Clifford Models of Image Transforms

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It is well known that a combined rotation of a solid can be conveniently described in terms of quaternion algebra operations [6] (screw transform).

Clifford algebras (CAs) (quaternion algebra is a particular case of these algebras) arose as a result of a natural wish of mathematicians to embed a vector space into such an algebraic structure where different classes of linear operators would be described in terms of operations of this structure in a possibly simple form.

Currently, efficient application of CAs in computer science is mainly based on two factors.

First, the substantive formulation of many problems in robotics, computer vision, etc. has either direct physical (mechanical) origin or clear physical analogies [1, 2, 7].

Second, specific Clifford algebras are formally employed as convenient computational models to represent the data and to improve the efficiency of some algorithms for digital signal processing [3, 4].

In the former case, CA properties related to geometric clearness dominate. In the latter case, structure properties associated with the existence of a sufficiently broad group of CA automorphisms are dominant.

The problems of analysis and digital processing of multidimensional signals (images) have a clear physical origin. Solution of these problems requires a considerable body of computations and, therefore, needs an efficient algorithmic support.

Let us emphasize the most significant specific feature of image transforms. If an image is interpreted as a subset of a three-dimensional space (a pair of coordinates and a function of brightness), then image transforms are transforms with an invariant plane (a plane of arguments). In other words, the coordinates of a vector from a space associated with an image are not equivalent. Due to this nonequivalence, various types of Clifford algebras employed in mechanical problems substantially differ from CAs that adequately describe image transforms.

As demonstrated in [5], a rather broad class of image transforms is described by a generalized screw transform of an eight-dimensional CA—an algebra of second-order dualized matrices.

In this paper, we consider relations between different groups of image transforms and representation of these image transforms in terms of screw transforms in the relevant CAs (in Clifford image models).

Some definitions. Consider a d-dimensional vector space V with a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$. Let $\mathbb{G}_{p,q,r}$ be an associative $\mathbb{R}$-algebra with a dimensionality equal to 2^d and a basis

$$\{E_{\alpha_1\ldots\alpha_d} = \mathbf{e}_1^{\alpha_1}\ldots\mathbf{e}_d^{\alpha_d}, \alpha_k = 0, 1; \mathbf{e}_k^0 = 1, \mathbf{e}_k^1 = \mathbf{e}_k\}.$$

Multiplication of elements in this algebra is assumed to be induced by the rules of multiplication of the elements $\mathbf{e}_k$:

$$\mathbf{e}_k^2 = \beta_k, \quad \mathbf{e}_k\mathbf{e}_s = -\mathbf{e}_s\mathbf{e}_k.$$

The Clifford algebra of quadratic form is

$$B(t_1, \ldots, t_d) = \beta_1 t_1^2 + \ldots + \beta_d t_d^2. \qquad (1)$$

Here, $p + q + r = d$, where p and q are the positive and negative inertia indices of the form (1) and r is the number of zero β_k.

Let us define the reverse $\tilde{S}$ of an element

$$S = \sum_{\alpha_1, \ldots, \alpha_d = 0}^{1} x_{\alpha_1\ldots\alpha_d} E_{\alpha_1\ldots\alpha_d}$$

as an $\mathbb{R}$-linear continuation of the mapping

$$\tilde{E}_{\alpha_1\ldots\alpha_d} = \mathbf{e}_d^{\alpha_d}\ldots\mathbf{e}_1^{\alpha_1}.$$

Screw transform of an element $\mathbf{J} \in \mathbb{G}_{p,q,r}$ is defined as the transform

$$\mathbf{J}' = \tilde{S}\mathbf{J}S + \mathbf{z}_0, \quad \mathbf{z}_0 \in \mathbb{G}_{p,q,r}. \qquad (2)$$

An image with t attributes is defined as a subset of a $(t+2)$-dimensional affine space

$$I = \{(X, Y; A_1, \ldots, A_t): (X, Y) \in \Omega \subset \mathbb{R}^2\},$$

where $A_1 = A_1(X, Y), \ldots, A_t = A_t(X, Y)$ are some real functions (image attributes, e.g., brightness, local time, intensity of color components, etc.), and variables

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(X, Y) are interpreted as the coordinates of a point in an affine plane.

Let $\mathbb{G} = \mathbb{G}_{p,q,r}$ be a Clifford algebra, $\mathbb{A}^{t+2}$ is a $(t+2)$-dimensional affine space, and mapping Ψ: $\mathbb{G} \longrightarrow \mathbb{A}$ is defined.

The complete original $\{\mathbf{J}\}$ of an image I with respect to Ψ will be referred to as the Clifford model of the image I, and mapping Ψ will be called the interpretation of the model.

