Computing Invariants in Computer Vision using Geometric Algebra

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Abstract

A central task of computer vision is to automatically recognize objects in real-world scenes. The parameters defining image and object spaces can vary due to lighting conditions, camera calibration and viewing position. It is therefore desirable to look for geometric properties of the object which remain invariant under such changes in the observation parameters. The study of such geometric invariance is a field of active research. In this paper we present *geometric algebra* as a complete framework for the theory and computation of invariants in computer vision and compare it with the currently popular Grassmann-Cayley (or Double) algebra. While this paper will only deal with the algebraic invariants formed from points and lines, other types of invariants, such as differential and moment invariants, can also be treated in the geometric algebra. In particular, Lie groups and Lie algebras are natural parts of the framework taking the forms of spin groups and bivector algebras. We hope to show that geometric algebra is a very elegant language for expressing all the ideas of projective geometry and provides us with a system in which real computer implementations are straightforward. Using these techniques we will look at the formation of 3D projective invariants in terms of both points and lines in multiple images and their implementation on simulated and real data. We will also give a new form of the 3D projective invariant in terms of the image points from 3 views and the trilinear tensor.

Categories: Computer vision; invariants; Clifford algebra; Grassmann-Cayley algebra; projective geometry; 3D projective invariants.

1 Introduction

Geometric algebra is a coordinate-free approach to geometry based on the algebras of Grassmann [?] and Clifford [?]. The algebra is defined on a space whose elements are called *multivectors*; a multivector is a linear combination of objects of different type, e.g. scalars and vectors. It has an associative and fully invertible product called the **geometric** or **Clifford** product. The existence of such a product and the calculus associated with the geometric algebra give the system tremendous power. Some preliminary applications of geometric algebra in the field of computer vision have already been given [?, ?, ?], and here we would like to extend the discussion of geometric invariance given in [?, ?]. Geometric algebra provides a very natural language for projective geometry and has all the necessary equipment for the tasks which the Grassmann-Cayley algebra is currently used for. The Grassmann-Cayley or double algebra [?, ?] is a system for computations with subspaces of finite-dimensional vector spaces. While it expresses the ideas of projective geometry, such as the meet and join, very elegantly, it lacks an inner (regressive) product (although an inner product *can* be defined, it is not a natural part of its structure) and some other key concepts which we will discuss later.

The next section will give a brief introduction to geometric algebra and to some of the associated linear algebra framework. For a more complete introduction see [?] and for other brief summaries see [?, ?, ?]. Given this background we can look at the familiar concepts of projective space and homogeneous coordinates, outline the formulation of projective geometry in the geometric algebra and introduce the concept of the *projective split*. We then deal with projective transformations and illustrate the formation of the 1D, 2D and 3D cross-ratios which are algebraic projective invariants in this framework. We will illustrate the comparisons between our methods and those of the Grassmann-Cayley algebra by considering 3D projective invariants and discussing the implementation of such invariants using only image points and lines.

2 Geometric Algebra: an outline

The algebras of Clifford and Grassmann are well known to pure mathematicians, but were long ago abandoned by physicists in favour of the vector algebra of Gibbs, which is indeed what is most commonly used today in most areas of physics. The approach to Clifford algebra we adopt here was pioneered in the 1960's by David Hestenes [?] who has, since then, worked on developing his version of Clifford algebra – which will be referred to as *geometric algebra* – into a unifying language for mathematics and physics.

2.1 Basic Definitions

Let $\mathcal{G}_n$ denote the geometric algebra of n-dimensions – this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition – this is the **geometric** or **Clifford**

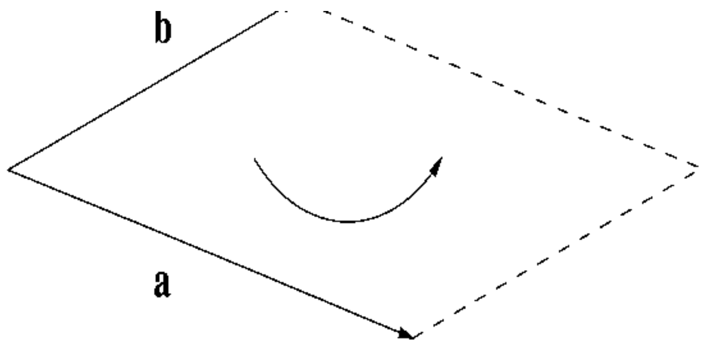

Figure 1: The directed area, or bivector, $\boldsymbol{a}\wedge\boldsymbol{b}$.

product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written $\boldsymbol{ab}$ and can be expressed as a sum of its symmetric and antisymmetric parts

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}, \tag{1}$$

where the inner product $\boldsymbol{a}\cdot\boldsymbol{b}$ and the outer product $\boldsymbol{a}\wedge\boldsymbol{b}$ are defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \tag{2}$$

$$\boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}). \tag{3}$$

The inner product of two vectors is the standard *scalar* or *dot* product and produces a scalar. The outer or wedge product of two vectors is a new quantity we call a **bivector**. We think of a bivector as a directed area in the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, formed by sweeping $\boldsymbol{a}$ along $\boldsymbol{b}$ – see Figure 1. Thus, $\boldsymbol{b} \wedge \boldsymbol{a}$ will have the opposite orientation making the wedge product anticommutative as given in equation (3). The outer product is immediately generalizable to higher dimensions – for example, $(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c}$, a **trivector**, is interpreted as the oriented volume formed by sweeping the area $\boldsymbol{a}\wedge\boldsymbol{b}$ along vector $\boldsymbol{c}$ – see Figure 2. The outer product of k vectors is a k-vector or k-blade, and such a quantity is said to have *grade k*. A **multivector** is *homogeneous* if it contains terms of only a single grade. The notation $\langle M\rangle_k$ is used to denote the k-grade part of the multivector M and $\langle M\rangle$ denotes the scalar part of M. For a product of multivectors, the operation of taking the scalar part satisfies the cyclic reordering property

$$\langle A....BCD\rangle = \langle DA....BC\rangle = \langle CDA.....B\rangle. \tag{4}$$

The operation of *reversion* reverses the order of vectors in any multivector. The reverse of A is written as $\tilde{A}$ so that

$$(\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3.....\boldsymbol{a}_k)^{\tilde{}} = \boldsymbol{a}_k......\boldsymbol{a}_3\boldsymbol{a}_2\boldsymbol{a}_1. \tag{5}$$

The geometric algebra provides a means of manipulating multivectors which allows us to keep track of different grade objects simultaneously – much as one does with complex number operations. In a space of 3 dimensions we can construct a trivector $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$, but no 4-vectors exist since there is no possibility of sweeping the volume element $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ over a 4th dimension. The highest grade element in a space is called the **pseudoscalar**. The unit pseudoscalar is denoted by I and will be seen to be crucial when discussing duality.

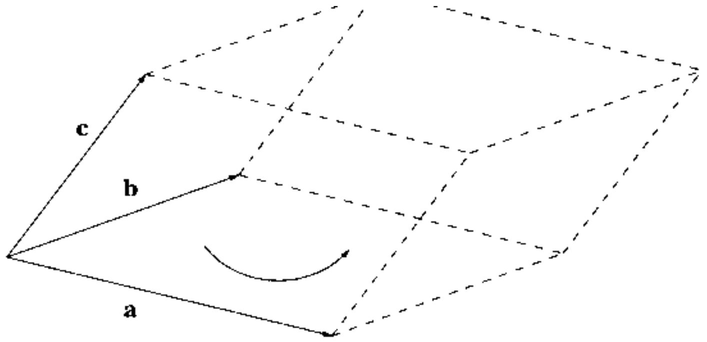

Figure 2: The oriented volume, or trivector, $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$.

In a space of dimension n we can have homogeneous multivectors of grade 0 (scalars), grade 1 (vectors), grade 2 (bivectors), etc... up to grade n. Multivectors containing only even-grade elements are termed *even* and those containing only odd-grade elements are termed *odd*. The geometric product can, of course, be defined for any two multivectors. Considering two homogeneous multivectors A_r and B_s of grades r and s respectively, it is clear that the geometric product of such multivectors will contain parts of various grades. It can be shown [?] that the geometric product of A_r and B_s can be written as

$$A_r B_s = \langle AB\rangle_{r+s} + \langle AB\rangle_{r+s-2} + \ldots + \langle AB\rangle_{|r-s|}. \tag{6}$$

For non-homogeneous multivectors we again simply apply the distributive rule over their homogeneous parts. Using the above we can now generalize the definitions of inner and outer products given in equations (2) and (3). For two homogeneous multivectors A_r and B_s, we define the inner and outer products as

$$A_r \cdot B_s = \langle A_r B_s\rangle_{|r-s|} \tag{7}$$

$$A_r \wedge B_s = \langle A_r B_s\rangle_{r+s}. \tag{8}$$

Thus, the inner product produces an $|r-s|$-vector – which means it effectively reduces the grade of B_s by r; and the outer product gives an $r+s$-vector, therefore increasing the grade of B_s by r. This is an extension of the general principle that dotting with a vector lowers the grade of a multivector by 1 and wedging with a vector raises the grade of a multivector by 1.

2.2 The Geometric Algebra of 3-D Space

In an n-dimensional space we can introduce an orthonormal basis of vectors $\{\sigma_i\}$ $i = 1, ..., n$, such that $\sigma_i \cdot \sigma_j = \delta_{ij}$. This leads to a basis for the entire algebra:

$$1, \quad \{\sigma_i\}, \quad \{\sigma_i\wedge\sigma_j\}, \quad \{\sigma_i\wedge\sigma_j\wedge\sigma_k\}, \quad \ldots, \quad \sigma_1\wedge\sigma_2\wedge\ldots\wedge\sigma_n. \tag{9}$$

Note that we shall not use bold symbols for these basis vectors. Any multivector can be expressed in terms of this basis, although much of the power of geometric algebra results from being able to carry out procedures in a basis-free manner. The basis for the 3-D space has $2^3 = 8$ elements given by:

$$\underbrace{1}_{scalar}, \underbrace{\{\sigma_1, \sigma_2, \sigma_3\}}_{vectors}, \underbrace{\{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\}}_{bivectors}, \underbrace{\{\sigma_1\sigma_2\sigma_3\} \equiv i}_{trivector}. \tag{10}$$

It can easily be verified that the trivector or pseudoscalar $\sigma_1\sigma_2\sigma_3$ squares to -1 and commutes with all multivectors in the 3-D space. We therefore give it the symbol i; noting that this is not the uninterpreted commutative scalar imaginary j used in quantum mechanics and engineering.

Multiplication of the three basis vectors $\{\sigma_i\}$ by i results in the three basis bivectors;

$$\sigma_1\sigma_2 = i\sigma_3 \quad \sigma_2\sigma_3 = i\sigma_1 \quad \sigma_3\sigma_1 = i\sigma_2. \tag{11}$$

These simple bivectors rotate vectors in their own plane by 90°, e.g. $(\sigma_1\sigma_2)\sigma_2 = \sigma_1$, $(\sigma_2\sigma_3)\sigma_2 = -\sigma_3$ etc. Identifying the $\boldsymbol{i}$,$\boldsymbol{j}$,$\boldsymbol{k}$ of the quaternion algebra with $i\sigma_1, -i\sigma_2, i\sigma_3$, we see that the famous Hamilton relations are recovered

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{12}$$

Since the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are really bivectors it comes as no surprise that they represent 90° rotations in orthogonal directions and provide a system well-suited for the representation of 3-D rotations. In geometric algebra a **rotor**, R, is an even-grade element of the algebra which satisfies $R\tilde{R} = 1$. If $\mathcal{A} = \{a_0, a_1, a_2, a_3\}$ represents a quaternion, then the rotor which performs the same rotation is simply given by

$$R = a_0 + a_1(i\sigma_1) - a_2(i\sigma_2) + a_3(i\sigma_3). \tag{13}$$

The quaternion algebra is therefore seen to be a subset of the geometric algebra of 3-space.

2.3 Reflections and Rotations

Any rotation can be formed by a pair of reflections. It can be shown straightforwardly that the result of reflecting a vector $\boldsymbol{a}$ in the plane perpendicular to a unit vector $\boldsymbol{n}$ is

$$\boldsymbol{a}_\perp - \boldsymbol{a}_\parallel = -\boldsymbol{nan} \tag{14}$$

where $\boldsymbol{a}_\perp$ and $\boldsymbol{a}_\parallel$ respectively denote parts of $\boldsymbol{a}$ perpendicular and parallel to $\boldsymbol{n}$. Thus, a reflection of $\boldsymbol{a}$ in the plane perpendicular to $\boldsymbol{n}$, followed by a reflection in the plane perpendicular to a unit vector $\boldsymbol{m}$ results in a new vector

$$-\boldsymbol{m}(-\boldsymbol{nan})\boldsymbol{m} = (\boldsymbol{mn})\boldsymbol{a}(\boldsymbol{nm}) = R\boldsymbol{a}\tilde{R}. \tag{15}$$

The multivector $R = \boldsymbol{mn}$ is a rotor. Rotors combine in a straightforward manner, i.e. a rotation R_1 followed by a rotation R_2 is equivalent to an overall rotation R where $R = R_2R_1$. The transformation $\boldsymbol{a} \mapsto R\boldsymbol{a}\tilde{R}$ is a very general way of handling rotations; it works for multivectors of any grade and in spaces of any dimension.

2.4 Formulation of Projective Geometry

Here we will outline the approach pioneered by Hestenes for using geometric algebra to discuss the algebra of incidence. The basic projective geometry operations of meet and join will be shown to be easily expressible in terms of standard operations within the geometric algebra. For a more extended discussion we refer the reader to [?].

A geometric algebra $\mathcal{G}_n$ can be written as $\mathcal{G}(p,q)$ where p and q are the dimensions of the maximal subspaces with positive and negative signatures respectively (the signature of a vector $\boldsymbol{a}$ is positive, negative or zero according as $\boldsymbol{a}^2 > 0, < 0, = 0$). While the results of a projective geometry theory should be independent of signature, for real applications we find it useful to specify the signature to facilitate actual computations. We will see later that we adopt the standard Euclidean signature $\mathcal{G}(3,0)$ for ordinary space, $\mathcal{E}^3$, but that we are forced to adopt a signature of $\mathcal{G}(1,3)$ for the 4-dimensional space we associate with the projective space.

