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A Lie group approach to steerable filters

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Abstract

Recently, Freeman and Adelson (1991) and Simoncelli et al. (1992) published an approach to steer filters in their orientation and scale by Fourier decompositions. We present a generalization of their formalism based on Lie group theory. Within this framework we especially clarify the following points: (1) the possible scope of steerability by Fourier decompositions, (2) approximate steerability with a limited number of basis functions, (3) the singularity that occurs when steering the scale.

Keywords: Steerable filters; Lie groups

1. Introduction

The performance of many filter-based early vision methods can be improved by using the responses of the filters in a continuum of orientations, scales and other parameters. This has been demonstrated by several authors (Freeman and Adelson, 1991; Perona, 1992; Simoncelli et al., 1992; Michaelis and Sommer, 1994). Recently, Freeman and Adelson (1991) introduced steerability to calculate the responses of filters in a continuum of orientations. Although similar concepts were applied by others before, Freeman and Adelson first addressed this problem explicitly and brought it to the attention of the computer vision community. There are several related methods that share a common and well developed mathematical background. Nevertheless, steerability has not yet established as a standard tool in early vision. Therefore, we emphasize to investigate the specific demands of steerability in more detail.

Let $F_\alpha(\boldsymbol{x})$ denote a filter, with $\boldsymbol{x} \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ the parameter of the deformation (translation, rotation, dilation). Steerability refers to the reconstruction of all deformed filters by a small number (M) of so-called *basis functions* $A_k(\boldsymbol{x})$:

$$F_\alpha(\boldsymbol{x}) = \sum_{k=1}^{M} b_k(\alpha)\, A_k(\boldsymbol{x}). \tag{1}$$

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In this article the term 'steerability' is applied to all deformations, not just to rotations. We call the weights $b_k(\alpha)$ *interpolation functions*. If α is viewed as a variable, steerability means an $\alpha, \boldsymbol{x}$-separable decomposition of $F(\alpha, \boldsymbol{x})$. This allows the calculation of the responses to all deformed filters with the computational burden of only M projections $\langle A_k \mid I \rangle, k = 1, \ldots, M$. I denotes the image and $\langle \cdot \mid \cdot \rangle$ the usual scalar product.

Freeman and Adelson (1991) steer orientation using rotated copies of F as basis functions ($A_k = F_{\alpha_k}$). Their interpolation functions are derived by a Fourier decomposition of F with respect to the angular coordinate of a polar representation. Simoncelli et al. (1992) generalized this approach to steerability for other deformations, especially dilations. However, some open questions remain: (1) What is the motivation for the choice of the Fourier basis? (2) What about approximate steerability with a limited number of basis functions? (3) How, in the case of steering scale, is the singularity treated adequately?

Recently Beil (1994) published an approach to steer orientation in which he applies the invariance theory of tensor calculus. The filters and steering equations are synthesized from basic invariant elements. This approach is close to the Fourier decomposition method from its practical implications. Perona (1992) proposed a completely different approach. The basis functions are obtained by a singular value decomposition (SVD). These basis functions are optimal in the sense that they guarantee the best approximation of the deformed filters with a fixed number of basis functions. An evaluation of this approach in comparison to the Lie group method may be found in (Michaelis, 1995).

In this paper, Lie group theory provides a basis for the deeper understanding and generalization of the Fourier decomposition approach of Freeman and Adelson and Simoncelli et al. The eigenfunctions of the generating operator of the deformation Lie group supply the basis functions, while the eigenvalues define the interpolation functions. The basis functions are orthogonal, they are not deformed copies of the filter. The filters can be approximated by a limited number of basis functions. As a particular property of the Lie group approach, the approximated filters are exactly steerable. We are also concerned with the 'scale-singularity' mentioned above.

2. The translational Lie group

The Fourier decomposition method for steering orientation of Freeman and Adelson can be reformulated in a way that allows the generalization to other deformations. For this, we have to consider that rotations can be viewed as translations with respect to the (periodic) angular coordinate φ. The generalization is achieved by a reformulation of steering translations based on Lie group theory.