The basic problem. Let $\mathbb{T}$ be a certain group of transforms of an affine (X, Y)-plane:

$$\tau(X, Y) = (X', Y'), \quad \tau \in \mathbb{T}.$$

It is required to describe a possibly broader class of image transforms

$$I = (X, Y; A_1, \ldots, A_t) \mapsto I' = (X', Y'; A_1', \ldots, A_t'),$$

such a model, and such an interpretation that screw transform (2) yields

$$\Psi(\mathbf{J}) = I, \quad \Psi(\mathbf{J}') = I', \quad (X', Y') = \tau(X, Y).$$

In this paper, we analyze some examples of the description of image transforms in the form (2).

Example 1. Study [5] considers a group of affine transforms of a plane, an eight-dimensional algebra $\mathbb{G}_{2,0,1}$,

$$\mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = 1, \quad \mathbf{e}_3^2 = 0$$

and a 1-attribute image $I = \{X, Y; F(X, Y)\}$.

Let $\mathbb{T}$ be a group of affine transforms of a plane,

$$\mathbf{J} = X\mathbf{e}_1 + Y\mathbf{e}_2 + F\mathbf{e}_3.$$

Then, for any $\tau \in \mathbb{T}$, there exists such $S \in \mathbb{G}$ that

(X', Y') linearly depend on (X, Y)

and

F' linearly depends on $(X, Y; F)$.

In particular, explicit formulas for orthogonal transforms of an (X, Y)-plane, scaling, and Frechét derivative were obtained in [5].

Example 2. Let I be a 2-attribute image,

$$I = \{(X, Y; A_1, A_2)\}, \quad \mathbf{J} = A_1 + X\mathbf{e}_1 + Y\mathbf{e}_2 + A_2\mathbf{e}_3.$$

Then, upon the application of a screw transform (2),

(X', Y') linearly depend on (X, Y),

A_1' linearly depends on $(X, Y; A_1)$,

and

A_2' linearly depends on $(X, Y; A_1, A_2)$.

Here, A_1 can be interpreted as a local time in the reading (transmission) of an image, and A_2 is a function of brightness.

Example 3. Suppose that $\mathbb{G} = \mathbb{G}_{0,2,1}$ (dualized quaternions), $\mathbb{T}$ is a group of projective transforms of a plane, and

$$I = \{(X, Y; F(X, Y))\}, \quad \mathbf{J} = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + F\mathbf{e}_3,$$

where $(x_0 : x_1 : x_2)$ are the homogeneous coordinates of the points in the projective plane and (X, Y) are the corresponding inhomogeneous coordinates.

Example 4. Suppose that $\mathbb{G} = \mathbb{G}_{0,1,3}$, $\mathbf{e}_1^2 = -1$, $\mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_4^2 = 0$; R, G, and B are the intensities of the main color components of an image

$$I = \{(X, Y; R, G, B)\},$$

$$\mathbf{J} = X + Y\mathbf{e}_1 + R\mathbf{e}_2 + G\mathbf{e}_3 + B\mathbf{e}_4,$$

etc.

We will thoroughly analyze the relation between the dimensionality of invariant spaces subject to image transformation and the signature (p, q, r) of the Clifford algebra.

We will also consider a recursive process for constructing Clifford algebras (the generalized Cayley–Dickson process) and examine the possibility of describing local properties of an image in terms of a (local) screw transform.

REFERENCES

1. Bayro-Corrochano, E. and Sommer, G., Object Modelling and Collision Avoidance Using Clifford Algebra, *Computer Analysis of Image and Pattern, Proc. CAIP'95, LNCS*, Hlavac, V. and Sara, R., Eds, 1995, vol. 970, pp. 699–704.
2. Bayro-Corrochano, Ed. and Lasenby, J., A Unified Language for Computer Vision and Robotics, *Algebraic Frames for the Perception–Action Cycle, LNCS*, Springer, vol. 1315, pp. 219–233.
3. Chernov, V.M., Arithmetic Methods in the Theory of Discrete Orthogonal Transforms, *Proc. SPIE*, 1995, vol. 2363, pp. 134–141.
4. Chernov, V.M., Discrete Orthogonal Transforms with Data Representation in Composition Algebras, *Proc. 9th Scandinavian Conf. on Image Analysis*, 1995, pp. 357–364.
5. Chernov, A.V. and Chernov, V.M., On Hestenes' Formalization for Describing Linear Image Transforms, *Algebraic Frames for the Perception–Action Cycle, LNCS*, Springer, vol. 1315, pp. 164–175.
6. Clifford, W.K., Applications of Grassmann's Extensive Algebra, *Am. J. Math.*, 1878, vol. 1, pp. 350–358.
7. Hestenes, D., *New Foundation for Classical Mechanics*, Dordrecht: Redel, 1986.