We have seen that in Euclidean spaces of 2 and 3 dimensions the unit pseudoscalar, I, squares to -1. In $\mathcal{G}(1,3)$ it is easy to see that this is also the case. If γ_i, $i = 1,2,3,4$ are our basis vectors in the 4D space, and ${\gamma_j}^2 = -1$ for j=1,2,3 and $\gamma_4^2 = +1$, then I^2 is given by

$$(\gamma_1\gamma_2\gamma_3\gamma_4)(\gamma_1\gamma_2\gamma_3\gamma_4) = (\gamma_2\gamma_3\gamma_4)(\gamma_2\gamma_3\gamma_4) = -(\gamma_3\gamma_4)(\gamma_3\gamma_4) = -1. \tag{16}$$

The sign of I^2 depends on the signature of the space. In a given space any pseudoscalar P can be written as $P = \alpha I$ where α is a scalar. If I^{-1} is the inverse of I, so that $II^{-1} = 1$, then,

$$PI^{-1} = \alpha II^{-1} = \alpha \equiv [P] \tag{17}$$

where we have defined the **bracket** of the pseudoscalar P, $[P]$, as its magnitude, arrived at by multiplication on the right by I^{-1}. We will see later that this bracket is precisely the bracket of the Grassmann-Cayley algebra. The sign of the bracket does not depend on the signature of the space and as such it has been a useful quantity for the non-metrical applications of projective geometry.

To introduce the concepts of duality which are so important in projective geometry, we define the dual A^* of an r-vector A as

$$A^* = AI^{-1}. \tag{18}$$

We use the notation A^* to relate these ideas of duality to the notion of a *Hodge dual* in differential geometry. Note that in general I^{-1} may not commute with A. From the definition of the unit pseudoscalar we see that the dual of an r-vector is an $(n-r)$-vector (e.g. duality of lines ($r = 1$) and planes ($n - r = 3 - 1$ in 3-space). In an n-dimensional space, if A is an r-vector and B is an s-vector then using the fact that $BI^{-1} = B\cdot I^{-1}$ (using equation (6) since BI^{-1} must be of grade $(n-s)$) and the identity

$$A_r\cdot(B_s\cdot C_t) = (A_r\wedge B_s)\cdot C_t \quad \text{for} \quad r+s \leq t, \tag{19}$$

we can write

$$A\cdot(BI^{-1}) = A\cdot(B\cdot I^{-1}) = (A\wedge B)\cdot I^{-1} = (A\wedge B)I^{-1}. \tag{20}$$

Using the definition of the dual we therefore have

$$A\cdot B^* = (A\wedge B)^*. \tag{21}$$

Equation (??) illustrates the duality of the inner and outer products. If $r + s = n$, then $A\wedge B$ is the highest grade part of AB, i.e. the pseudoscalar part, and it then follows that

$$[A\wedge B] = (A\wedge B)I^{-1} = A\cdot B^*. \tag{22}$$

In this case we can express the bracket in terms of duals and as such, relate the inner and outer products to non-metrical quantities. It is via this route that the inner product, which is normally associated with a metric, is used in a non-metrical theory such as projective geometry. We note at this point that since we have reduced duality to a simple multiplication by an element of the algebra, there is no need to introduce a special operator or any concept of a different space.

In an n-dimensional geometric algebra one can define the **join** $J = A \bigwedge B$ of an r-vector, A, and an s-vector, B, by

$$J = A\wedge B \qquad \text{if } A \text{ and } B \text{ are linearly independent.} \tag{23}$$

If A and B are not linearly independent the join is not given simply by the wedge but by the subspace that they span. J can be interpreted as a *common dividend of lowest grade* and is defined up to a scale factor. It is easy to see that if $(r + s) \geq n$ then J will be the pseudoscalar for the space. In what follows we will use $\bigwedge$ for the join only when the blades A and B are not linearly independent, otherwise we will use the ordinary exterior product, $\wedge$.

If A and B have a common factor (i.e. there exists a k-vector C such that $A = A'C$ and $B = B'C$ for some A', B') then we can define the 'intersection' or **meet** of A and B as $A \vee B$ where [?]

$$(A \vee B)^* = A^*\wedge B^*. \tag{24}$$

That is, the dual of the meet is given by the join of the duals. In equation (??) we must be slightly careful to specify what space we take the dual of $(A \vee B)$ with respect to. The dual of $(A \vee B)$ is understood to be taken with respect to the *join* of A and B. In most cases of practical interest this join will be the whole space and the meet is therefore easily computed so that we can use equation (??) to obtain a more useful expression for the meet as follows

$$A \vee B = (A^*\wedge B^*)I = (A^*\wedge B^*)(I^{-1}I)I = (A^*\cdot B) \tag{25}$$

We therefore have the very simple and readily computed relation of $A \vee B = (A^*\cdot B)$. The above concepts are discussed further in [?].

2.5 Linear Algebra

In [?] Hestenes has attempted to make obvious the intimate relationship between linear algebra and projective geometry – a relationship which, he claims, has been obscured by the pursuit of the two areas as separate disciplines. In this section we will give a brief review of the geometric algebra approach to linear algebra with the aim of covering the material needed in later sections. More detailed reviews can be found in [?, ?].

Consider a linear function f which maps vectors to vectors in the same space. We can extend f to act linearly on multivectors via the **outermorphism**, $\underline{f}$, defining the action of $\underline{f}$ on blades by

$$\underline{f}(a_1 \wedge a_2 \wedge \ldots \wedge a_r) = \underline{f}(a_1) \wedge \underline{f}(a_2) \wedge \ldots \wedge \underline{f}(a_r). \tag{26}$$

We use the term outermorphism because $\underline{f}$ preserves the grade of any r-vector it acts on. The action of $\underline{f}$ on general multivectors is then defined through linearity. $\underline{f}$ must therefore satisfy:

$$\begin{aligned} \underline{f}(a_1 \wedge a_2) &= \underline{f}(a_1) \wedge \underline{f}(a_2) \\ \underline{f}(A_r) &= \langle \underline{f}(A_r) \rangle_r \\ \underline{f}(\alpha_1 a_1 + \alpha_2 a_2) &= \alpha_1 \underline{f}(a_1) + \alpha_2 \underline{f}(a_2). \end{aligned} \tag{27}$$

Given this definition it is easy to see that the outermorphism of a product of two linear functions is the product of the outermorphisms, i.e. if $f(a) = f_2(f_1(a))$, then we can write $\underline{f} = \underline{f}_2 \underline{f}_1$.

The adjoint $\overline{f}$ of a linear function $\underline{f}$ can be defined by the property

$$\underline{f}(a) \cdot b = a \cdot \overline{f}(b) \tag{28}$$

for vectors a and b. If $\underline{f} = \overline{f}$ the function is *self-adjoint* and its matrix representation will be symmetric.

Since the outermorphism preserves grade, we know that the pseudoscalar of the space must be mapped onto some multiple of itself. The scale factor in this mapping is the **determinant** of $\underline{f}$;

$$\underline{f}(I) = \det(\underline{f}) I. \tag{29}$$

This is much simpler than many definitions of the determinant. Using this definition, most properties of determinants can be established with little effort.

3 Projective Space and Projective Transformations

Since about the mid 1980's most of the computer vision literature discussing geometry and invariants has used the language of projective geometry – indeed in the appendix of [?] the authors argue very eloquently that many of the problems in computer vision (particularly in invariant theory) would be difficult to solve without analytic projective geometry. As any point on a ray from the optical centre of a camera will map to the same point in the camera image plane it is easy to see why a 2D view of a 3D world might well be best expressed in projective space. In classical projective geometry one defines a 3D space, $\mathcal{P}^3$, whose points are in 1 − 1 correspondence with lines through the origin in a 4D space, R^4. Similarly, k-dimensional subspaces of $\mathcal{P}^3$ are identified with $(k+1)$-dimensional subspaces of R^4. Such projective views can provide very elegant descriptions of the geometry of incidence (intersections, unions etc.), but in order to carry out any real computations one is forced to introduce some sort of basis and associated metric. From a mathematical viewpoint the projective space, $\mathcal{P}^3$, would have no metric, the basis and metric are introduced in the associated 4D space. In this 4D space a coordinate description of a projective point is conventionally brought about by using *homogeneous coordinates*. The usefulness of the projective description of space is often only realised with the introduction of such homogeneous coordinates. What we are aiming to do in this paper is to provide a new way of looking at the problems of computer vision, using a system in which the algebra is clear and computations are completely straightforward and well-defined. Much of the conventional mathematical apparatus of projective geometry will not be needed.

3.1 The Projective Split

Points in real 3D space will be represented by vectors in $\mathcal{E}^3$, a 3D space with a Euclidean metric. Since any point on a line through some origin O will be mapped to a single point in the image plane, we will find it useful to associate a point in $\mathcal{E}^3$ with a line in a 4D space, R^4. In these two distinct but related spaces we define basis vectors: $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in R^4 and $(\sigma_1, \sigma_2, \sigma_3)$ in $\mathcal{E}^3$. We identify R^4 and $\mathcal{E}^3$ with the geometric algebras of 4 and 3 dimensions, $\mathcal{G}_4$ and $\mathcal{G}_3$. We require that vectors, bivectors and trivectors in R^4 will represent points, lines and planes in $\mathcal{E}^3$. Suppose we choose γ_4 as a selected direction in R^4, we can then define a mapping which associates the bivectors $\gamma_i\gamma_4$, $i = 1, 2, 3$, in R^4 with the vectors σ_i, $i = 1, 2, 3$, in $\mathcal{E}^3$;

$$\sigma_1 \equiv \gamma_1\gamma_4, \quad \sigma_2 \equiv \gamma_2\gamma_4, \quad \sigma_3 \equiv \gamma_3\gamma_4. \tag{30}$$

To preserve the Euclidean structure of the spatial vectors, $\{\sigma_i\}$, (i.e. $\sigma_i^2 = +1$) it is easy to see that we are forced to assume a non-Euclidean metric for the basis vectors in R^4. We choose to use $\gamma_4^2 = +1$, $\gamma_i = -1$, $i = 1, 2, 3$. It is interesting to note here that this is precisely the metric structure of Minkowski spacetime used in studies of relativistic physics. This process of associating the quantities in the higher dimensional space with quantities in the lower dimensional space is an application of what Hestenes calls the **projective split**. In the version of geometric algebra used for physics (spacetime algebra) we have the same 4D space with a Minkowski metric ($\mathcal{G}(1,3)$) and basis vectors $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$. The projective split is there called a **spacetime split** where γ_0, the time axis, is chosen as the preferred direction and performs the same function as γ_4 in R^4.

For a vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ in R^4 the projective split is obtained by taking the geometric product of $\mathbf{X}$ and γ_4;

$$\mathbf{X}\gamma_4 = \mathbf{X}\cdot\gamma_4 + \mathbf{X}\wedge\gamma_4 = X_4\left(1 + \frac{\mathbf{X}\wedge\gamma_4}{X_4}\right) \equiv X_4(1 + \boldsymbol{x}). \tag{31}$$

Note that $\boldsymbol{x}$ contains terms of the form $\gamma_1\gamma_4$, $\gamma_2\gamma_4$, $\gamma_3\gamma_4$ or, via the associations in equation (??), terms in $\sigma_1, \sigma_2, \sigma_3$. We can therefore think of the vector $\boldsymbol{x}$ as a vector in $\mathcal{E}^3$ which is associated with the bivector $\mathbf{X}\wedge\gamma_4/X_4$ in R^4.

If we start with a vector $\boldsymbol{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ in $\mathcal{E}^3$, we can represent this in R^4 by the vector $\mathbf{X} = X_1\gamma_1 + X_2\gamma_2 + X_3\gamma_3 + X_4\gamma_4$ such that

$$\begin{aligned} \boldsymbol{x} &= \frac{\mathbf{X}\wedge\gamma_4}{X_4} = \frac{X_1}{X_4}\gamma_1\gamma_4 + \frac{X_2}{X_4}\gamma_2\gamma_4 + \frac{X_3}{X_4}\gamma_3\gamma_4 \\ &= \frac{X_1}{X_4}\sigma_1 + \frac{X_2}{X_4}\sigma_2 + \frac{X_3}{X_4}\sigma_3, \end{aligned} \tag{32}$$

$\Rightarrow x_i = \frac{X_i}{X_4}$, for $i = 1, 2, 3$. The process of representing $\boldsymbol{x}$ in a higher dimensional space can therefore be seen to be equivalent to using **homogeneous coordinates**, $\mathbf{X}$, for $\boldsymbol{x}$. Thus, in this geometric algebra formulation we postulate distinct spaces in which we represent ordinary 3D quantities and their 4D projective counterparts, together with a well-defined way of moving between these spaces. It may be worth noting that for all of the issues addressed here, the non-Euclidean nature of R^4 will have no effect, but the presence of a null structure (vectors which square to zero) may have interesting consequences for other problems.

3.2 Projective transformations

It is well known that there are various advantages to working in homogeneous coordinates. For example, general displacements can be expressed in terms of a single matrix and some non-linear transformations in $\mathcal{E}^3$ become linear transformations in R^4 – indeed, historically this has been the main motivation for working in homogeneous coordinates [?].

