Synthesis of Fast Algorithms for Discrete Fourier–Clifford Transform

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Conventional methods of discrete spectral analysis, widely applied in digital signal processing, mainly allow one to analyze "averaged" energy properties of these signals but possess a low sensitivity with respect to local geometric properties.

This disadvantage can be clearly seen in the case when multidimensional signals should be analyzed. Specifically, a d-dimensional discrete Fourier transform (DFT-d)

$$\hat{x}(m_1, \ldots, m_d) = \sum_{n_1=0}^{N-1} \cdots \sum_{n_d=0}^{N-1} x(n_1, \ldots, n_d) \prod_{k=1}^{d} \exp\left\{2\pi i \frac{m_k n_k}{N}\right\} \qquad (1)$$

of an input array of the size N^d has the following basis functions:

$$h_{m_1, \ldots, m_d}(n_1, \ldots, n_d) = \prod_{k=1}^{d} \exp\left\{2\pi i \frac{m_k n_k}{N}\right\}. \qquad (2)$$

The values of these functions belong to a finite set of Nth roots of unity. Such a disproportion between the numbers of freedom degrees of the input signal and the values of basis functions practically excludes the possibility to analyze local geometric properties of a signal (excluding elementary geometric properties) by means of spectral techniques.

This paper considers one of possible approaches that permit one to overcome the difficulties described above. We will consider analogues of discrete Fourier transform where the values of basis functions belong to finite-dimensional algebras of a more general form than the two-dimensional algebra of complex numbers.

In our previous studies, we considered different particular cases of such transforms. In [5, 6], a discrete transform with values belonging to a noncommutative algebra of quaternions $\mathbb{H}$,

$$Q(m_1, m_2) = \sum_{n_1, n_2 = 0}^{N-1} \omega_1^{m_1 n_1} x(n_1, n_2) \omega_2^{m_2 n_2} \qquad (3)$$

(here, $\omega_1 = \exp\{2\pi i/N\}$, $\omega_2 = \exp\{2\pi j/N\} \in \mathbb{H}$), was employed as an auxiliary transform for the construction of efficient algorithms of complex DFT-2. In [3, 4], a (continuous) quaternion analogue of the Fourier transform,

$$F(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2) e^{2\pi i x_1 y_1} e^{2\pi j x_2 y_2} dx_1 dx_2 \qquad (4)$$

was applied to solve some practical problems with substantial anisotropy with respect to coordinates.

The purpose of this study is to consider a discrete transform of a d-dimensional signal with values from a 2^d-dimensional Clifford algebra, to synthesize efficient (fast) algorithms for the computation of the Fourier–Clifford spectrum, and to investigate the actual speed of these algorithms.

Let V be a d-dimensional vector space with a basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$. Consider an associative 2^d-dimensional $\mathbb{R}$-algebra with a basis

$$E_{\alpha_1 \ldots \alpha_d} = \mathbf{e}_1^{\alpha_1} \ldots \mathbf{e}_2^{\alpha_d}, \quad \alpha_k = 0, 1,$$
$$\mathbf{e}_k^0 = 1, \quad \mathbf{e}_k^1 = \mathbf{e}_k,$$

where multiplication rules for arbitrary elements are induced by multiplication rules for the elements $\mathbf{e}_k$:

$$\mathbf{e}_k^2 = -1, \quad \mathbf{e}_k \mathbf{e}_s = -\mathbf{e}_s \mathbf{e}_k \qquad (5)$$

(a Clifford algebra of quadratic form is

$$B(t_1, \ldots, t_d) = -t_1^2 - \ldots - t_d^2;$$

e.g., see [1]).

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Definition. The transform of a real d-dimensional array $\{x(n_1, \ldots, n_d)\}$ defined by the equality

$$\hat{x}(m_1, \ldots, m_d) = \sum_{(n_1, \ldots, n_d) = 0}^{N-1} x(n_1, \ldots, n_d) \prod_{k=1}^{d} \exp\left\{2\pi \frac{\mathbf{e}_k m_k n_k}{N}\right\} \quad (6)$$

will be referred to as discrete Fourier–Clifford transform (DFCT).

Representing the basis functions of DFCT as products of Nth roots of unity lying in *different* realizations of the field of complex numbers $\mathbb{C}_k = \mathbb{R}(\mathbf{e}_k)$, we can adequately analyze input signals anisotropic along different coordinate axes.

The fact that the multiplication of the elements of a Clifford algebra is noncommutative somewhat complicates the synthesis of fast DFCT algorithms. To be more specific, construction of an analogue of a row–column DFT-2 algorithm presents no difficulties. Synthesis of substantially multidimensional analogues of DFT-d (such as Rayword algorithms, vector-radix algorithms, etc.) requires the use of explicit formulas for a distributive law in its noncommutative modification. With $d = 2$ (i.e., in the case of a quaternion DFCT), quaternions lying in $\mathbb{C}_1 = \mathbb{R}(i)$ and $\mathbb{C}_2 = \mathbb{R}(j)$ can be placed outside the brackets as pre- and postmultipliers, respectively, as it was shown in [2, 8].

In this paper, we present explicit formulas for noncommutative distributive laws (taking outside the brackets up and down and so on), which makes it possible to synthesize direct analogues of multidimensional DFTs.

Specifically, we consider the following fast DFCT algorithms:

—row–column algorithms;

—radix-two algorithms;

—radix-four algorithms;

—vector-radix algorithms (e.g., split-radix algorithms);

—split-radix algorithms with a radix of fractional order [7].

We also provide a comparative analysis of the computational complexity of these algorithms.

A special attention is focused on algorithms that take into account the reality of input (for forward DFCT) and output (for backward DFCT) signals (DFCT combination, algorithms similar to those considered in [8]).

We also discuss some problems of practical importance that can be efficiently solved with the use of DFCT.

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