{B}_l) \equiv \beta_l \Phi^B_{il}$ and $\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}' = \delta_m(\mathbf{C}_0 \wedge \mathbf{C}_j \wedge \mathbf{C}_m) \equiv \delta_m \Phi^C_{jm}$,where we have now renamed the planes Φ_{11} etc. as given above. The constraint in equation (13) can now be written as

$$\alpha_k(\mathbf{A}_0 \wedge \mathbf{A}_k) \wedge \{\beta_l \delta_m (\Phi^B_{il} \vee \Phi^C_{jm})\} = 0 \quad (14)$$

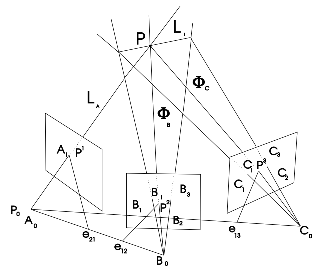

Fig. 1. The joint of a line with the resultant line of two intersecting planes.

which can be put into the tensor form

$$\tilde{T}^{ij}_{klm}\alpha_k\beta_l\delta_m = 0 \tag{15}$$

where

$$\tilde{T}^{ij}_{klm} = [\mathbf{A}_0\mathbf{A}_k(\Phi^B_{il} \vee \Phi^C_{jm})]. \tag{16}$$

This is equivalent to the *trilinear constraint*.

4.3 Line correspondences between three cameras

Suppose we have world points P_1 and P_2, whose R^4 representations are $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. The line L_{12} joining $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can be expressed as $L_{12} = \boldsymbol{P}_1 \wedge \boldsymbol{P}_2$. L_{12} projects down to lines in the three image planes we shall call $L^A_{12} = \mathbf{A}'_1 \wedge \mathbf{A}'_2$, $L^B_{12} = \mathbf{B}'_1 \wedge \mathbf{B}'_2$ and $L^C_{12} = \mathbf{C}'_1 \wedge \mathbf{C}'_2$.

As before, we can expand $\mathbf{A}'_i$ as $\alpha^i_1\mathbf{A}_1 + \alpha^i_2\mathbf{A}_2 + \alpha^i_3\mathbf{A}_3$. L^A_{12} can then be expanded in terms of the 'basis bivectors' L^A_k as follows $L^A_{12} = l_k L^A_k$, where $L^A_1 = \mathbf{A}_2 \wedge \mathbf{A}_3$, $L^A_2 = \mathbf{A}_3 \wedge \mathbf{A}_1$ and $L^A_3 = \mathbf{A}_1 \wedge \mathbf{A}_2$ and $l_1 = \alpha^1_2\alpha^2_3 - \alpha^1_3\alpha^2_2$ etc. Similarly we have $L^B_{12} = l'_l L^B_l$ and $L^C_{12} = l''_m L^C_m$. To arrive at a constraint between the lines in the image planes we note that the line $L_{12} = \boldsymbol{P}_1 \wedge \boldsymbol{P}_2$ can be expressed as the meet of the planes $(\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2)$ and $(\mathbf{C}_0 \wedge \mathbf{C}'_1 \wedge \mathbf{C}'_2)$. Also $L^A_{12} = \mathbf{A}'_1 \wedge \mathbf{A}'_2$ can be written as the meet of planes $(\mathbf{A}_0 \wedge \boldsymbol{P}_1 \wedge \boldsymbol{P}_2)$ and $(\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3)$. We therefore have the identity

$$\mathbf{A}'_1 \wedge \mathbf{A}'_2 = \{\mathbf{A}_0 \wedge \{(\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2) \vee (\mathbf{C}_0 \wedge \mathbf{C}'_1 \wedge \mathbf{C}'_2)\}\} \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3). \tag{17}$$