If a general point (x, y, z) in 3-D space is projected onto an image plane, the coordinates (x', y') in the image plane will be related to (x, y, z) via a transformation of the form:

$$x' = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{33}$$

The transformation is therefore non-linear and expressible as the ratio of two linear transformations. To make this non-linear transformation in $\mathcal{E}^3$ into a linear transformation in R^4 we define a linear function $\underline{f}_p$ mapping vectors onto vectors in R^4 such that the action of $\underline{f}_p$ on the basis vectors $\{\gamma_i\}$ is given by

$$\begin{aligned} \underline{f}_p(\gamma_1) &= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \tilde{\alpha}\gamma_4 \\ \underline{f}_p(\gamma_2) &= \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \tilde{\beta}\gamma_4 \\ \underline{f}_p(\gamma_3) &= \delta_1\gamma_1 + \delta_2\gamma_2 + \delta_3\gamma_3 + \tilde{\delta}\gamma_4 \\ \underline{f}_p(\gamma_4) &= \epsilon_1\gamma_1 + \epsilon_2\gamma_2 + \epsilon_3\gamma_3 + \tilde{\epsilon}\gamma_4 \end{aligned} \tag{34}$$

A general point P in $\mathcal{E}^3$ given by $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ becomes the point $\mathbf{X} = (X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4)$ in R^4, where $x = X/W$, $y = Y/W$, $z = Z/W$. We can then see that $\underline{f}_p$ maps $\mathbf{X}$ onto $\mathbf{X}'$ where

$$\mathbf{X}' = \sum_{i=1}^{3}\{(\alpha_i X + \beta_i Y + \delta_i Z + \epsilon_i W)\gamma_i\} + (\tilde{\alpha}X + \tilde{\beta}Y + \tilde{\delta}Z + \tilde{\epsilon}W)\gamma_4 \tag{35}$$

The vector $\boldsymbol{x}' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3$ in $\mathcal{E}^3$ corresponds to $\mathbf{X}'$, where x' is given by

$$x' = \frac{\alpha_1 X + \beta_1 Y + \delta_1 Z + \epsilon_1 W}{\tilde{\alpha}X + \tilde{\beta}Y + \tilde{\delta}Z + \tilde{\epsilon}W} = \frac{\alpha_1 x + \beta_1 y + \delta_1 z + \epsilon_1}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{36}$$

Similarly we have

$$y' = \frac{\alpha_2 x + \beta_2 y + \delta_2 z + \epsilon_2}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}, \quad z' = \frac{\alpha_3 x + \beta_3 y + \delta_3 z + \epsilon_3}{\tilde{\alpha}x + \tilde{\beta}y + \tilde{\delta}z + \tilde{\epsilon}}. \tag{37}$$

Note that in general we would take $\alpha_3 = f\tilde{\alpha}$, $\beta_3 = f\tilde{\beta}$ etc. so that $z' = f$ (focal length), independent of the point chosen. Via this means the non-linear transformation in $\mathcal{E}^3$ becomes a linear transformation, $\underline{f}_p$, in R^4. We will see later that use of the linear function $\underline{f}_p$ makes the invariant nature of various quantities very easy to establish.

3.3 Algebra in projective space

There has been much recent interest in the use of the Grassmann-Cayley or double algebra as an elegant means of formulating the algebra of incidence [?, ?, ?, ?]. Here we will show briefly how the main algebraic results of the Grassmann-Cayley algebra arise naturally when we express projective geometry in geometric algebra. It should be stressed here that we can recover a very similar algebraic formalism while remaining totally within a mathematical system that is applicable to standard 3D problems – it is not necessary to introduce different quantities (e.g. extensors) or different rules for manipulation of elements of the algebra.

Consider three non-collinear points, P_1, P_2, P_3, represented by vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ in $\mathcal{E}^3$ and by vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in R^4. The line L_{12} joining points P_1 and P_2 can be expressed in R^4 by the following bivector,

$$L_{12} = \mathbf{X}_1 \wedge \mathbf{X}_2. \tag{38}$$

Since $\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3 = 0$ if $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ are collinear (for $\mathbf{Y}_i$ in R^4), any point P, represented in R^4 by $\mathbf{X}$, on the line through P_1 and P_2, will satisfy

$$\mathbf{X} \wedge L_{12} = \mathbf{X} \wedge \mathbf{X}_1 \wedge \mathbf{X}_2 = 0. \tag{39}$$

This is therefore the equation of this line in R^4. In general such an equation is telling us that $\mathbf{X}$ belongs to the subspace spanned by $\mathbf{X}_1$ and $\mathbf{X}_2$, i.e. that

$$\mathbf{X} = \alpha_1 \mathbf{X}_1 + \alpha_2 \mathbf{X}_2 \tag{40}$$

for some α_1, α_2. For completeness, we will go through the argument which relates equation (??) in R^4 to the standard equation of the line through P_1 and P_2 in $\mathcal{E}^3$, namely

$$\boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \tag{41}$$

with $\lambda_1 + \lambda_2 = 1$. Taking (equation (??)) $\wedge \gamma_4$ and (equation (??)) $\cdot \gamma_4$ and dividing the results gives

$$\frac{\mathbf{X} \wedge \gamma_4}{\mathbf{X} \cdot \gamma_4} = \frac{1}{\{\alpha_1(\mathbf{X}_1 \cdot \gamma_4) + \alpha_2(\mathbf{X}_2 \cdot \gamma_4)\}} \{\alpha_1 \mathbf{X}_1 \wedge \gamma_4 + \alpha_2 \mathbf{X}_2 \wedge \gamma_4\}. \tag{42}$$

Via the projective split we can identify $\boldsymbol{x} = \frac{\mathbf{X} \wedge \gamma_4}{\mathbf{X} \cdot \gamma_4}$, $\boldsymbol{x}_1 = \frac{\mathbf{X}_1 \wedge \gamma_4}{\mathbf{X}_1 \cdot \gamma_4}$, $\boldsymbol{x}_2 = \frac{\mathbf{X}_2 \wedge \gamma_4}{\mathbf{X}_2 \cdot \gamma_4}$, so that equation (??) becomes

$$\boldsymbol{x} = \frac{1}{\{\alpha_1(\mathbf{X}_1 \cdot \gamma_4) + \alpha_2(\mathbf{X}_2 \cdot \gamma_4)\}} \{\alpha_1(\mathbf{X}_1 \cdot \gamma_4)\boldsymbol{x}_1 + \alpha_2(\mathbf{X}_2 \cdot \gamma_4)\boldsymbol{x}_2\} \tag{43}$$

or, $\boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2$ with $\lambda_1 + \lambda_2 = 1$, as required. We have laboured this rather obvious case to emphasize how the projective split brings about the necessary reduction of the number of free parameters from two in R^4 to one in $\mathcal{E}^3$.

Similarly, the plane Φ_{123} passing through points P_1, P_2, P_3 is expressed by the following trivector in R^4

$$\Phi_{123} = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \tag{44}$$

which further motivates the previous statement concerning three collinear R^4 vectors. In $\mathcal{E}^3$ there are generally three types of intersections we wish to consider; the intersection of a line and a plane, a plane and a plane, and a line and a line. We will look at each of these three cases individually, but for each of them we will require the following general result, giving the inner product of an r-blade, $A_r = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge ..... \wedge \boldsymbol{a}_r$, and an s-blade, $B_s = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge ..... \wedge \boldsymbol{b}_s$ (for $s \leq r$)

$$B_s \cdot (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge ..... \wedge \boldsymbol{a}_r) = \sum_{\boldsymbol{j}} \epsilon(j_1 j_2 .... j_r) B_s \cdot (\boldsymbol{a}_{j_1} \wedge \boldsymbol{a}_{j_2} \wedge .... \wedge \boldsymbol{a}_{j_s}) \boldsymbol{a}_{j_s+1} \wedge ..... \wedge \boldsymbol{a}_{j_r} \tag{45}$$

where we sum over all combinations $\boldsymbol{j} = (j_1, j_2, ..., j_r)$ such that $j_1 < j_2 < .... < j_r$. $\epsilon(j_1 j_2 ... j_r) = +1$ if $\boldsymbol{j}$ is an even permutation of $(1, 2, 3, ..., r)$ and -1 if it is an odd permutation. See [?] for further discussion of this result.

3.3.1 Intersection of a line and a plane

Consider a line $A = \mathbf{X}_1 \wedge \mathbf{X}_2$ intersecting a plane $\Phi = \mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3$ – all vectors are in R^4. The intersection point is expressible using the *meet* operation

$$A \vee \Phi = (\mathbf{X}_1 \wedge \mathbf{X}_2) \vee (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3). \tag{46}$$

Using the definition of the meet given in equation (??) we have (noting that the dual is taken with respect to the join, which is, in this case, the whole space)

$$A \vee \Phi = A^* \cdot \Phi. \tag{47}$$

Note that the pseudoscalar for R^4, which we shall call I_4 if any ambiguity is possible, squares to -1 and commutes with bivectors but anticommutes with vectors and trivectors in $\mathcal{G}(1,3)$, and that $I_4^{-1} = -I_4$. This leads to

$$A^* \cdot \Phi = (AI^{-1}) \cdot \Phi = -(AI) \cdot \Phi. \tag{48}$$

According to equation (??) we can then expand the meet as

$$\begin{aligned} A \vee \Phi &= -(AI) \cdot (\mathbf{Y}_1 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3) \\ &= -[\{(AI) \cdot (\mathbf{Y}_2 \wedge \mathbf{Y}_3)\}\mathbf{Y}_1 + \{(AI) \cdot (\mathbf{Y}_3 \wedge \mathbf{Y}_1)\}\mathbf{Y}_2 + \{(AI) \cdot (\mathbf{Y}_1 \wedge \mathbf{Y}_2)\}\mathbf{Y}_3] \end{aligned} \tag{49}$$

Noting that $(AI) \cdot (\mathbf{Y}_i \wedge \mathbf{Y}_j)$ is a scalar, we can evaluate the above by taking scalar parts. For example, $(AI) \cdot (\mathbf{Y}_2 \wedge \mathbf{Y}_3) = \langle I(\mathbf{X}_1 \wedge \mathbf{X}_2)(\mathbf{Y}_2 \wedge \mathbf{Y}_3) \rangle = I(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3)$. From the definition of the bracket given earlier, we can see that if $P = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3$, then $[P] = (\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{Y}_2 \wedge \mathbf{Y}_3) I_4^{-1}$. If we therefore write $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]$ as a shorthand for the magnitude of the pseudoscalar formed from the four vectors, then we can readily see that the meet reduces to

$$A \vee \Phi = [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_2\mathbf{Y}_3]\mathbf{Y}_1 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_3\mathbf{Y}_1]\mathbf{Y}_2 + [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_1\mathbf{Y}_2]\mathbf{Y}_3 \tag{50}$$

giving the intersection point (vector in R^4). Note that this is precisely the expansion of the meet that would result from the analysis in the Grassmann-Cayley algebra [?, ?]. We can see that we must identify the r-extensors of the Grassmann-Cayley algebra with r-blades in our geometric algebra. Also, the definition of the bracket of four vectors in R^4 as the magnitude of the pseudoscalar formed from the outer product of the vectors is equivalent to its definition as the determinant of the four vectors in the Grassmann-Cayley algebra.

From the definition of the bracket given above it is easy to show that for $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]$ the equivalent bracket in $\mathcal{E}^3$ is formed by evaluating the following volume

$$(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1) I_3^{-1} \tag{51}$$

where, as before, $\boldsymbol{x}_i = \frac{\mathbf{X}_i \wedge \gamma_4}{\mathbf{X}_i \cdot \gamma_4}$. This is most easily seen by the following argument;

$$\begin{aligned} P = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 &= \langle \mathbf{X}_1 \gamma_4 \gamma_4 \mathbf{X}_2 \mathbf{X}_3 \gamma_4 \gamma_4 \mathbf{X}_4 \rangle_4 \\ &= W_1 W_2 W_3 W_4 \langle (1 + \boldsymbol{x}_1)(1 - \boldsymbol{x}_2)(1 + \boldsymbol{x}_3)(1 - \boldsymbol{x}_4) \rangle_4 \end{aligned}$$

where $W_i = \mathbf{X}_i \cdot \gamma_4$ from equation (??). A pseudoscalar part is produced by taking the product of three spatial vectors (there are no (spatial bivector)×(spatial vector) terms), i.e.

$$\begin{aligned} P &= W_1 W_2 W_3 W_4 \langle -\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3 - \boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{x}_4 + \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_4 + \boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4 \rangle_4 \\ &= W_1 W_2 W_3 W_4 \langle (\boldsymbol{x}_2 - \boldsymbol{x}_1)(\boldsymbol{x}_3 - \boldsymbol{x}_1)(\boldsymbol{x}_4 - \boldsymbol{x}_1) \rangle_4 \\ &= W_1 W_2 W_3 W_4 \{(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1)\}. \end{aligned} \tag{52}$$

If the $W_i = 1$, we can summarize the above relationships between the brackets of 4 points in R^4 and $\mathcal{E}^3$ as follows

$$[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} \propto \{(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1)\} I_3^{-1}. \tag{53}$$

3.3.2 Intersection of two planes

We now consider the intersection of two planes $\Phi_1 = \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3$ and $\Phi_2 = \mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3$. The meet of Φ_1 and Φ_2 is given by

$$\Phi_1 \vee \Phi_2 = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3) \vee (\mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3). \tag{54}$$

As before, this can be expanded as

$$\begin{aligned}\Phi_1 \vee \Phi_2 &= \Phi_1{}^{*}\cdot(\mathbf{Y}_1\wedge\mathbf{Y}_2\wedge\mathbf{Y}_3)\\ &= -\left[\{(\Phi_1 I)\cdot\mathbf{Y}_1\}(\mathbf{Y}_2\wedge\mathbf{Y}_3) + \{(\Phi_1 I)\cdot\mathbf{Y}_2\}(\mathbf{Y}_3\wedge\mathbf{Y}_1) + \{(\Phi_1 I)\cdot\mathbf{Y}_3\}(\mathbf{Y}_1\wedge\mathbf{Y}_2)\right].\end{aligned}$$

Again, the join is the whole space and so the dual is easily formed. Following the arguments of the previous section we can show that $(\Phi_1 I)\cdot\mathbf{Y}_i \equiv -[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_i]$, so that the meet is

$$\Phi_1 \vee \Phi_2 = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_1](\mathbf{Y}_2\wedge\mathbf{Y}_3) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_2](\mathbf{Y}_3\wedge\mathbf{Y}_1) + [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{Y}_3](\mathbf{Y}_1\wedge\mathbf{Y}_2), \tag{55}$$

producing a line of intersection (bivector in R^4). If one identifies the 2-extensors of the Grassmann-Cayley algebra with bivectors in the geometric algebra, the above expansion is seen to be the same as the expressions given in [?].

