Computing 3D Projective Invariants from Points and Lines

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Abstract. In this paper we will look at some 3D projective invariants for both point and line matches over several views and, in the case of points, give explicit expressions for forming these invariants in terms of the image coordinates. We discuss whether such invariants are useful by looking at their formation on simulated data.

1 Introduction

The following sections will derive the forms for some 3D projective invariants using the system of *Geometric Algebra* (GA). Geometric algebra is a coordinate-free approach to geometry based on the algebras of Grassmann [5] and Clifford [3] – the approach we adopt here was pioneered by David Hestenes [7,6]. We will use the GA framework for projective geometry and geometric invariance outlined in [9,10]. A more extensive discussion of the formation of invariants in GA is given in [11].

2 Geometric Algebra – a Brief Outline

An n-dimensional geometric algebra is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford** product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ and can be expressed as a sum of its symmetric ($\boldsymbol{a}\cdot\boldsymbol{b}$) and antisymmetric ($\boldsymbol{a}\wedge\boldsymbol{b}$) parts

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}. \tag{1}$$

The inner product of two vectors is the standard *scalar* or *dot* product. The outer or wedge product of two vectors is a new quantity we call a **bivector**. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$. Thus, $\boldsymbol{b}\wedge\boldsymbol{a}$ will have the opposite orientation making the wedge product anticommutative. The outer product is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a **trivector**, is interpreted as the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$. The outer product of k vectors is a k-vector, which has *grade* k. Sums of objects of different grades are called *multivectors* and GA provides a system in which we can efficiently manipulate multivectors. In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I.

3 Projective Space and the Projective Split

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$, a 3D space with a Euclidean metric. Since any point on a line through some origin O will be mapped to a single point in the image plane, we associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. We then define basis vectors: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in R^4 and $(\sigma_1, \sigma_2, \sigma_3)$ in $\mathcal{E}^3$ and identify R^4 and $\mathcal{E}^3$ with the geometric algebras of 4 and 3 dimensions. We require that vectors, bivectors and trivectors in R^4 will represent points, lines and planes in $\mathcal{E}^3$. Choosing γ_4 as a selected direction in R^4, we can then define a mapping which associates the bivectors $\gamma_i\gamma_4$, $i = 1, 2, 3$, in R^4 with the vectors σ_i, $i = 1, 2, 3$, in $\mathcal{E}^3$. This process of association is called the **projective split**. To ensure $\sigma_i^2 = +1$ we are forced to assume a non-Euclidean metric for the basis vectors in R^4. We choose to use $\gamma_4^2 = +1$, $\gamma_i = -1$, $i = 1, 2, 3$.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is obtained by taking the geometric product of $\mathbf{X}$ and γ_4. This leads to the association of the vector $\boldsymbol{x}$ in $\mathcal{E}^3$ with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4 so that

$$\boldsymbol{x} = \frac{X_1}{X_4}\gamma_1\gamma_4 + \frac{X_2}{X_4}\gamma_2\gamma_4 + \frac{X_3}{X_4}\gamma_3\gamma_4 = \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3, \tag{2}$$

which $\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1, 2, 3$. The process of representing $\boldsymbol{x}$ in a higher dimensional space can therefore be seen to be equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$.

3.1 Projective Geometry and Algebra in Projective Space

We now look at the basic projective geometry operations of *meet* and *join*, and briefly discuss algebra in projective space. For more detail the reader is referred to [8,11,9]

Any pseudoscalar P can be written as $P = \alpha I$ where α is a scalar, so that

$$PI^{-1} = \alpha II^{-1} = \alpha \equiv [P]. \tag{3}$$

This *bracket* is precisely the bracket of the Grassmann-Cayley algebra. We then define the dual, A^*, of an r-vector A as

$$A^* = AI^{-1}. \tag{4}$$

The **join** $J = A \bigwedge B$ of an r-vector A and an s-vector B by

$$J = A\wedge B \qquad \text{if } A \text{ and } B \text{ are linearly independent,} \tag{5}$$

while it can be shown that the *meet* of A and B can be written as

$$A \vee B = (A^*\wedge B^*)I = (A^*\wedge B^*)(I^{-1}I)I = (A^*\cdot B). \tag{6}$$

The join and meet can be used to describe lines and planes and to intersect these quantities. Consider three non-collinear points, P_1, P_2, P_3, represented by vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in $\mathcal{E}^3$ and by vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in R^4. The line L_{12} joining points P_1 and P_2, and the plane Φ_{123} passing through points P_1, P_2, P_3, can be expressed in R^4 by the following bivector and trivector respectively:

$$L_{12} = \mathbf{X}_1\wedge\mathbf{X}_2 \qquad \Phi_{123} = \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3. \tag{7}$$

In $\mathcal{E}^3$ the intersection of a line and a plane, two planes and two lines can be dealt with entirely using the meet operation. Details and derivations are given in [11].