Let $\{\mathcal{L}_\alpha \mid \alpha \in \mathbb{R}\}$ denote the Lie group of translation operators in x-direction. $F(x) \equiv F_0(x)$ denotes the original filter. The deformation is defined by

$$\mathcal{L}_\alpha F(x) = F_\alpha(x) := F(x + \alpha). \tag{2}$$

The identity operator is $\mathcal{L}_0 = \mathbf{1}$. The group multiplication table f is defined by

$$\mathcal{L}_\beta \mathcal{L}_\alpha F(x) = \mathcal{L}_{f(\beta,\alpha)} F(x) = F(x + \alpha + \beta) \quad \text{with } f(\beta, \alpha) = \beta + \alpha. \tag{3}$$

For very small deformations, $\varepsilon \to 0$, a first-order expansion is possible that defines the generating operator $\hat{\mathcal{L}}$ of the Lie group:

$$\mathcal{L}_\varepsilon \approx \mathbf{1} + \varepsilon \hat{\mathcal{L}}. \tag{4}$$

We calculate the generating operator for translations by a first-order Taylor expansion of the translated function:

$$F(x + \varepsilon) = F(x) + \varepsilon \partial_x F(x) \implies \hat{\mathcal{L}} = \partial_x. \tag{5}$$

The finite operator $\mathcal{L}_\alpha$ can be derived from $\hat{\mathcal{L}}$ by repeating many small deformations by $\varepsilon = \alpha/n$:

$$\mathcal{L}_\alpha = \lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\hat{\mathcal{L}}\right)^n = e^{\alpha\hat{\mathcal{L}}} . \tag{6}$$

The eigenfunctions of $\hat{\mathcal{L}}$ are exponentials, e^{zx}, with $z \in \mathbb{C}$. We can restrict ourselves to the functions e^{jkx} with $k \in \mathbb{R}$ because they form a complete basis for all square integrable functions. An eigenfunction of the generating operator $\hat{\mathcal{L}}$ is also an eigenfunction of the finite translations $\mathcal{L}_\alpha$ but for another eigenvalue. From (6) we conclude:

$$\hat{\mathcal{L}}\, e^{jkx} = jk\, e^{jkx} \implies \mathcal{L}_\alpha\, e^{jkx} = e^{jk\alpha}\, e^{jkx} = e^{jk(x+\alpha)} . \tag{7}$$

By definition, every eigenfunction e^{jkx} is exactly steerable by multiplication with the corresponding eigenvalue $e^{jk\alpha}$. In the language of Lie group theory, exact steerability corresponds to the fact that the eigenfunctions span *invariant subspaces*. All elements of an invariant subspace remain within this subspace under arbitrary deformations $\mathcal{L}_\alpha$. In the language of steerability this implies that a basis of an invariant subspace steers all functions of this subspace without any error. *Irreducible subspaces* are invariant subspaces that, in addition, cannot be divided into smaller invariant subspaces. Therefore, the irreducible subspaces characterize exactly steerable filters. All irreducible subspaces of any one-parameter Lie group are one-dimensional. They are spanned by the eigenfunctions of the generating operators.

2.1. Steerability by Fourier decomposition

Eq. (7) establishes the motivation of the Fourier decomposition approach by Lie group theory. An arbitrary filter $F(x)$ is translated by its decomposition into the eigenfunctions e^{jkx}, i.e. by a Fourier decomposition:

$$\begin{aligned} F(x) &= \sum_{k=-\infty}^{\infty} C_k\, e^{jkx}, \quad \text{with } C_k = \frac{1}{2\pi}\int F(x)\, e^{-jkx}\, dx, \\ F_\alpha(x) &= \sum_{k=-\infty}^{\infty} C_k\, e^{jk\alpha}\, e^{jkx} =: \sum_{k=-\infty}^{\infty} b_k(\alpha) A_k(x). \end{aligned} \tag{8}$$

Comparing (8) with the definition of steerability in (1), it is obvious that the basis functions are the eigenfunctions: $A_k(x) = e^{jkx}$. The interpolation functions are the eigenvalues: $b_k(\alpha) = e^{jk\alpha}$. The weights C_k can be absorbed in either of both functions. In (8) we wrote a sum over k instead of an integral. This is important because in practice we have to restrict the filters and translations to bounded intervals (respectively make them periodic) to avoid a continuum of basis functions. Then, α becomes a periodic parameter, the Lie group becomes compact, and the eigenfunctions are square integrable. More precisely, the basis functions are $A_k(x) = e^{j2\pi kx/L}$, with L the length of the interval. Without loss of generality we assume $L = 2\pi$ in the following.