3.3.3 Intersection of two lines

Two lines will intersect at a point only if they are coplanar, this will mean that their representations in R^4, $A = \mathbf{X}_1\wedge\mathbf{X}_2$, and $B = \mathbf{Y}_1\wedge\mathbf{Y}_2$ will satisfy

$$A\wedge B = 0 \tag{56}$$

since it is then not possible to form any 4-volume. We see therefore that any one vector is expressible as a linear combination of the other three vectors. We then need to work only in a 2D Euclidean space, $\mathcal{E}^2$, which has an associated 3D projective counterpart, R^3. The intersection point is given by

$$A \vee B = A^{*}\cdot B = -(AI_3)\cdot(\mathbf{Y}_1\wedge\mathbf{Y}_2) = -\left\{((AI_3)\cdot\mathbf{Y}_1)\mathbf{Y}_2 - ((AI_3)\cdot\mathbf{Y}_2)\mathbf{Y}_1\right\} \tag{57}$$

where I_3 is the pseudoscalar for R^3. Once again we evaluate $((AI_3)\cdot\mathbf{Y}_i)$ by taking scalar parts

$$(AI_3)\cdot\mathbf{Y}_i = \langle\mathbf{X}_1\mathbf{X}_2 I_3\mathbf{Y}_i\rangle = I_3\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_i = -[\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_i]. \tag{58}$$

The meet can therefore be written as

$$A \vee B = [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_1]\mathbf{Y}_2 - [\mathbf{X}_1\mathbf{X}_2\mathbf{Y}_2]\mathbf{Y}_1 \tag{59}$$

where the bracket $[\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3]$ in R^3 is understood to mean $(\mathbf{A}_1\wedge\mathbf{A}_2\wedge\mathbf{A}_3)I_3{}^{-1}$.

3.3.4 Implementation of the algebra

The approach we have outlined here has expressed the projective nature of the situation by postulating a 4-dimensional geometric algebra with an imposed Minkowski metric, $\mathcal{G}(1,3)$ – recall that the reason for adopting such a metric was to ensure that the 3D spatial vectors to which the 4D vectors are related via the projective split, are Euclidean. This 4D algebra has been implemented using the computer algebra package MAPLE and all of the operations and expressions derived here are easily evaluated. Another advantage of having the quantities we are dealing with as elements of a geometric algebra is that one can then (in any dimension) use the framework for dealing with rotations outlined in section (2.3) and deal with other classes of structure, such as Lie groups, within the algebra.

4 Invariance using Geometric Algebra

In this section we will use the framework established so far to show how standard geometric invariants can be expressed both elegantly and concisely using geometric algebra. Here we will look only at algebraic quantities which are invariant under projective transformations, arriving at these invariants in a way which can be naturally generalized from 1D to 2D to 3D.

4.1 1-D and 2-D Projective Invariants from a Single View

The 1-D Cross-Ratio

The '*fundamental projective invariant*' of points on a line is the so-called **cross-ratio**, ρ, defined as

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)},$$

where $t_1 = |PA|$, $t_2 = |PB|$, $t_3 = |PC|$, $t_4 = |PD|$ – see Figure ??. It is fairly easy to show that for the projection through O of the collinear points A, B, C, D onto any line, ρ remains constant. For this 1D case, any point q on the line L can be written as $\boldsymbol{q} = t\sigma_1$ relative to P, where σ_1 is a unit vector in the direction of L. We then move up a dimension to a 2D space, with basis vectors (γ_1, γ_2), we will call R^2 in which $\boldsymbol{q}$ is represented by the vector $\mathbf{Q}$;

$$\mathbf{Q} = T\gamma_1 + S\gamma_2$$

where, as before, we associate $\boldsymbol{q}$ with the bivector

$$\frac{\mathbf{Q}\wedge\gamma_2}{\mathbf{Q}\cdot\gamma_2} = \frac{T}{S}\gamma_1\gamma_2 \equiv \frac{T}{S}\sigma_1$$

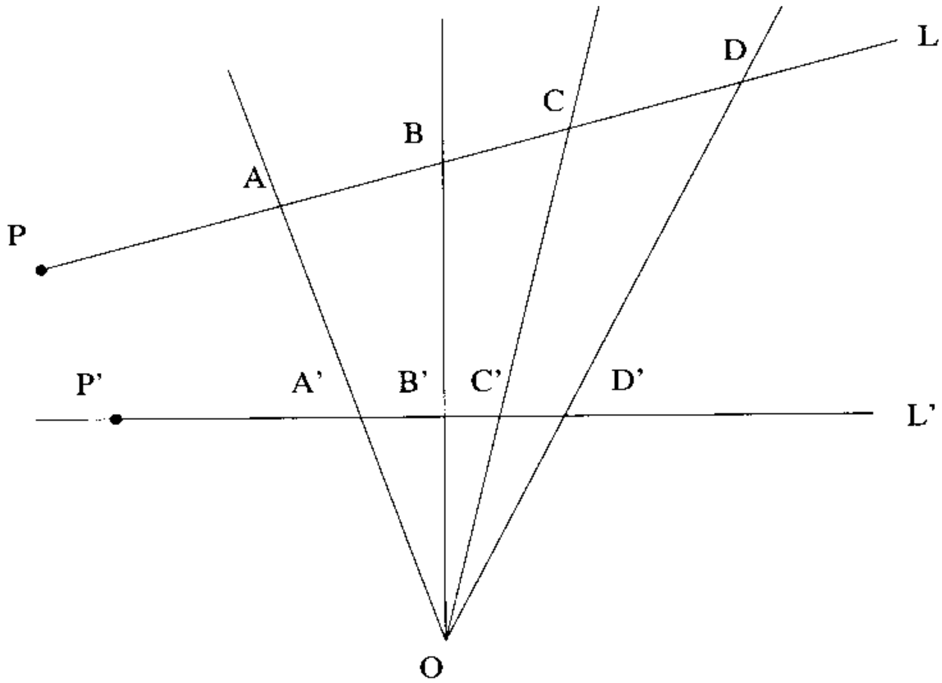

Figure 3: Formation of the 1D cross-ratio

so that $t = T/S$. When a point on line L is projected onto another line L', the distances t and t', where $t' = P'A'$ etc., are related by a projective transformation of the form

$$t' = \frac{\alpha t + \beta}{\tilde{\alpha} t + \tilde{\beta}}. \tag{60}$$

This non-linear transformation in $\mathcal{E}^1$ can be made into a linear transformation in R^2 by defining the linear function $\underline{f}_1$ mapping vectors onto vectors in R^2;

$$\begin{aligned} \underline{f}_1(\gamma_1) &= \alpha_1\gamma_1 + \tilde{\alpha}\gamma_2 \\ \underline{f}_1(\gamma_2) &= \beta_1\gamma_1 + \tilde{\beta}\gamma_2. \end{aligned}$$

Consider 2 vectors $\mathbf{X}_1, \mathbf{X}_2$ in R^2. Form the bivector

$$\mathcal{S}_1 = \mathbf{X}_1 \wedge \mathbf{X}_2 = \lambda_1 I_2$$

where $I_2 = \gamma_1\gamma_2$ is the pseudoscalar for R^2. We now look at how $\mathcal{S}_1$ transforms under $\underline{f}_1$:

$$\mathcal{S}_1' = \mathbf{X}_1' \wedge \mathbf{X}_2' = \underline{f}_1(\mathbf{X}_1 \wedge \mathbf{X}_2) = (\det \underline{f}_1)(\mathbf{X}_1 \wedge \mathbf{X}_2). \tag{61}$$

This last step follows since a linear function must map a pseudoscalar onto a multiple of itself, this multiple being the determinant of the function (equation ??). Suppose that we now take 4 points on the line L whose corresponding vectors in R^2 are $\{\mathbf{X}_i\}$, $i = 1, .., 4$, and consider the ratio $\mathcal{R}_1$ of 2 wedge products;

$$\mathcal{R}_1 = \frac{\mathbf{X}_1 \wedge \mathbf{X}_2}{\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{62}$$

Then, under $\underline{f}_1$, $\mathcal{R}_1 \to \mathcal{R}_1'$, where

$$\mathcal{R}_1' = \frac{\mathbf{X}_1' \wedge \mathbf{X}_2'}{\mathbf{X}_3' \wedge \mathbf{X}_4'} = \frac{(\det \underline{f}_1)\mathbf{X}_1 \wedge \mathbf{X}_2}{(\det \underline{f}_1)\mathbf{X}_3 \wedge \mathbf{X}_4}. \tag{63}$$

$\mathcal{R}_1$ is therefore invariant under $\underline{f}_1$. However, we want to express our invariants in terms of distances on the 1D line; for this we must consider how the bivector $\mathcal{S}_1$ in R^2 projects down to $\mathcal{E}^1$.

$$\begin{aligned} \mathbf{X}_1 \wedge \mathbf{X}_2 &= (T_1\gamma_1 + S_1\gamma_2) \wedge (T_2\gamma_1 + S_2\gamma_2) \\ &= (T_1 S_2 - T_2 S_1)\gamma_1\gamma_2 \\ &\equiv S_1 S_2 (T_1/S_1 - T_2/S_2) I_2 \\ &= S_1 S_2 (t_1 - t_2) I_2. \end{aligned} \tag{64}$$

Note that we could also have arrived at this result via the method outlined in equation (??). In order to form a projective invariant which is independent of the choice of the arbitrary scalars S_i, we must then take *ratios* of the bivectors $\mathbf{X}_i \wedge \mathbf{X}_j$ (so that $\det \underline{f}_1$ cancels) and *multiples* of such ratios so that the S_i's cancel. More precisely, consider the following expression

$$Inv_1 = \frac{(\mathbf{X}_3 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_2) I_2^{-1}}{(\mathbf{X}_4 \wedge \mathbf{X}_1) I_2^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_2) I_2^{-1}}$$

Then, in terms of distances along the lines, under the projective transformation $\underline{f}_1$, Inv_1 goes to Inv_1' where

$$Inv_1' = \frac{S_3 S_1 (t_3 - t_1) S_4 S_2 (t_4 - t_2)}{S_4 S_1 (t_4 - t_1) S_3 S_2 (t_3 - t_2)} = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_4 - t_1)(t_3 - t_2)}, \tag{65}$$

which is independent of the S_i's and is indeed the 1D classical projective invariant, the **cross-ratio.** Deriving the cross-ratio in this way enables us to easily generalize it to form invariants in higher dimensions.

The 2-D generalization of the Cross-Ratio

For points in a plane we again move up to a space with one higher dimension which we shall call R^3. Let a point P in the plane M be described by the vector $\boldsymbol{x}$ in $\mathcal{E}^2$ where $\boldsymbol{x} = x\sigma_1 + y\sigma_2$. In R^3 this point will be represented by $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3$ where $x = X/Z$ and $y = Y/Z$. As described in Section 3.2, we can define a general projective transformation via a linear function $\underline{f}_2$ mapping vectors to vectors in R^3 such that;

$$\begin{aligned} \underline{f}_2(\gamma_1) &= \alpha_1\gamma_1 + \alpha_2\gamma_2 + \tilde{\alpha}\gamma_3 \\ \underline{f}_2(\gamma_2) &= \beta_1\gamma_1 + \beta_2\gamma_2 + \tilde{\beta}\gamma_3 \\ \underline{f}_2(\gamma_3) &= \delta_1\gamma_1 + \delta_2\gamma_2 + \tilde{\delta}\gamma_3. \end{aligned} \tag{66}$$

Consider 3 vectors (representing non-collinear points) $\mathbf{X}_i$, $i = 1, 2, 3$, in R^3 and form the trivector

$$\mathcal{S}_2 = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 = \lambda_2 I_3 \tag{67}$$

where $I_3 = \gamma_1\gamma_2\gamma_3$ is the pseudoscalar for R^3. As before, under the projective transformation given by $\underline{f}_2$, S_2 transforms to S_2' where

$$S_2' = \det\underline{f}_2 S_2. \tag{68}$$

Therefore, the ratio of any trivectors is invariant under $\underline{f}_2$. To project down into $\mathcal{E}^2$, using the fact that $\mathbf{X}_i\gamma_3 = Z_i(1+\boldsymbol{x}_i)$ under the projective split and $\gamma_3^2 = 1$, we can write

$$\begin{aligned} S_2 I_3^{-1} &= \langle \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3 I_3^{-1}\rangle \\ &= \langle \mathbf{X}_1\gamma_3\gamma_3\mathbf{X}_2\mathbf{X}_3\gamma_3\gamma_3\gamma_3 I_3^{-1}\rangle \\ &= Z_1 Z_2 Z_3 \langle (1+\boldsymbol{x}_1)(1-\boldsymbol{x}_2)(1+\boldsymbol{x}_3)\gamma_3 I_3^{-1}\rangle. \end{aligned} \tag{69}$$

Where the $\boldsymbol{x}_i$ represent vectors in $\mathcal{E}^2$. We can only form a scalar part from the expression within the brackets by taking products of a vector, 2 spatial vectors and I_3^{-1}, i.e.

$$S_2 I_3^{-1} = Z_1 Z_2 Z_3 \langle (\boldsymbol{x}_1\boldsymbol{x}_3 - \boldsymbol{x}_1\boldsymbol{x}_2 - \boldsymbol{x}_2\boldsymbol{x}_3)\gamma_3 I_3^{-1}\rangle = Z_1 Z_2 Z_3 \{(\boldsymbol{x}_2 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_3 - \boldsymbol{x}_1)\} I_2^{-1}. \tag{70}$$

As $I_3^2 = -1$, we have $I_3^{-1} = \gamma_3\gamma_2\gamma_1$ so that $\gamma_3 I_3^{-1} = \gamma_2\gamma_1 = (\gamma_2\gamma_3)(\gamma_3\gamma_1) = -\sigma_2\sigma_1 = -I_2^{-1}$, which accounts for the identification in equation (??). It is then clear that we should take multiples of such ratios so that the arbitrary scalars Z_i cancel. For 4 points in a plane, there are only 4 possible combinations of $Z_i Z_j Z_k$ and we cannot have a situation where we multiply two ratios of the form $\mathbf{X}_i\wedge\mathbf{X}_j\wedge\mathbf{X}_k$ together and have all the Z's cancelling. For 5 coplanar points $\{\mathbf{X}_i\}$, $i = 1,..,5$, there are several ways of achieving the desired cancellation, for example

$$Inv_2 = \frac{(\mathbf{X}_5\wedge\mathbf{X}_4\wedge\mathbf{X}_3)I_3^{-1}(\mathbf{X}_5\wedge\mathbf{X}_2\wedge\mathbf{X}_1)I_3^{-1}}{(\mathbf{X}_5\wedge\mathbf{X}_1\wedge\mathbf{X}_3)I_3^{-1}(\mathbf{X}_5\wedge\mathbf{X}_2\wedge\mathbf{X}_4)I_3^{-1}}.$$

According to equation (??) we can interpret this ratio in $\mathcal{E}^2$ as

$$Inv_2 = \frac{(\boldsymbol{x}_5 - \boldsymbol{x}_4)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_3)I_2^{-1}(\boldsymbol{x}_5 - \boldsymbol{x}_2)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_1)I_2^{-1}}{(\boldsymbol{x}_5 - \boldsymbol{x}_1)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_3)I_2^{-1}(\boldsymbol{x}_5 - \boldsymbol{x}_2)\wedge(\boldsymbol{x}_5 - \boldsymbol{x}_4)I_2^{-1}} = \frac{A_{543}A_{521}}{A_{513}A_{524}} \tag{71}$$

where $\frac{1}{2}A_{ijk}$ is the area of the triangle defined by the 3 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k$. This invariant is regarded as the 2D generalization of the 1D cross-ratio.