4 3-D Projective Invariants from Multiple Views

Given six general 3D points P_i, $i = 1,..,6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4, we can form 3D projective invariants. One such invariant is

$$Inv1 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{8}$$

If one can express the bracket $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ in terms of the **image coordinates** of the points, then this invariant will be readily computable. Some recent work which has addressed this problem has utilized the Grassmann-Cayley (CG) algebra [2,4]. In [2] invariants were computed from a pair of images in terms of the image coordinates and the fundamental matrix, $\boldsymbol{F}$, using the CG-algebra. Despite the clarity of the derivations in [2], some degree of confusion has arisen when subsequent workers have tried to implement these invariants with real data [4]. In the following sections we will look at how we would derive, using the GA formalism, explicit expressions for the invariants in terms of the experimental data and discuss why this confusion has arisen.

Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1} = (\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)I_4{}^{-1}. \tag{9}$$

The quantities $(\mathbf{X}_1 \wedge \mathbf{X}_2)$ and $(\mathbf{X}_3 \wedge \mathbf{X}_4)$ represent the lines joining points P_1 & P_2, and P_3 & P_4. $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ are the centres of projection of the two cameras and the two camera image planes are defined by the two sets of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$. The projection of points $\{P_i\}$ through the centres of projection onto the image planes are given by the vectors $\{\boldsymbol{a}'_i\}$ and $\{\boldsymbol{b}'_i\}$. Note that the vectors, $\boldsymbol{a}_i, \boldsymbol{b}_i$, etc., are vectors in $\mathcal{E}^3$; we let the representations of these vectors in R^4 be $\mathbf{A}_i, \mathbf{B}_i, \mathbf{A}'_i, \mathbf{B}'_i$..., etc.

It can be shown [11] that we are able to reproduce the result given in [2], namely that it is possible to write the bracket of the 4 points (in R^4) as

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] \equiv [\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}], \tag{10}$$

where $\mathbf{A}'_{1234}$ is the 4D representation of $\boldsymbol{a}'_{1234}$, the intersection of the lines joining $\{\boldsymbol{a}'_1 \text{ \& } \boldsymbol{a}'_2\}$ and $\{\boldsymbol{a}'_3 \text{ \& } \boldsymbol{a}'_4\}$. In [11] equation (10) is obtained by splitting up the bracket into $\mathbf{X}_1 \wedge \mathbf{X}_2$ and $\mathbf{X}_3 \wedge \mathbf{X}_4$, and then expressing each of these lines (bivectors) as the meet of two planes (trivectors). When we take ratios of brackets to form invariants, the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ must occur in the numerator and denominator so the factors, due to the choices of the γ_4 components, cancel. In the case of $Inv1$ given in equation (8), we have

$$Inv1 = \frac{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)\}I_4{}^{-1}\{(\mathbf{X}_4\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_4\wedge\mathbf{X}_5)\}I_4{}^{-1}\{(\mathbf{X}_3\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}} \tag{11}$$

so we see that this decomposition rule has been obeyed. Consider now the invariant which can be thought of as arising from 4 points and a line (since the line $\mathbf{X}_1\wedge\mathbf{X}_2$ appears in each bracket), namely

$$Inv2 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]}. \tag{12}$$

We note that we can simply rearrange equation (12) into the form of equation (8) and decompose into bivectors to obtain the following

$$Inv2 \equiv \frac{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1423}\mathbf{B}'_{1423}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1526}\mathbf{B}'_{1526}]}{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1523}\mathbf{B}'_{1523}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1426}\mathbf{B}'_{1426}]}. \tag{13}$$