Using the expansions in terms of the line coefficients this reduces to

$$l_k L^A_k = \{\mathbf{A}_0 \wedge \{l'_l \Phi^B_l \vee l''_m \Phi^C_m\}\} \vee (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3) \tag{18}$$

where $\Phi_1^B = \mathbf{B}_0 \wedge \mathbf{B}_2 \wedge \mathbf{B}_3$, etc.. Using the definition of the meet we can form the expression $l_k L_k^A = l'_l l''_m \{[\mathbf{A}_0 \{\Phi_l^B \vee \Phi_m^C\} \mathbf{A}_n] L_n^A\}$. From this it is clear that the relationship between the l's is

$$l_k = l'_l l''_m [\mathbf{A}_0 \{\Phi_l^B \vee \Phi_m^C\} \mathbf{A}_k] = l'_l l''_m T_{klm}$$

from the definition of $\tilde{T}$ in terms of Hartley's tensor.

5 Projective Invariants

This section shows the power of geometric algebra by computing a well known invariant which results when we consider six 3D points P_i, $i = 1, .., 6$ in general position, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in E^3 and R^4 respectively. 3D projective invariants can be formed from these points, and an example of such an invariant is

$$Inv = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{19}$$

It will be highly desirable to compute the brackets $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ simply in terms of **image coordinates** of points P_i, P_j, P_k, P_l, in order to compute this invariant straightforwardly. Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4{}^{-1} = (\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4) I_4{}^{-1}. \tag{20}$$

The quantities $(\mathbf{X}_1 \wedge \mathbf{X}_2)$ and $(\mathbf{X}_3 \wedge \mathbf{X}_4)$ represent the line joining points P_1 and P_2, and P_3 and P_4. Let the projection of points $\{P_i\}$ through the centres of projection onto the image planes be given by the vectors $\{\boldsymbol{a}'_i\}$ and $\{\boldsymbol{b}'_i\}$ which are ordinary vectors in E^3. The representations of these vectors in R^4 will be $\mathbf{A}_i, \mathbf{B}_i, \mathbf{A}'_i, \mathbf{B}'_i ...$, etc. In [2] it is shown that the bracket of these 4 points (in R^4) can be equated as

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] \equiv [\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]. \tag{21}$$

Thus, when we take ratios of brackets to form our invariants we must ensure that, if we want to express the brackets in the form of equation (21), the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ must occur in the numerator and denominator so that these arbitrary factors cancel. In the case of Inv, we have

$$Inv = \frac{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_3 \wedge \mathbf{X}_4)\} I_4{}^{-1} \{(\mathbf{X}_4 \wedge \mathbf{X}_5) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}{\{(\mathbf{X}_1 \wedge \mathbf{X}_2) \wedge (\mathbf{X}_4 \wedge \mathbf{X}_5)\} I_4{}^{-1} \{(\mathbf{X}_3 \wedge \mathbf{X}_4) \wedge (\mathbf{X}_2 \wedge \mathbf{X}_6)\} I_4{}^{-1}}. \tag{22}$$

The problem of how to express such invariants in observed coordinates still remains. This is not a trivial task as this invariant has been derived in 3D using the 4D definition of the fundamental matrix and it needs to be transferred to 3D. Expanding the bracket in equation(21) by expressing the intersection points in terms of the $\mathbf{A}$'s and $\mathbf{B}$'s and defining a matrix $\tilde{\boldsymbol{F}}$ such that

$$\tilde{F}_{ij} = [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_i\mathbf{B}_j] \tag{23}$$

and vectors $\boldsymbol{\alpha}_{1234} = (\alpha_{1234,1}, \alpha_{1234,2}, \alpha_{1234,3})$ and $\boldsymbol{\beta}_{1234} = (\beta_{1234,1}, \beta_{1234,2}, \beta_{1234,3})$. It is easy to see that we can write $S_{1234} = \boldsymbol{\alpha}^T{}_{1234}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1234}$. The ratio