4.2 3-D Projective Invariants from Multiple Views

4.2.1 The 3-D generalization of the Cross-Ratio

When considering general points in $\mathcal{E}^3$ we have seen that we move up one dimension to work in the 4D space R^4. The point $\boldsymbol{x} = x\sigma_1 + y\sigma_2 + z\sigma_3$ in $\mathcal{E}^3$ is written as $\mathbf{X} = X\gamma_1 + Y\gamma_2 + Z\gamma_3 + W\gamma_4$, where $x = X/W$, $y = Y/W$, $z = Z/W$. As before, a non-linear projective transformation in $\mathcal{E}^3$ becomes a linear transformation, described by the linear function $\underline{f}_3$, say, in R^4.

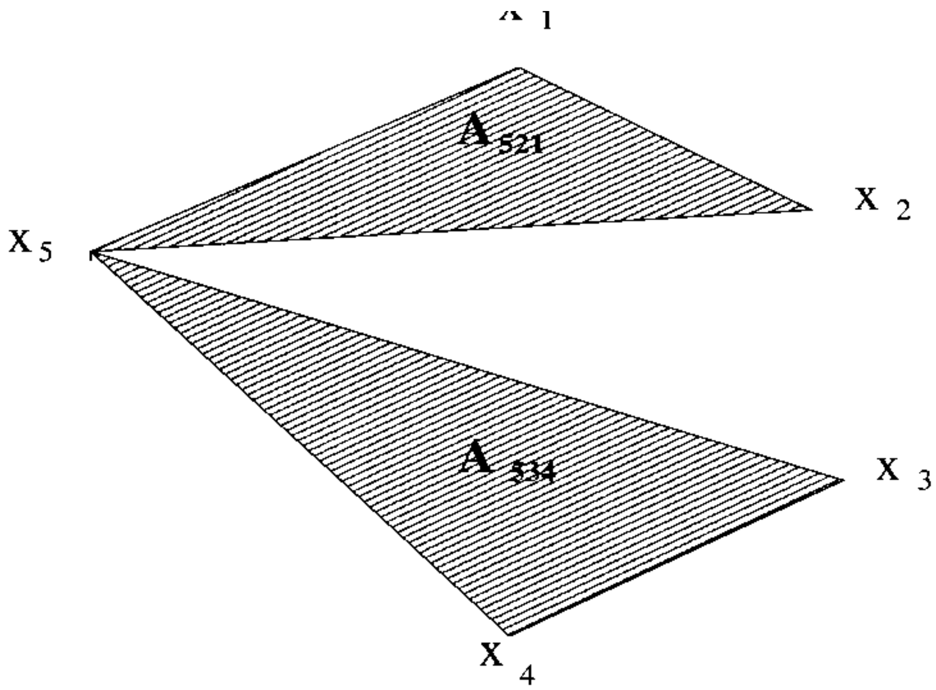

Figure 4: The areas A_{521} and A_{543}

Consider 4 vectors in R^4, $\{\mathbf{X}_i\}$, $i = 1,..,4$. Form the 4-vector

$$\mathcal{S}_3 = \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 = \lambda_3 I_4 \tag{72}$$

where $I_4 = \gamma_1\gamma_2\gamma_3\gamma_4$ is the pseudoscalar for R^4. As before, $\mathcal{S}_3$ transforms to $\mathcal{S}_3'$ under $\underline{f}_3$;

$$\mathcal{S}_3' = \mathbf{X}_1' \wedge \mathbf{X}_2' \wedge \mathbf{X}_3' \wedge \mathbf{X}_4' = \det \underline{f}_3 \mathcal{S}_3. \tag{73}$$

The ratio of any two 4-vectors is therefore invariant under $\underline{f}_3$ and we must take multiples of such ratios to ensure the arbitrary scale factors W_i cancel. With 5 general points there are 5 possible ways of forming the combinations $W_iW_jW_kW_l$ and it is then simple to show that one cannot take multiples of ratios such that the W factors cancel. For 6 points one can, however, do this, and an example of such an invariant is

$$Inv_3 = \frac{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} (\mathbf{X}_4 \wedge \mathbf{X}_5 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}{(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5) I_4^{-1} (\mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_2 \wedge \mathbf{X}_6) I_4^{-1}}. \tag{74}$$

Using the arguments of the previous sections we can write

$$(\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4) I_4^{-1} \equiv W_1W_2W_3W_4\{(\boldsymbol{x}_2 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_3 - \boldsymbol{x}_1) \wedge (\boldsymbol{x}_4 - \boldsymbol{x}_1)\} I_3^{-1}. \tag{75}$$

The invariant Inv_3 is therefore the 3D equivalent of the 1D cross-ratio and consists of ratios of volumes;

$$Inv_3 = \frac{V_{1234} V_{4526}}{V_{1245} V_{3426}}, \tag{76}$$

where V_{ijkl} is the volume of the solid formed by the 4 vertices $\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k, \boldsymbol{x}_l$.

Conventionally all of these invariants are well known but above we have outlined a general process for generating projective invariants in any dimension which is straightforward and simple.

4.2.2 3D projective invariants in terms of image coordinates

Suppose we have six general 3D points P_i, $i = 1,..,6$, represented by vectors $\{\boldsymbol{x}_i, \mathbf{X}_i\}$ in $\mathcal{E}^3$ and R^4 respectively. We have seen in Section 4.2.1 that 3D projective invariants can be formed from these points, and an example of such an invariant is

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_4\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_5][\mathbf{X}_3\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{77}$$

This is simply equation (??) rewritten in terms of brackets. If it is possible to express the bracket $[\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k\mathbf{X}_l]$ in terms of the **image coordinates** of points P_i, P_j, P_k, P_l, then this invariant will be readily computable in practice. Some of the most recent work which has addressed this problem has utilized the Grassmann-Cayley algebra [?, ?, ?]. It has been shown [?] that it is not possible to compute general 3D invariants from a single image and in [?] Carlsson discussed the computation of such invariants from a pair of images in terms of the image coordinates and the fundamental matrix, $\boldsymbol{F}$, using the Grassmann-Cayley algebra. Subsequent work by Csurka and Faugeras [?] discussed corrections to Carlsson's expressions by including a series of scale factors. We want to form invariants from only image coordinates and the fundamental matrix, but in the analyses referred to above, both coordinates and matrix are quantities derived in *projective space.* Despite the clarity of the derivations in [?], some degree of confusion has arisen when subsequent workers have tried to implement these invariants with real data. We need to be completely clear about how one forms *real* invariants which are formed from the *observed* image coordinates and the fundamental matrix calculated from these *observed* coordinates. Here we will translate the approach of Carlsson into the geometric algebra framework in order to clarify these issues. We will also give the explicit expressions for forming 3D projective invariants from the experimental data and note the relation between these expressions and previous work.

Consider the scalar S_{1234} formed from the bracket of 4 points

$$S_{1234} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4] = (\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1} = (\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)I_4^{-1}. \tag{78}$$

The quantities $(\mathbf{X}_1 \wedge \mathbf{X}_2)$ and $(\mathbf{X}_3 \wedge \mathbf{X}_4)$ represent the line joining points P_1 and P_2, and P_3 and P_4. We let $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ be the centres of projection of the two cameras and suppose that the two camera image planes can be defined by the two sets of vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$. Let the projection of points $\{P_i\}$ through the centres of projection onto the image planes be given by the vectors $\{\boldsymbol{a}'_i\}$ and $\{\boldsymbol{b}'_i\}$. Note that this notation follows that of Carlsson [?] but that our vectors, $\boldsymbol{a}_i, \boldsymbol{b}_i$, etc. are ordinary vectors in $\mathcal{E}^3$. We then let the representations of these vectors in R^4 be $\mathbf{A}_i, \mathbf{B}_i, \mathbf{A}'_i, \mathbf{B}'_i$..., etc.– see figure ??.

As shown in figure ??, a world point $\mathbf{X}$ projects onto points $\mathbf{A}'$ and $\mathbf{B}'$ in the two image planes. In figure ?? the epipoles (the intersections of the line joining the optical centres with the image planes) in the A and B planes are denoted by $\boldsymbol{E_{AB}}$ and $\boldsymbol{E_{BA}}$. It is clear that the points $\mathbf{A}_0, \mathbf{B}_0, \mathbf{A}', \mathbf{B}'$ are coplanar. We have seen previosuly that the wedge of these four vectors must therefore vanish:

$$\mathbf{A}_0\wedge\mathbf{B}_0\wedge\mathbf{A}'\wedge\mathbf{B}' = 0. \tag{79}$$

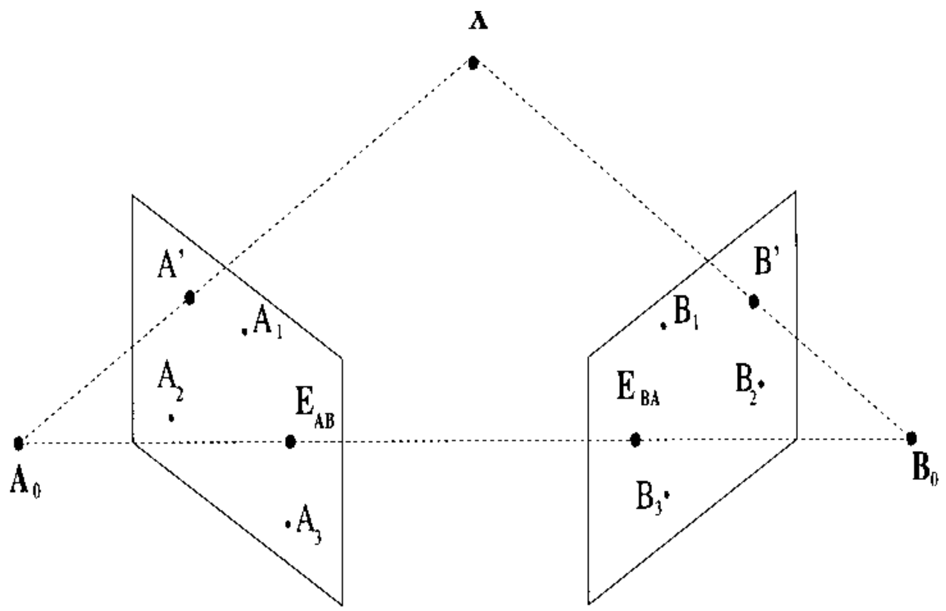

Figure 5: The projections of a world point $\mathbf{X}$ onto two image planes A and B are shown together with the epipoles, $\boldsymbol{E}_{AB}$ and $\boldsymbol{E}_{BA}$, and three points which define each plane, $(\mathbf{A}_i)$ and $(\mathbf{B}_i)$ for $i = 1, 2, 3$. All vectors are in R^4.

Now, if we let $\mathbf{A}' = \alpha_i \mathbf{A}_i$ and $\mathbf{B}' = \beta_j \boldsymbol{b}_j$, then equation (??) can be written as

$$\alpha_i \beta_j \{\mathbf{A}_0 \wedge \mathbf{B}_0 \wedge \mathbf{A}_i \wedge \mathbf{B}_j\} = 0. \tag{80}$$

Defining $\tilde{F}_{ij} = \{\mathbf{A}_0 \wedge \mathbf{B}_0 \wedge \mathbf{A}_i \wedge \mathbf{B}_j\} I^{-1}$ gives us

$$\tilde{F}_i j \alpha_i \beta_j = 0 \tag{81}$$

which is the well-known relationship between the components of the fundamental matrix, $\tilde{F}$, and the image coordinates.