So far the invariants have been derived in 4D using the 4D definition of the fundamental matrix; we therefore need to correctly transfer the expression to 3D. Expanding the bracket in equation (10) by expressing the intersection points in terms of the $\mathbf{A}$s and $\mathbf{B}$s ($\mathbf{A}'_i = \alpha_{ij}\mathbf{A}_j$ and $\mathbf{B}'_i = \beta_{ij}\mathbf{B}_j$) and defining a matrix $\tilde{\boldsymbol{F}}$ such that

$$\tilde{F}_{ij} = [\mathbf{A}_0\mathbf{B}_0\mathbf{A}_i\mathbf{B}_j] \tag{14}$$

and vectors $\boldsymbol{\alpha}_{1234} = (\alpha_{1234,1}, \alpha_{1234,2}, \alpha_{1234,3})$ and $\boldsymbol{\beta}_{1234} = (\beta_{1234,1}, \beta_{1234,2}, \beta_{1234,3})$, it is easy to see that we can write $S_{1234} = \boldsymbol{\alpha}^T{}_{1234}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1234}$ [2]. The ratio

$$Inv1 = \frac{(\boldsymbol{\alpha}^T{}_{1234}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1234})(\boldsymbol{\alpha}^T{}_{4526}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{4526})}{(\boldsymbol{\alpha}^T{}_{1245}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1245})(\boldsymbol{\alpha}^T{}_{3426}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{3426})} \tag{15}$$

is therefore an invariant. We now wish to express $Inv1$ in terms of the observed image coordinates and the fundamental matrix calculated from these coordinates. A point P_i will be projected onto points $\boldsymbol{a}'_i$ and $\boldsymbol{b}'_i$ in image planes 1 and 2, which can be written as

$$\boldsymbol{a}'_i = \boldsymbol{a}_1 + \lambda_i(\boldsymbol{a}_2 - \boldsymbol{a}_1) + \mu_i(\boldsymbol{a}_3 - \boldsymbol{a}_1) = \delta_{i1}\boldsymbol{a}_1 + \delta_{i2}\boldsymbol{a}_2 + \delta_{i3}\boldsymbol{a}_3 \tag{16}$$

so that $\sum_{j=1}^3 \delta_{ij} = 1$. Similarly, we have $\boldsymbol{b}'_i = \epsilon_{i1}\boldsymbol{b}_1 + \epsilon_{i2}\boldsymbol{b}_2 + \epsilon_{i3}\boldsymbol{b}_3$ (so that $\sum_{j=1}^3 \epsilon_{ij} = 1$). Using the projective split we can now write the α_{ij}'s and β_{ij}'s in terms of the δ_{ij}'s and ϵ_{ij}'s:

$$\alpha_{ij} = \frac{\mathbf{A}'_i \cdot \gamma_4}{\mathbf{A}_j \cdot \gamma_4}\delta_{ij} \qquad \beta_{ij} = \frac{\mathbf{B}'_i \cdot \gamma_4}{\mathbf{B}_j \cdot \gamma_4}\epsilon_{ij} \tag{17}$$

The 'fundamental' matrix $\tilde{\boldsymbol{F}}$ is such that $\boldsymbol{\alpha}^T{}_i\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{\boldsymbol{i}} = 0$, if $\boldsymbol{\alpha}_{\boldsymbol{i}}$ and $\boldsymbol{\beta}_{\boldsymbol{i}}$ correspond to the same world point P_i. Given more than eight pairs of corresponding observed points in the two planes, $(\boldsymbol{\delta}_{\boldsymbol{i}}, \boldsymbol{\epsilon}_{\boldsymbol{i}})$, $i = 1, .., 8$, we can form an 'observed' fundamental matrix $\boldsymbol{F}$ such that

$$\boldsymbol{\delta}^T{}_i\boldsymbol{F}\boldsymbol{\epsilon}_i = 0. \tag{18}$$

This $\boldsymbol{F}$ can be found by some method such as the Longuet-Higgins 8-point algorithm [12] or, more correctly, by some method which gives an $\boldsymbol{F}$ which has the true structure [13]. Therefore, if we define $\tilde{\boldsymbol{F}}$ by

$$\tilde{F}_{kl} = (\mathbf{A}_k \cdot \gamma_4)(\mathbf{B}_l \cdot \gamma_4)F_{kl} \tag{19}$$

then it follows from equation (17) that

$$\alpha_{ik}\tilde{F}_{kl}\beta_{il} = (\mathbf{A}'_i \cdot \gamma_4)(\mathbf{B}'_i \cdot \gamma_4)\delta_{ik}F_{kl}\epsilon_{il}. \tag{20}$$

Therefore an $\tilde{\boldsymbol{F}}$ defined as in equation (19) will also act as a fundamental matrix in R^4.