$$Inv = \frac{(\boldsymbol{\alpha}^T{}_{1234}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1234})(\boldsymbol{\alpha}^T{}_{4526}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{4526})}{(\boldsymbol{\alpha}^T{}_{1245}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1245})(\boldsymbol{\alpha}^T{}_{3426}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{3426})} \tag{24}$$

is therefore seen to be an invariant. Now if we define $\tilde{\boldsymbol{F}}$ by

$$\tilde{F}_{kl} = (\mathbf{A}_k \cdot \gamma_4)(\mathbf{B}_l \cdot \gamma_4)F_{kl} \tag{25}$$

then it follows that

$$\alpha_{ik}\tilde{F}_{kl}\beta_{il} = (\mathbf{A}'_i \cdot \gamma_4)(\mathbf{B}'_i \cdot \gamma_4)\delta_{ik}F_{kl}\epsilon_{il}. \tag{26}$$

If $\boldsymbol{F}$ is estimated then an $\tilde{\boldsymbol{F}}$ defined as in equation (25) will also act as a fundamental matrix in R^4. Now let us look again at the invariant Inv. According to the above, we can write the invariant as

$$Inv = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})\phi_{1234}\phi_{4526}}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})\phi_{1245}\phi_{3426}} \tag{27}$$

where

$$\phi_{pqrs} = (\mathbf{A}'_{pqrs} \cdot \gamma_4)(\mathbf{B}'_{pqrs} \cdot \gamma_4). \tag{28}$$

By expressing $\mathbf{A}'_{1234}$ as the intersection of the line joining $\mathbf{A}'_1$ and $\mathbf{A}'_2$ with the plane through $\mathbf{A}_0, \mathbf{A}'_3, \mathbf{A}'_4$ we can projective split and equate terms to give

$$\frac{(\mathbf{A}'_{1234} \cdot \gamma_4)(\mathbf{A}'_{4526} \cdot \gamma_4)}{(\mathbf{A}'_{3426} \cdot \gamma_4)(\mathbf{A}'_{1245} \cdot \gamma_4)} = \frac{\mu_{1245}(\mu_{3426} - 1)}{\mu_{4526}(\mu_{1234} - 1)}. \tag{29}$$

The values of μ are readily obtainable from the images. The factors $\mathbf{B}'_{pqrs} \cdot \gamma_4$ are found in a similar way so that if $\boldsymbol{b}'_{1234} = \lambda_{1234}\boldsymbol{b}'_4 + (1 - \lambda_{1234})\boldsymbol{b}'_3$ etc., the overall expression for the invariant becomes

$$Inv = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})}\frac{\mu_{1245}(\mu_{3426} - 1)}{\mu_{4526}(\mu_{1234} - 1)} \cdot \frac{\lambda_{1245}(\lambda_{3426} - 1)}{\lambda_{4526}(\lambda_{1234} - 1)}. \tag{30}$$

Thus, to summarize, given the coordinates of a set of 6 corresponding points in the two image planes (where these 6 points are projections from arbitrary world points but with the assumption that they are not coplanar) we can form 3D projective invariants provided we have some estimate of $\boldsymbol{F}$.

6 Conclusions

This paper has focused in the geometry and algebra of 3D and 4D spaces which are necessary for the representation and manipulation of basic geometric entities required in robotics and computer vision. In the case of the hand-eye calibration when the 3D representation is extended to the 4-D space using the motor algebra the problem of computing the unknown motion becomes linear. In computer vision geometric algebra does not need to invoke the standard concepts of classical projective geometry, all that is needed is the idea of the *projective split* relating the quantities in R^4 to quantities in our 3D world and the algebra of incidence. The case of computing linear constraints and invariants using n uncalibrated cameras is elegantly carried out. Here the duality principle and projective split help to reduce the complexity of the computation. It have been seen by the problems treated in this paper that geometric algebra indeed has a powerful representation capability and a strong geometric basis. That is why the authors believe that it is a competitive language to provide a unified approach [6] for the design and implementation of visual guided autonomous systems.

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