From this geometric interpretation of $\tilde{F}$, which was first given by Carlsson, [?], we can immediately see how we might express the 3D projective invariants in terms of $\tilde{F}$ and the image coordinates. We start by considering two world points $\mathbf{X}_1$ and $\mathbf{X}_2$ projected into two images planes as shown in figure ??. We can express the line L_{12} joining these two world points as the intersection of the two planes $\Phi_A = \mathbf{A}_0 \wedge L^A{}_{12}$ and $\Phi_B = \mathbf{B}_0 \wedge L^B{}_{12}$, where $L^A{}_{12}$ and $L^B{}_{12}$ are the lines joining the projections of the world points in the two image planes, see figure ??. We can therefore write the line L_{12} as the intersection of two planes

$$L_{12} = \mathbf{X}_1 \wedge \mathbf{X}_2 = (\mathbf{A}_0 \wedge L^A{}_{12}) \vee (\mathbf{B}_0 \wedge L^B{}_{12}).$$

(82)

Now, we can obviously extend the above to give an expression for the 4-vector $\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4$:

$$\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 = [(\mathbf{A}_0 \wedge L^A{}_{12}) \vee (\mathbf{B}_0 \wedge L^B{}_{12})] \wedge [(\mathbf{A}_0 \wedge L^A{}_{34}) \vee (\mathbf{B}_0 \wedge L^B{}_{34})]. \tag{83}$$

Using the fact that

$$(\Phi_1 \vee \Phi_2)\wedge(\Phi_3 \vee \Phi_4) = -(\Phi_1 \vee \Phi_3)\wedge(\Phi_2 \vee \Phi_4) \tag{84}$$

(this result is proved in Appendix A), we can rewrite the 4-vector as

$$\begin{aligned} \mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4 &= [(\mathbf{A}_0\wedge L^A{}_{12}) \vee (\mathbf{A}_0\wedge L^A{}_{34})]\wedge[(\mathbf{B}_0\wedge L^B{}_{12}) \vee (\mathbf{B}_0\wedge L^B{}_{34})] \\ &= (\mathbf{A}_0\wedge\mathbf{A}_{1234})\wedge(\mathbf{B}_0\wedge\mathbf{B}_{1234}) \end{aligned} \tag{85}$$

where $\mathbf{A}_{1234}$ is the R^4 representation of the point $\boldsymbol{a}_{1234}$, the point in the first image plane which is the intersection of the lines joining points $\{\boldsymbol{a}'_1$ and $\boldsymbol{a}'_2\}$ and $\{\boldsymbol{a}'_3$ and $\boldsymbol{a}'_4\}$ and similarly for $\mathbf{B}_{1234}$ – see figure ??. The equality of the second line of equation (??) with the first can easily be seen by noting that the two planes $(\mathbf{A}_0\wedge L^A{}_{12})$ and $(\mathbf{A}_0\wedge L^A{}_{34})$ must intersect at a line passing through $\mathbf{A}_0$ and the intersection of $L^A{}_{12}$ and $L^A{}_{34}$.

Now suppose we write $\mathbf{A}_{1234}$ and $\mathbf{B}_{1234}$ in terms of the basis vectors:

$$\mathbf{A}'_{1234} = \alpha_{1234,1}\mathbf{A}_1 + \alpha_{1234,2}\mathbf{A}_2 + \alpha_{1234,3}\mathbf{A}_3. \tag{86}$$

$$\mathbf{B}'_{1234} = \beta_{1234,1}\mathbf{B}_1 + \beta_{1234,2}\mathbf{B}_2 + \beta_{1234,3}\mathbf{B}_3. \tag{87}$$

Then it is clear from our definition of $\tilde{F}$ above that we can now rewrite equation (??) as

$$\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4 = \mathbf{A}_0\wedge\mathbf{B}_0\wedge\mathbf{A}_{1234}\wedge\mathbf{A}_{1234} = \tilde{F}_{ij}\alpha_{1234,i}\beta_{1234,j}. \tag{88}$$

Note here that the bracket $[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]$ has been equated to $[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1234}\mathbf{B}'_{1234}]$ by the process of splitting up the bracket into two parts, $\mathbf{X}_1\wedge\mathbf{X}_2$ and $\mathbf{X}_3\wedge\mathbf{X}_4$ and then expressing each of these lines (bivectors) as the meet of two planes (trivectors). During this process, since we are working in R^4, we are effectively ignoring any scale factors due to the arbitrary choices of the γ_4 components. Thus, when we take ratios of brackets to form our invariants we must ensure that, if we want to express the brackets in the form of equation (??), the same decomposition of $\mathbf{X}_i \wedge \mathbf{X}_j$ must occur in the numerator and denominator so that these arbitrary factors cancel. In the case of Inv_3, we have

$$Inv_3 = \frac{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)\}I_4{}^{-1}\{(\mathbf{X}_4\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_4\wedge\mathbf{X}_5)\}I_4{}^{-1}\{(\mathbf{X}_3\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4{}^{-1}} \tag{89}$$

so we see that this decomposition rule has been obeyed. Now, in the past it has been claimed that the invariant of 6 points which can be thought of as arising from 4 points and a line, namely

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]} \tag{90}$$

is not invariant when expressed in Carlsson's terms; the solution being to include various correcting scale factors. If we were to decompose the expression as given in equation (??) in the manner outlined previously, we would have

$$Inv_3 = \frac{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_4)\}I_4{}^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_5\wedge\mathbf{X}_6)\}I_4{}^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_3\wedge\mathbf{X}_5)\}I_4{}^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_2)\wedge(\mathbf{X}_4\wedge\mathbf{X}_6)\}I_4{}^{-1}}. \tag{91}$$

It is clear that the same bivectors do *not* appear in both the numerator and denominator and therefore there will not be the required cancelling of scale factors. However, suppose we simply rearrange equation (??) in the following way;

$$Inv_3' = \frac{[\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_5\mathbf{X}_2\mathbf{X}_3][\mathbf{X}_1\mathbf{X}_4\mathbf{X}_2\mathbf{X}_6]}. \tag{92}$$

This can always be done since interchanging vectors in $\mathbf{X}_i \wedge \mathbf{X}_j \wedge \mathbf{X}_k \wedge \mathbf{X}_l$ simply changes the sign of the pseudoscalar. Now, the decomposition would look like

$$Inv_3' = \frac{\{(\mathbf{X}_1\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_3)\}I_4^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_5)\wedge(\mathbf{X}_2\wedge\mathbf{X}_3)\}I_4^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_4)\wedge(\mathbf{X}_2\wedge\mathbf{X}_6)\}I_4^{-1}}. \tag{93}$$

We now see that the *same* bivectors appear in both numerator and denominator and therefore that all scale factors should cancel. Writing

$$Inv_3 = \frac{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_5\mathbf{X}_6]}{[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_5][\mathbf{X}_1\mathbf{X}_2\mathbf{X}_4\mathbf{X}_6]} = \frac{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1423}\mathbf{B}'_{1423}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1526}\mathbf{B}'_{1526}]}{[\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1523}\mathbf{B}'_{1523}][\mathbf{A}_0\mathbf{B}_0\mathbf{A}'_{1426}\mathbf{B}'_{1426}]} \tag{94}$$

where $\mathbf{A}_{ijkl}$ is the point in R^4 corresponding to the intersection point $\boldsymbol{a}'_{ijkl}$ as defined previously, will indeed produce an invariant. We have verified this via simulated views of a number of points in MAPLE.

However, the problem of how to express such invariants in *observed* coordinates, still remains. Let us write the ratio Inv_3 as follows

$$Inv_3 = \frac{(\boldsymbol{\alpha}^T{}_{1234}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1234})(\boldsymbol{\alpha}^T{}_{4526}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{4526})}{(\boldsymbol{\alpha}^T{}_{1245}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{1245})(\boldsymbol{\alpha}^T{}_{3426}\tilde{\boldsymbol{F}}\boldsymbol{\beta}_{3426})} \tag{95}$$

where the we define the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ by $\boldsymbol{\alpha}_{1234} = (\alpha_{1234,1}, \alpha_{1234,2}, \alpha_{1234,3})$ and $\boldsymbol{\beta}_{1234} = (\beta_{1234,1}, \beta_{1234,2}, \beta_{1234,3})$. This was the expression obtained by Carlsson [?] – however the derivation in [?] was considerably more involved than that given here. Note that equation (??) is invariant whatever values of the γ_4 components of the vectors $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i$ etc. are chosen.

There is indeed no doubt that Inv_3 an invariant – the confusion seems to arise when we attempt to express Inv_3 in terms of what we actually observe, the 3D image coordinates and the fundamental matrix calculated from these image coordinates. A point P_i will be projected onto a point in image plane 1, say $\boldsymbol{a}'_i$, which can be written as

$$\boldsymbol{a}'_i = \boldsymbol{a}_1 + \lambda_i(\boldsymbol{a}_2 - \boldsymbol{a}_1) + \mu_i(\boldsymbol{a}_3 - \boldsymbol{a}_1) = \delta_{i1}\boldsymbol{a}_1 + \delta_{i2}\boldsymbol{a}_2 + \delta_{i3}\boldsymbol{a}_3 \tag{96}$$

where $\delta_{i1} = 1 - (\lambda_i + \mu_i)$, $\delta_{i2} = \lambda_i$ and $\delta_{i3} = \mu_i$, so that $\sum_{j=1}^{3}\delta_{ij} = 1$. We have seen that in R^4, $\boldsymbol{a}'_i$ is represented by $\mathbf{A}'_i$ and that $\mathbf{A}'_i$ and $\boldsymbol{a}'_i$ are connected via the projective split so that

$$\boldsymbol{a}'_i = \frac{\mathbf{A}'_i\wedge\gamma_4}{\mathbf{A}'_i\cdot\gamma_4} = \frac{1}{\mathbf{A}'_i\cdot\gamma_4}\{\alpha_{i1}(\mathbf{A}_1\wedge\gamma_4) + \alpha_{i2}(\mathbf{A}_2\wedge\gamma_4) + \alpha_{i3}(\mathbf{A}_3\wedge\gamma_4)\}. \tag{97}$$

We can therefore write the α_{ij}'s in terms of the δ_{ij}'s:

$$\alpha_{ij} = \frac{\mathbf{A}'_i\cdot\gamma_4}{\mathbf{A}_j\cdot\gamma_4}\delta_{ij}. \tag{98}$$

Similarly, if P_i projects onto a point $\boldsymbol{b}'_i$ in the second image plane which we write as $\boldsymbol{b}'_i = \epsilon_{i1}\boldsymbol{b}_1 + \epsilon_{i2}\boldsymbol{b}_2 + \epsilon_{i3}\boldsymbol{b}_3$ (so that $\sum_{j=1}^{3}\epsilon_{ij} = 1$) with a representation in R^4 given by $\mathbf{B}'_i = \beta_{i1}\mathbf{B}_1 + \beta_{i2}\mathbf{B}_2 + \beta_{i3}\mathbf{B}_3$, then the β_{ij}'s are related to the ϵ_{ij}'s by

$$\beta_{ij} = \frac{\mathbf{B}'_i\cdot\gamma_4}{\mathbf{B}_j\cdot\gamma_4}\epsilon_{ij}. \tag{99}$$

The 'fundamental' matrix $\tilde{\boldsymbol{F}}$ is such that $\boldsymbol{\alpha}^T{}_i\tilde{\boldsymbol{F}}\boldsymbol{\beta}_i = 0$, if $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ are the vectors of coefficients of the points in planes 1 and 2 produced by the same world point P_i. Now, given more than eight pairs of corresponding observed points in the two planes, $(\boldsymbol{\delta}_i, \boldsymbol{\epsilon}_i)$, $i = 1,..,8$, we can form an 'observed' fundamental matrix $\boldsymbol{F}$ such that

$$\boldsymbol{\delta}^T{}_i\boldsymbol{F}\boldsymbol{\epsilon}_i = 0. \tag{100}$$

This $\boldsymbol{F}$ can be found by some method such as the Longuet-Higgins 8-point algorithm [?] or, more correctly, by some method which gives an $\boldsymbol{F}$ which has the true structure [?]. If we define $\tilde{\boldsymbol{F}}$ by

$$\tilde{F}_{kl} = (\mathbf{A}_k\cdot\gamma_4)(\mathbf{B}_l\cdot\gamma_4)F_{kl} \tag{101}$$

then if follows from equations (??),(??) that

$$\alpha_{ik}\tilde{F}_{kl}\beta_{il} = (\mathbf{A}'_i\cdot\gamma_4)(\mathbf{B}'_i\cdot\gamma_4)\delta_{ik}F_{kl}\epsilon_{il}. \tag{102}$$

If $\boldsymbol{F}$ is formed using a method such as the Longuet-Higgins algorithm, then an $\tilde{\boldsymbol{F}}$ defined as in equation (??) will also act as a fundamental matrix in R^4.

Now let us look again at the invariant Inv_3. According to the above, we can write the invariant as

$$Inv_3 = \frac{(\boldsymbol{\delta}^T{}_{1234}\boldsymbol{F}\boldsymbol{\epsilon}_{1234})(\boldsymbol{\delta}^T{}_{4526}\boldsymbol{F}\boldsymbol{\epsilon}_{4526})\phi_{1234}\phi_{4526}}{(\boldsymbol{\delta}^T{}_{1245}\boldsymbol{F}\boldsymbol{\epsilon}_{1245})(\boldsymbol{\delta}^T{}_{3426}\boldsymbol{F}\boldsymbol{\epsilon}_{3426})\phi_{1245}\phi_{3426}} \tag{103}$$

where

$$\phi_{pqrs} = (\mathbf{A}'_{pqrs}\cdot\gamma_4)(\mathbf{B}'_{pqrs}\cdot\gamma_4). \tag{104}$$

We see therefore that the ratio of the $\boldsymbol{\delta}^T\boldsymbol{F}\boldsymbol{\epsilon}$ terms which parallels the expression for the invariant in R^4, but uses only the observed coordinates and the estimated fundamental matrix, will not be an invariant. Instead, we need to include the factors ϕ_{1234} etc., which do not cancel. These factors can be found straightforwardly in terms of observable quantities; in Appendix B it is shown that we can write

$$\frac{(\mathbf{A}'_{1234}\cdot\gamma_4)(\mathbf{A}'_{4526}\cdot\gamma_4)}{(\mathbf{A}'_{3426}\cdot\gamma_4)(\mathbf{A}'_{1245}\cdot\gamma_4)} = \frac{\mu_{1245}(\mu_{3426}-1)}{\mu_{4526}(\mu_{1234}-1)} \tag{105}$$

and

$$\frac{(\mathbf{B}'_{1234}\cdot\gamma_4)(\mathbf{B}'_{4526}\cdot\gamma_4)}{(\mathbf{B}'_{3426}\cdot\gamma_4)(\mathbf{B}'_{1245}\cdot\gamma_4)} = \frac{\lambda_{1245}(\lambda_{3426}-1)}{\lambda_{4526}(\lambda_{1234}-1)}. \tag{106}$$

Here, μ and λ are defined by expanding the images points as follows. Since $\boldsymbol{a}'_r$, $\boldsymbol{a}'_s$ and $\boldsymbol{a}'_{pqrs}$ are collinear we can write

$$\boldsymbol{a}'_{pqrs} = \mu_{pqrs}\boldsymbol{a}'_s + (1-\mu_{pqrs})\boldsymbol{a}'_r \tag{107}$$

$$b'_{pqrs} = \lambda_{pqrs} b'_s + (1 - \lambda_{pqrs}) b'_r. \tag{108}$$

While the above has adopted the approach of forming all invariants in 4D and then finding the equivalent expression in 3D, the approach outlined in [?] gave the invariant in the form of equation (??), **but** did indeed *define* $\boldsymbol{\alpha}_{1234}$ as follows:

$$\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 \wedge \boldsymbol{\alpha}_3\boldsymbol{\alpha}_4 \tag{109}$$

where the '$\wedge$' in this equation is the *meet* of the Cayley-Grassmann algebra. Thus, $\boldsymbol{\alpha}_{1234}$ is **not** the homogeneous coordinate vector of the intersection point of the two lines in the image plane joining $\mathbf{A}'_1$ & $\mathbf{A}'_2$ and $\mathbf{A}'_3$ & $\mathbf{A}'_4$, but rather some multiple of that vector, given by equation (??). It can be shown that computing the invariant using equation (??) and the corresponding expressions for the other intersection points, produces exactly those correction factors arrived at by us in equations (??). It is therefore likely that the past confusion over the formation of the invariants has been soley due to the misinterpretation of the nature of the quantities $\boldsymbol{\alpha}_{ijkl}$ and $\boldsymbol{\beta}_{ijkl}$; however, the derivation we have presented here is totally unambiguous and, by clearly distinguishing between 3- and 4D quantities, cannot be misinterpreted. Indeed, in the following section where we discuss invariants involving line coordinates, we will see more clearly how equations (??) and (??) are related.