According to the above, we can write the invariant as

$$Inv1 = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})\phi_{1234}\phi_{4526}}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})\phi_{1245}\phi_{3426}} \tag{21}$$

where $\phi_{pqrs} = (\mathbf{A}'_{pqrs}\cdot\gamma_4)(\mathbf{B}'_{pqrs}\cdot\gamma_4)$. The ratio of the $\boldsymbol{\delta}^T\boldsymbol{F}\boldsymbol{\epsilon}$ terms using only the observed coordinates and the estimated fundamental matrix, will therefore not be an invariant – one must include the factors ϕ_{1234} etc. It is easy to show [11] that these factors can be formed as follows:

Since $\boldsymbol{a}'_3$, $\boldsymbol{a}'_4$ and $\boldsymbol{a}'_{1234}$ are collinear we can write $\boldsymbol{a}'_{1234} = \mu_{1234}\boldsymbol{a}'_4 + (1-\mu_{1234})\boldsymbol{a}'_3$. Then, by expressing $\mathbf{A}'_{1234}$ as the intersection of the line joining $\mathbf{A}'_1$ and $\mathbf{A}'_2$ with the plane through $\mathbf{A}_0, \mathbf{A}'_3, \mathbf{A}'_4$ we can projective split and equate terms to give

$$\frac{(\mathbf{A}'_{1234}\cdot\gamma_4)(\mathbf{A}'_{4526}\cdot\gamma_4)}{(\mathbf{A}'_{3426}\cdot\gamma_4)(\mathbf{A}'_{1245}\cdot\gamma_4)} = \frac{\mu_{1245}(\mu_{3426}-1)}{\mu_{4526}(\mu_{1234}-1)}. \tag{22}$$

We obtain the values of μ from the images. The factors $\mathbf{B}'_{pqrs}\cdot\gamma_4$ are found in a similar way so that if $\boldsymbol{b}'_{1234} = \lambda_{1234}\boldsymbol{b}'_4 + (1-\lambda_{1234})\boldsymbol{b}'_3$ etc., the overall expression for the invariant becomes

$$Inv1 = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})}\frac{\mu_{1245}(\mu_{3426}-1)}{\mu_{4526}(\mu_{1234}-1)}\cdot\frac{\lambda_{1245}(\lambda_{3426}-1)}{\lambda_{4526}(\lambda_{1234}-1)}. \tag{23}$$

While the above has adopted the approach of forming all invariants in 4D and then finding the equivalent expression in 3D, the approach outlined in [2] gave the invariant in the form of equation (15), **but** did indeed *define* $\boldsymbol{\alpha}_{1234}$ as follows:

$$\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\wedge\boldsymbol{\alpha}_3\boldsymbol{\alpha}_4 \tag{24}$$

where the '$\wedge$' in this equation is the *meet* of the Cayley-Grassmann algebra. Thus, $\boldsymbol{\alpha}_{1234}$ is **not** the homogeneous coordinate vector of the intersection point of the two lines in the image plane joining $\mathbf{A}'_1$ & $\mathbf{A}'_2$ and $\mathbf{A}'_3$ & $\mathbf{A}'_4$, but rather some multiple of that vector, given by equation (24). It can be easily shown that computing the invariant using equation (24) and the corresponding expressions for the other intersection points, produces exactly those correction factors arrived at by us in equation (23). It is therefore likely that the past confusion over the formation of the invariants has been soley due to the misinterpretation of the nature of the quantities $\boldsymbol{\alpha}_{ijkl}$ and $\boldsymbol{\beta}_{ijkl}$; however, the derivation we have presented here is totally unambiguous and, by clearly distinguishing between 3- and 4D quantities, cannot be misinterpreted.