Thus, to summarize, given the coordinates of a set of 6 corresponding points in the two images planes (where these 6 points are projections from arbitrary world points but with the assumption that they are not coplanar) we can form 3D projective invariants provided we have some estimate of $\boldsymbol{F}$.

4.3 Invariants from lines

Consider again the invariant discussed in the previous section. In terms of world *lines* we can rewrite the invariant of 6 points given in equation (??) as an invariant of 4 lines

$$Inv_3 = \frac{\{(\mathbf{X}_1\wedge\mathbf{X}_3)\wedge(\mathbf{X}_2\wedge\mathbf{X}_4)\}I_4^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_6)\wedge(\mathbf{X}_2\wedge\mathbf{X}_5)\}I_4^{-1}}{\{(\mathbf{X}_1\wedge\mathbf{X}_3)\wedge(\mathbf{X}_2\wedge\mathbf{X}_5)\}I_4^{-1}\{(\mathbf{X}_1\wedge\mathbf{X}_6)\wedge(\mathbf{X}_2\wedge\mathbf{X}_4)\}I_4^{-1}} \tag{110}$$

$$= \frac{\{L_{13}\wedge L_{24}\}I_4^{-1}\{L_{16}\wedge L_{25}\}I_4^{-1}}{\{L_{13}\wedge L_{25}\}I_4^{-1}\{L_{16}\wedge L_{24}\}I_4^{-1}}. \tag{111}$$

As in equation (??) we can express each line in the above expression as the intersection of two planes, e.g.

$$L_{13}\wedge L_{24} = \{\mathbf{A}_0\wedge(L^A{}_{13} \vee L^A{}_{24})\}\wedge\{\mathbf{B}_0\wedge(L^B{}_{13} \vee L^B{}_{24})\} \tag{112}$$

where, as before, $L^A{}_{ij}$ is the line in the A-plane joining the projections of $\mathbf{X}_i$ and $\mathbf{X}_j$. Now, as well as having a basis $\{\mathbf{A}_i\}$ in the A-plane, we can define a *bivector basis* in the A-plane. This bivector basis is L_{Ai}, $i = 1, 2, 3$, given by

$$L_{A1} = \mathbf{A}_2\wedge\mathbf{A}_3 \tag{113}$$

$$L_{A2} = \mathbf{A}_3\wedge\mathbf{A}_1 \tag{114}$$

$$L_{A3} = \mathbf{A}_1\wedge\mathbf{A}_2. \tag{115}$$

Any bivector in the plane can therefore be expanded in terms of these basis bivectors

$$L^A = l_i L_{Ai}. \tag{116}$$

Similarly, we define basis bivectors for the B-plane; $\{L_{Bj}\}, j = 1,2,3. Now, we can expand equation (??) by$ $l_i^{A13} L_{Ai}$, as follows

$$L_{13} \wedge L_{24} = l_i^{A13} l_j^{A24} l_k^{B13} l_n^{B24} \{\mathbf{A}_0 \wedge (L_{Ai} \vee L_{Aj})\} \wedge \{\mathbf{B}_0 \wedge (L_{Bk} \vee L_{Bn})\}. (117)$$

Since $L_{A1} \vee L_{A2} = \mathbf{A}_3$, $L_{A2} \vee L_{A3} = \mathbf{A}_1$, and $L_{A3} \vee L_{A1} = \mathbf{A}_2$ (with $L_{Ai} \vee L_{Ai} = 0$), we see that we can write $L_{Aj} \vee L_{Ak} = 0$ as

$$L_{Aj} \vee L_{Ak} = \epsilon_{ijk} \mathbf{A}_i \tag{118}$$

from which it follows that the intersection of any two lines in the A-plane can be expressed as

$$L_p^A \vee L_q^A = \epsilon_{ijk} l_j^p l_k^q \mathbf{A}_i \equiv \{\boldsymbol{l}_p \boldsymbol{l}_q\}_i \mathbf{A}_i \tag{119}$$

where is the standard vector product and $\boldsymbol{l}_p$ is the vector $[l_1^p, l_2^p, l_3^p]$. We are thus able to write

$$L_{13} \wedge L_{24} = (\boldsymbol{l}^{A13} \boldsymbol{l}^{B24})_i (\boldsymbol{l}^{B13} \boldsymbol{l}^{B24})_j \{\mathbf{A}_0 \wedge \mathbf{A}_i \wedge \mathbf{B}_0 \wedge \mathbf{B}_j\} \equiv \tilde{F}_{ij} (\boldsymbol{l}^{A13} \boldsymbol{l}^{B24})_i (\boldsymbol{l}^{B13} \boldsymbol{l}^{B24})_j. \tag{120}$$

We can cast equation (??) in terms of *observed* quantities using the projective split by rewriting the line coordinates in terms of the point coordinates:

$$\begin{aligned} L^{A13} &= \mathbf{A}_1' \wedge \mathbf{A}_3' = (\alpha_{1i} \mathbf{A}_i) \wedge (\alpha_{3j} \mathbf{A}_j) \\ &= \epsilon_{ijk} \alpha_{1i} \alpha_{3j} L_k^A \\ &= (\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_3)_k L_k^A = \equiv l_k^{A13} L_k^A. \end{aligned} \tag{121}$$

Which tells us that $\boldsymbol{l}^{A13} = (\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_3)$ and therefore that our definition of line coordinates which uses the expansion in terms of the basis bivectors agrees with the conventional idea of homogeneous line coordinates. Now consider the term $\tilde{F}_{ij} (\boldsymbol{l}^{A13} \boldsymbol{l}^{B24})_i (\boldsymbol{l}^{B13} \boldsymbol{l}^{B24})_j$, we can rewrite the line coordinates in terms of points and use the projective split on these points to give

$$\tilde{F}_{ij} (\boldsymbol{l}^{A13} \boldsymbol{l}^{B24})_i (\boldsymbol{l}^{B13} \boldsymbol{l}^{B24})_j =$$

Doing the same things for the other factors enables us to write the above invariant as

Since $\mathbf{A}_{1234}$ and $\mathbf{B}_{1234}$ can be written as the intersection of lines $L_{12}^A \& L_{34}^A$ and $L_{12}^B \& L_{34}^B$ respectively, we are able to write equation (??) as

$$\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 = (\mathbf{A}_0 \wedge \mathbf{A}_{1234}) \wedge (\mathbf{B}_0 \wedge \mathbf{B}_{1234})$$

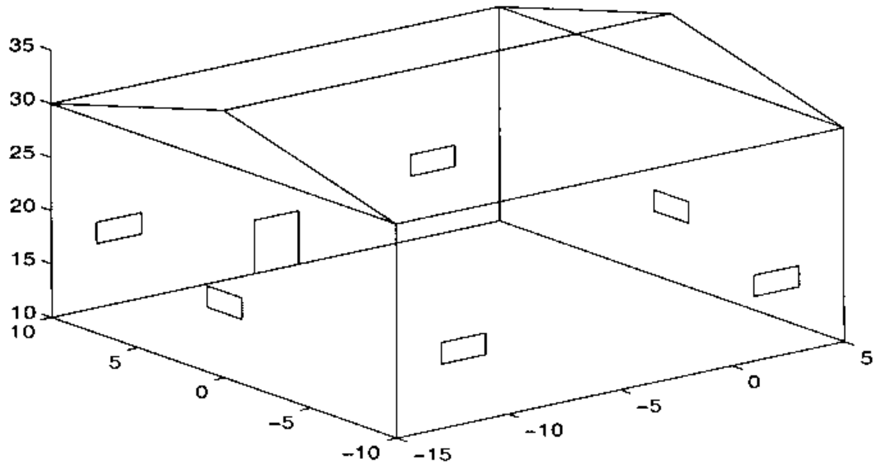

Figure 6: The wire frame house used in the simlations.

5 Experiments

Here we investigate the formation of the 3D projective invariants from sets of 6 matching image points – in particular we look at their stability in noisy environments.

The simulated data was a set of 38 points taken from the vertices of a wire-frame house, as shown in figure ?? and viewed from three different camera positions. From three sets of 6 points (non coplanar) we form $Inv1$ for each set over views 1 & 2, 1 & 3 and 2 & 3. During the simulations the world points are projected onto the image planes and then gaussian noise is added. Figure ?? shows results for the three sets of points chosen. In figure ??, a), c), e) we plot the value of the invariant with increasing noise. In a), c), and e) the invariant was formed using an $\boldsymbol{F}$ calculated via a linear least-squares method from a set of 30 matching points. Figure ?? b), d) and f) show the same invariants formed this time by taking the noisy point matches but the true value of $\boldsymbol{F}$ (i.e. that formed in the noiseless case). The true values of the invariants for the three sets of lines were 0.655, 0.402 and 8.99.

For small values of the noise the invariants can be calculated accurately. In greater noise large variations are possible for some invariants whereas other invariants are relatively robust. Figure ?? indicates that uncertainties in the calculation of $\boldsymbol{F}$ will significantly affect the invariant in some cases. It is also apparent that the formation of this invariant is more accurate between some pairs of views than between others. We should expect this since altering the view may mean that the 6 points move closer to some unstable or degenerate configuration. In summary it appears that the type of invariant described here, when used in isolation, may be useful for data which is not noisy but that the degradation in the presence of significant noise may render them ineffective for real images unless used in some sort of averaging process (i.e. several invariants are considered and an average value formed).

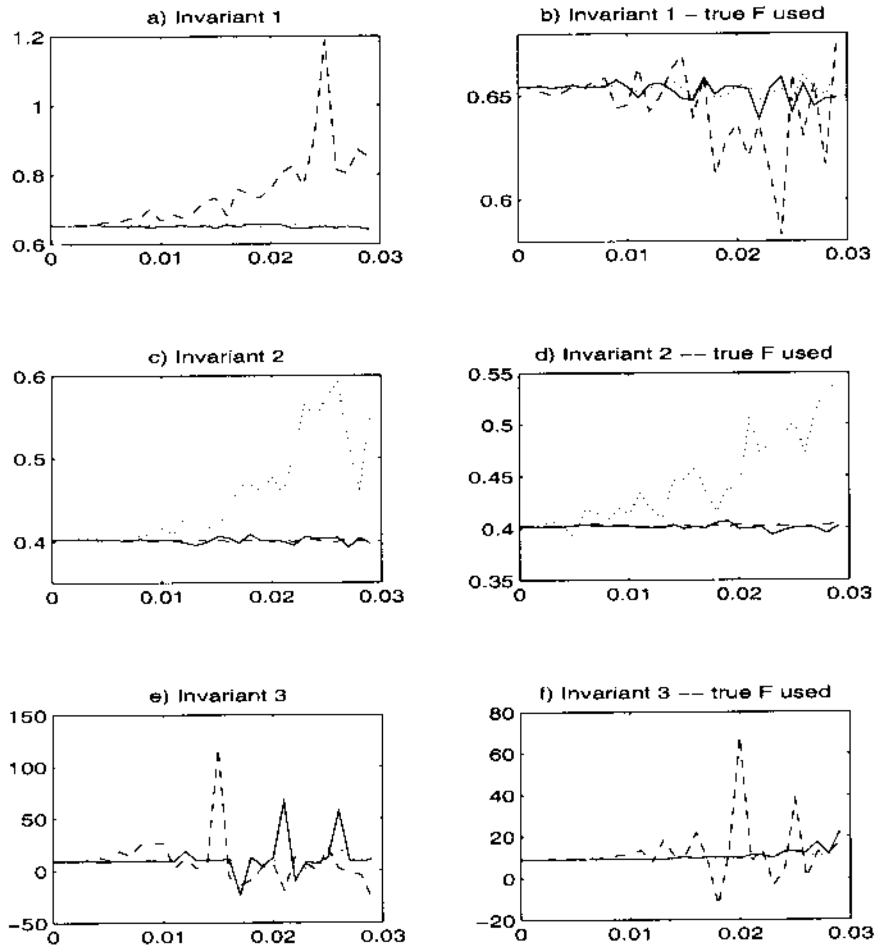

Figure 7: Plots showing the behaviour of the 3D invariant between three different pairs of views with increasing noise. The solid, dotted and dashed lines show the invariant formed between views 1 & 2, 1 & 3 and 2 & 3 respectively. The x-axis shows the standard deviation of the gaussian noise used.

6 Invariants from three views

The technique used to form the 3D projective invariants for 2 views can be straightforwardly extended to give expressions for invariants of 3 views. Consider the scenario shown in figure ??, which shows four world points, $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ (or two lines $\mathbf{X}_1\wedge\mathbf{X}_2$ and $\mathbf{X}_3\wedge\mathbf{X}_4$) projected into three camera planes, where we use the same notation as in section 4.2.2. As before, we can write

$$\begin{aligned}\mathbf{X}_1\wedge\mathbf{X}_2 &= (\mathbf{A}_0\wedge L^A{}_{12}) \vee (\mathbf{B}_0\wedge L^B{}_{12}) \\ \mathbf{X}_3\wedge\mathbf{X}_4 &= (\mathbf{A}_0\wedge L^A{}_{34}) \vee (\mathbf{C}_0\wedge L^C{}_{34}).(125)\end{aligned}$$

Once again, we can extend the above to give an expression for the 4-vector $\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4$: using the result in equation ??

$$\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4 = [(\mathbf{A}_0\wedge L^A{}_{12}) \vee (\mathbf{B}_0\wedge L^B{}_{12})]\wedge[(\mathbf{A}_0\wedge L^A{}_{34}) \vee (\mathbf{C}_0\wedge L^C{}_{34})]. \quad (126)$$

Now, using the result in equation ?? we can write,

$$\begin{aligned}\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4 &= [(\mathbf{A}_0\wedge L^A{}_{12}) \vee (\mathbf{A}_0\wedge L^A{}_{34})]\wedge[(\mathbf{B}_0\wedge L^B{}_{12}) \vee (\mathbf{B}_0\wedge L^C{}_{34})] \\ &= (\mathbf{A}_0\wedge\mathbf{A}_{1234})\wedge(\mathbf{B}_0\wedge L^B{}_{12}) \vee (\mathbf{C}_0\wedge L^C{}_{34}). \quad (127)\end{aligned}$$

where $\mathbf{A}_{1234}$ is as previously defined. Now, writing the lines $L^B{}_{12}$ and $L^C{}_{34}$ in terms of the line coordinates we have

$$\begin{aligned}L^B{}_{12} &= l^B{}_{12,j}L^B{}_j \\ L^C{}_{34} &= l^C{}_{34,j}L^C{}_j.\end{aligned}$$

Now, it has been shown, [?], that the trilinear tensor (which plays the part of the fundamental matrix for 3 views), can be written in geometric algebra as

$$T_{ijk} = \{(\mathbf{A}_0\wedge\mathbf{A}_i)\wedge[(\mathbf{B}_0\wedge L^B{}_j) \vee (\mathbf{C}_0\wedge L^C{}_k) \quad (128)$$

so that equation ?? reduces to

$$\mathbf{X}_1\wedge\mathbf{X}_2\wedge\mathbf{X}_3\wedge\mathbf{X}_4 = T_{ijk}\alpha_{1234}l^B{}_12l^C{}_34. \quad (129)$$

The invariant Inv_3 can then be expressed as

$$Inv_3 = \frac{(T_{ijk}\alpha_{1234}l^B{}_12}{)}T_{ijk}\alpha_{4526}l^B{}_45l^C{}_26(T_{ijk}\alpha_{1245}l^B{}_12l^C{}_45)T_{ijk}\alpha_{3426}l^B{}_34l^C{}_26. \quad (130)$$

Therefore we have an expression for invariants in three views which is a direct extension of the invariant for 2 views. When we form the above invariant from *observed* quantities we note, as before, that some correction factors will be necessary – equation (??) is given above in terms of R^4 quantities. Applications of this 3-view invariant will be given elsewhere.

7 Conclusions

We have presented an introduction to the techniques of geometric algebra and shown how they can be used in the algebra of incidence and in the formation and computation of invariants. For intersections of planes, lines etc. and for the discussion of projective transformations it is useful to work in a 4D space we have called R^4. We find that we do not need to invoke the standard concepts or machinery of classical projective geometry (although they can be used if desired), all that is needed is the idea of the *projective split* relating the quantities in R^4 to quantities in our 3D world. We believe that with this approach we can achieve everything that has currently been achieved with the standard approaches, but that we can do it in a more transparent and clear fashion. For real computations in the space R^4 we have a 4D geometric algebra with a Minkowski metric. We can therefore use the extensive symbolic algebra packages (for use with MAPLE) which have been developed for work in relativity, quantum mechanics and cosmology [?] using the spacetime algebra, also a 4D geometric algebra with a Minkowski metric. Analysing problems using geometric algebra provides the enormous advantage of working in a system which can be used for most areas of computer vision and which has very powerful associated linear algebra and calculus frameworks.

Acknowledgments

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8 Appendix A

We define U_{ij} to be the plane through the centre of projection of the first camera, $\mathbf{A}_0$, and the image points $\mathbf{A}'_i$ and $\mathbf{A}'_j$. Similarly, V_{ij} is defined as the plane through the centre of projection of the second camera, $\mathbf{B}_0$, and the image points $\mathbf{B}'_i$ and $\mathbf{B}'_j$. We can now express each of $\mathbf{X}_1 \wedge \mathbf{X}_2$ and $\mathbf{X}_3 \wedge \mathbf{X}_4$ as the meet of two planes; $U_{12} \vee V_{12}$ and $U_{34} \vee V_{34}$, respectively;

$$\mathbf{X}_1 \wedge \mathbf{X}_2 = (\mathbf{A}_0 \wedge \mathbf{A}'_1 \wedge \mathbf{A}'_2) \vee (\mathbf{B}_0 \wedge \mathbf{B}'_1 \wedge \mathbf{B}'_2) \equiv U_{12} \vee V_{12} \qquad (131)$$

$$\mathbf{X}_3 \wedge \mathbf{X}_4 = (\mathbf{A}_0 \wedge \mathbf{A}'_3 \wedge \mathbf{A}'_4) \vee (\mathbf{B}_0 \wedge \mathbf{B}'_3 \wedge \mathbf{B}'_4) \equiv U_{34} \vee V_{34}. \qquad (132)$$

$S_{1234} \equiv [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4]$ can therefore be written as $S_{1234} = \{(U_{12} \vee V_{12}) \wedge (U_{34} \vee V_{34})\} I_4^{-1}$. From the definition of the meet (noting that the dual is in this case taken with respect to the join which is the whole space) this equation can be simplified to

$$S_{1234} = \{[(U_{12}{}^* \wedge V_{12}{}^*) I_4] \wedge [(U_{34}{}^* \wedge V_{34}{}^*) I_4]\} I_4^{-1}. \qquad (133)$$

Using the fact that I_4 commutes with bivectors in the space and that $I_4{}^2 = 1$, the expression for S_{1234} can then be reduced to

$$S_{1234} = \{U_{12}{}^* \wedge V_{12}{}^* \wedge U_{34}{}^* \wedge V_{34}{}^*\} I_4^{-1}. \qquad (134)$$

The duals of the trivectors U_{ij} and V_{ij} are vectors and since the wedge product is anti-commutative, the order of the vectors in the last equation can be altered – in particular we can write

$$S_{1234} = -\{U_{12}^{*} \wedge U_{34}^{*} \wedge V_{12}^{*} \wedge V_{34}^{*}\} I_4^{-1}. \tag{135}$$

A comparison with equation (??) enables us to express the bracket as

$$S_{1234} = -\{(U_{12} \vee U_{34}) \wedge (V_{12} \vee V_{34})\} I_4^{-1}. \tag{136}$$

9 Appendix B

Since the intersection point $\boldsymbol{a}'_{pqrs}$ is collinear with $\{\boldsymbol{a}_p \& \boldsymbol{a}_q\}$ *and* $\{\boldsymbol{a}_r \& \boldsymbol{a}_s\}$, we can define the following μ's

$$\begin{aligned}
\boldsymbol{a}_{1234} &= \mu_{1234}\boldsymbol{a}'_4 + (1-\mu_{1234})\boldsymbol{a}'_3 \\
\boldsymbol{a}_{1245} &= \mu_{1245}\boldsymbol{a}'_5 + (1-\mu_{1245})\boldsymbol{a}'_4 \\
\boldsymbol{a}_{3426} &= \mu_{3426}\boldsymbol{a}'_3 + (1-\mu_{3426})\boldsymbol{a}'_4 \\
\boldsymbol{a}_{4526} &= \mu_{4526}\boldsymbol{a}'_5 + (1-\mu_{4526})\boldsymbol{a}'_4.
\end{aligned} \tag{137}$$

Also, we can formulate $\mathbf{A}'_{1234}$ as the intersection of the line joining $\mathbf{A}'_1$ and $\mathbf{A}'_2$ with the plane through $\mathbf{A}_0, \mathbf{A}'_3, \mathbf{A}'_4$ as follows;

$$\begin{aligned}
\mathbf{A}'_{1234} &= (\mathbf{A}'_1 \wedge \mathbf{A}'_2) \vee (\mathbf{A}_0 \wedge \mathbf{A}'_3 \wedge \mathbf{A}'_4) \\
&= [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}_0\mathbf{A}'_3]\mathbf{A}'_4 + [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4\mathbf{A}_0]\mathbf{A}'_3 + [\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3\mathbf{A}'_4]\mathbf{A}_0
\end{aligned} \tag{138}$$

The last term on the RHS is zero since $\mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_3$ and $\mathbf{A}'_4$ are coplanar. We can therefore write our intersection point as

$$\mathbf{A}'_{1234} = [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3]\mathbf{A}'_4 - [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4]\mathbf{A}'_3. \tag{139}$$

Via the projective split we know that $\boldsymbol{a}'_{1234}$ and $\mathbf{A}'_{1234}$ are related by

$$\boldsymbol{a}'_{1234} = \frac{\mathbf{A}'_{1234} \wedge \gamma_4}{\mathbf{A}'_{1234} \cdot \gamma_4}. \tag{140}$$

Substituting equation (??) into equation (??) then gives

$$\begin{aligned}
\boldsymbol{a}'_{1234} &= [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3]\frac{\mathbf{A}'_4 \wedge \gamma_4}{\mathbf{A}'_{1234} \cdot \gamma_4} - [\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4]\frac{\mathbf{A}'_3 \wedge \gamma_4}{\mathbf{A}'_{1234} \cdot \gamma_4} \\
&= \frac{\mathbf{A}'_4 \cdot \gamma_4}{\mathbf{A}'_{1234} \cdot \gamma_4}[\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_3]\boldsymbol{a}'_4 - \frac{\mathbf{A}'_3 \cdot \gamma_4}{\mathbf{A}'_{1234} \cdot \gamma_4}[\mathbf{A}_0\mathbf{A}'_1\mathbf{A}'_2\mathbf{A}'_4]\boldsymbol{a}'_3,
\end{aligned} \tag{141}$$

where we have used $\boldsymbol{a}'_i = \frac{\mathbf{A}'_i \wedge \gamma_4}{\mathbf{A}'_i \cdot \gamma_4}$. Equating this expression to equation (??) therefore gives us

$$\mu_{1234} = \frac{(\mathbf{A}'_4 \cdot \gamma_4)}{(\mathbf{A}'_{1234} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_3] \tag{142}$$

and

$$(\mu_{1234} - 1) = \frac{(\mathbf{A}'_3 \cdot \gamma_4)}{(\mathbf{A}'_{1234} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_4]. \tag{143}$$

Similarly we can write

$$\mu_{1245} = \frac{(\mathbf{A}'_5 \cdot \gamma_4)}{(\mathbf{A}'_{1245} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_4] \qquad (\mu_{1245} - 1) = \frac{(\mathbf{A}'_4 \cdot \gamma_4)}{(\mathbf{A}'_{1245} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_1, \mathbf{A}'_2, \mathbf{A}'_5]$$

$$\mu_{3426} = \frac{(\mathbf{A}'_4 \cdot \gamma_4)}{(\mathbf{A}'_{3426} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_2, \mathbf{A}'_6, \mathbf{A}'_3] \qquad (\mu_{3426} - 1) = \frac{(\mathbf{A}'_3 \cdot \gamma_4)}{(\mathbf{A}'_{3426} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_2, \mathbf{A}'_2, \mathbf{A}'_4]$$

$$\mu_{4526} = \frac{(\mathbf{A}'_5 \cdot \gamma_4)}{(\mathbf{A}'_{4526} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_2, \mathbf{A}'_6, \mathbf{A}'_4] \qquad (\mu_{4526} - 1) = \frac{(\mathbf{A}'_4 \cdot \gamma_4)}{(\mathbf{A}'_{4526} \cdot \gamma_4)}[\mathbf{A}_0, \mathbf{A}'_2, \mathbf{A}'_6, \mathbf{A}'_5]$$

Now consider the following combination of the μ's:

$$\begin{aligned} \frac{\mu_{1245}(\mu_{3426} - 1)}{\mu_{4526}(\mu_{1234} - 1)} &= \frac{\frac{(\mathbf{A}'_5 \gamma_4)}{\mathbf{A}_{1245}\gamma_4}[\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_4]\frac{(\mathbf{A}'_3 \gamma_4)}{\mathbf{A}_{3426}\gamma_4}[\mathbf{A}_0, \mathbf{A}_2, \mathbf{A}_6, \mathbf{A}_4]}{\frac{(\mathbf{A}'_5 \gamma_4)}{\mathbf{A}_{4526}\gamma_4}[\mathbf{A}_0, \mathbf{A}_2, \mathbf{A}_6, \mathbf{A}_4]\frac{(\mathbf{A}'_3 \gamma_4)}{\mathbf{A}_{1234}\gamma_4}[\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_4]} \\ &= \frac{(\mathbf{A}'_{1234} \cdot \gamma_4)(\mathbf{A}'_{4526} \cdot \gamma_4)}{(\mathbf{A}'_{3426} \cdot \gamma_4)(\mathbf{A}'_{1245} \cdot \gamma_4)}. \end{aligned} \tag{144}$$

A similar process can be repeated for the second image. This therefore tells us that the factors in equation (??) in terms of observable quantities are as follows:

$$\frac{\phi_{1234}\phi_{4526}}{\phi_{1245}\phi_{3426}} = \frac{\mu_{1245}(\mu_{3426} - 1)}{\mu_{4526}(\mu_{1234} - 1)} \frac{\lambda_{1245}(\lambda_{3426} - 1)}{\lambda_{4526}(\lambda_{1234} - 1)}. \tag{145}$$

Thus we are able to evaluate the scale factors we need. The values of the μ's and λ's can easily be found from equation (??).

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