5 3D Projective Invariants for Lines

Consider again the projective invariant $Inv2$. Splitting equation (12) into bivectors gives

$$Inv2 = \frac{[L_1\wedge L_2][L_3\wedge L_4]}{[L_1\wedge L_4][L_3\wedge L_2]} \tag{25}$$

where, $L_1 = \mathbf{X}_1\wedge\mathbf{X}_3$, $L_2 = \mathbf{X}_2\wedge\mathbf{X}_4$, $L_3 = \mathbf{X}_1\wedge\mathbf{X}_6$ and $L_4 = \mathbf{X}_2\wedge\mathbf{X}_5$. We thus have an invariant of four lines (provided the lines are not coplanar). Following the notation used in [1] we can express each of these lines as an intersection of planes:

$$L_1 = l_i^{13} l_j^{13'} (\Phi_i^A \vee \Phi_j^B) \qquad L_2 = l_k^{24} l_m^{24'} (\Phi_k^A \vee \Phi_m^B) \tag{26}$$

$$L_3 = l_n^{16} l_p^{16'} (\Phi_n^A \vee \Phi_p^B) \qquad L_4 = l_q^{25} l_r^{25'} (\Phi_q^A \vee \Phi_r^B). \tag{27}$$

In this expression $\Phi_1^A = \mathbf{A}_0 \wedge \mathbf{A}_2 \wedge \mathbf{A}_3$, $\Phi_2^A = \mathbf{A}_0 \wedge \mathbf{A}_3 \wedge \mathbf{A}_1$ etc., and the ls and l's are the line coordinates (equivalent to the homogeneous line coordinates) defined by,

$$\mathbf{A}'_1 \wedge \mathbf{A}'_2 = l_i^{12} L_i^A \qquad \mathbf{B}'_1 \wedge \mathbf{B}'_2 = l_i^{12'} L_i^B \tag{28}$$

where, $L_1^A = \mathbf{A}_2 \wedge \mathbf{A}_3$, $L_2^A = \mathbf{A}_3 \wedge \mathbf{A}_1$ and $L_3^A = \mathbf{A}_1 \wedge \mathbf{A}_2$ etc. We can now write $L_1 \wedge L_2$ as

$$L_1 \wedge L_2 = l_i^{13} l_j^{13'} l_k^{24} l_m^{24'} \{(\Phi_i^A \vee \Phi_j^B) \wedge (\Phi_k^A \vee \Phi_m^B)\} = S_{ijkm} l_i^{13} l_j^{13'} l_k^{24} l_m^{24'} \tag{29}$$

where we define the 4th rank tensor S_{ijkm} by

$$S_{ijkm} = \{(\Phi_i^A \vee \Phi_j^B) \wedge (\Phi_k^A \vee \Phi_m^B)\}. \tag{30}$$

It can be shown that S has only 9 independent elements which are, of course, the elements of $\boldsymbol{F}$. S relates pairs of intersecting lines in two images via the following equation;

$$S_{ijkm} l_i^{ab} l_j^{ab'} l_k^{ac} l_m^{ac'} = 0, \tag{31}$$

where l_i^{ab} are the line coordinates of the line joining points a and b in the first image etc. Thus, according to equations (25) and (29), given two pairs of intersecting lines ((13)&(16) and (24)&(25)), we can form the following 3D projective invariant:

$$Inv2 = \frac{(S_{ijkm} l_i^{13} l_j^{13'} l_k^{24} l_m^{24'})(S_{npqr} l_i^{16} l_j^{16'} l_k^{25} l_m^{25'})}{(S_{ijqr} l_i^{13} l_j^{13'} l_k^{25} l_m^{25'})(S_{npkm} l_i^{16} l_j^{16'} l_k^{24} l_m^{24'})}. \tag{32}$$

We note that the above is equivalent to the determination of the invariant of 4 lines given in [2]:

$$Inv2 = \frac{(\boldsymbol{l}^T{}_{12} \tilde{\boldsymbol{F}} \boldsymbol{l}'_{12})(\boldsymbol{l}^T{}_{34} \tilde{\boldsymbol{F}} \boldsymbol{l}'_{34})}{(\boldsymbol{l}^T{}_{14} \tilde{\boldsymbol{F}} \boldsymbol{l}'_{14})(\boldsymbol{l}^T{}_{32} \tilde{\boldsymbol{F}} \boldsymbol{l}'_{32})} \tag{33}$$

where $\boldsymbol{l}_{ij} = \boldsymbol{l}_j \times \boldsymbol{l}_i$ etc, with $\boldsymbol{l}_i$ the homogeneous line coordinates. A fuller discussion of the subject of 3D projective invariants from lines will be given elsewhere.

6 Experiments

Here we investigate the formation of the 3D projective invariants from sets of 6 matching image points – in particular we look at their stability in noisy environments.

The simulated data was a set of 38 points taken from the vertices of a wire-frame house and viewed from three different camera positions. From three sets of 6 points (non coplanar) we form $Inv1$ for each set over views 1 & 2, 1 & 3 and 2 & 3. During the simulations the world points are projected onto the image planes and then gaussian noise is added. Figure 1 shows results for the three sets of points chosen. In figure 1, a), c), e) we plot the value of the invariant with increasing noise. In a), c), and e) the invariant was formed using an $\boldsymbol{F}$ calculated via a linear least-squares method from a set of 30 matching points. Figure 1 b), d) and f) show the same invariants formed this time by taking the noisy point matches but the true value of $\boldsymbol{F}$ (i.e. that formed in the noiseless case). The true values of the invariants for the three sets of lines were 0.655, 0.402 and 8.99.

For small values of the noise the invariants can be calculated accurately. In greater noise large variations are possible for some invariants whereas other invariants are relatively robust. Figure 1 indicates that uncertainties in the calculation of $\boldsymbol{F}$ will significantly affect the invariant in some cases. It is also apparent that the formation of this invariant is more accurate between some pairs of views than between others. We should expect this since altering the view may mean that the 6 points move closer to some unstable or degenerate configuration. In summary it appears that the type of invariant described here may be useful for data which is not noisy but that the degradation in the presence of significant noise may render it ineffective for real images.

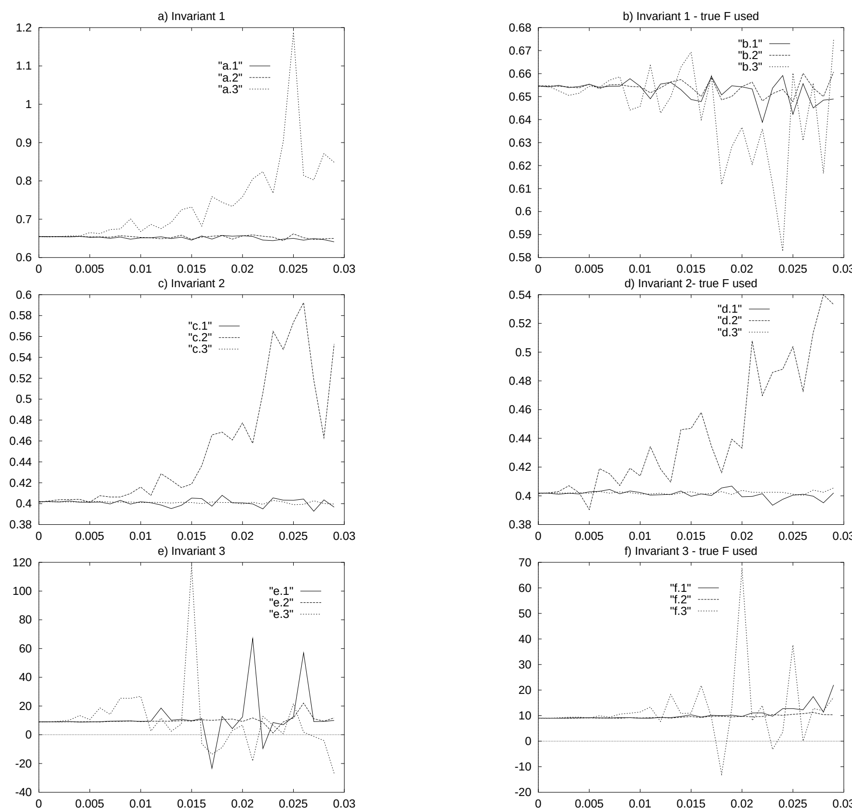

Fig. 1. Plots showing the behaviour of the 3D invariant between three different pairs of views with increasing noise. The solid, dashed and dotted lines show the invariant formed between views 1 & 2, 1 & 3 and 2 & 3 respectively (denoted by $a.1$, $a.2$, $a.3$ etc. in the key). The x-axis shows the standard deviation of the gaussian noise used.

7 Conclusions

This paper outlines a framework for projective geometry and the algebra of incidence which is then used to discuss the formation of projective invariants. Explicit expressions are given for one sort of 3D invariant using image points and the behaviour of this invariant is investigated for a variety of simulated scenarios. Such invariants may be useful in low noise, but in cases of greater uncertainty there may be too many problems for the invariants to be useful over a wide range of possible circumstances.

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