Freeman and Adelson use deformed filters as basis functions. The derivation of their steering equation, however, is based on the Fourier decomposition in (8). The Lie group approach is the appropriate mathematical framework to motivate their formulas because it allows the generalization to other deformations and it leads to a deeper understanding of the method.

2.2. Optimality of the basis functions

For periodic translations the Lie group basis functions e^{jkx} are the optimal ones with respect to the L^2 distance. 'Optimal' means that there are no better approximations for a fixed number of basis functions. The proof of this statement shows another interesting aspect of the Fourier transform in connection with steerability and related problems.

We define a matrix B that contains all translated and sampled filters:

$$B_{kl} := F_{\alpha_k}(x_l). \tag{9}$$

The filter is assumed to be periodic to avoid a continuum of basis functions. The interval where the filter is sampled is one period. The matrix B then is *circulant*. Hence, it can be diagonalized by a Fourier transform (for a proof see (Hall, 1979, Appendix B)). B is assumed to be square and of size $N \times N$, Λ is a diagonal matrix.

$$B = W\Lambda W^{-1}, \quad \text{with } W_{kl} = N^{-1}\,\mathrm{e}^{\mathrm{j}2\pi kl/N}. \tag{10}$$

For a diagonalizable matrix with orthogonal eigenfunctions the diagonalization is equivalent to a SVD. From the properties of the SVD we know that it is the best possible (row,column)-separable (i.e. $\boldsymbol{x}, \alpha$-separable) decomposition of the matrix in the L^2 sense. Approximating the matrix means approximating all steered filters simultaneously. The continuous case is approximated to an arbitrary precision for large N.

2.3. Approximate steerability

The computational effort is reduced if a limited number of basis functions is used. Which basis functions should be chosen? The basis functions are orthogonal and hence, their L^2 norms sum up to the norm of F_α. Therefore, the L^2-norm of a particular basis function reflects its importance for approximating the filters. From (8) the L^2 norm of the basis functions clearly is $||A_k|| = C_k$.

The basis functions can be obtained also by $A_k(x) = \frac{1}{2\pi}\int F(x+\alpha)\,\mathrm{e}^{-\mathrm{j}k\alpha}\,\mathrm{d}\alpha$. Calculating the norm of the basis functions using this expression yields an interesting result.

$$\begin{aligned} ||A_k||^2 &= \frac{1}{4\pi^2}\int\int F(x+\alpha)\,\mathrm{e}^{-\mathrm{j}k\alpha}\,\mathrm{d}\alpha \int F^*(x+\alpha')\,\mathrm{e}^{\mathrm{j}k\alpha'}\,\mathrm{d}\alpha'\,\mathrm{d}x \\ &\overset{(a)}{=} \frac{1}{4\pi^2}\int \mathrm{d}\alpha \int\int F(z)F^*(z+\beta)\,\mathrm{d}z\;\mathrm{e}^{\mathrm{j}k\beta}\,\mathrm{d}\beta = \frac{1}{2\pi}\int \langle F_\beta \mid F_0\rangle\,\mathrm{e}^{\mathrm{j}k\beta}\,\mathrm{d}\beta. \end{aligned} \tag{11}$$

In (a) we changed the order of integration and we applied the substitutions $z = x + \alpha$ and $\beta = \alpha' - \alpha$. The integrand does only depend on $\alpha' - \alpha$ and hence, we can factor out the α integration that merely gives a factor of 2π. Again, we assume the Lie group compact and the filter periodic.

As the result, the L^2 norm of A_k is given by the kth Fourier coefficient of the autocorrelation function $h(\alpha)$ of F:

$$h(\alpha) := \langle F_\alpha \mid F_0\rangle = \int F^*(x+\alpha)F(x)\,\mathrm{d}x. \tag{12}$$

The same result has been derived by Perona (1992) using the SVD. In wavelet theory $h(\alpha)$ is called the reproducing kernel which governs the sampling scheme for complete wavelet bases (Antoine et al., 1993).

3. Canonical parametrization and coordinates

The steering scheme for translations gains strength for the following reason: By appropriate transformations of the coordinates and the parameter, all one-parameter Lie groups are isomorphic to the translational group (Cohen, 1931; Hall, 1967). A one-parameter Lie group is defined by its elements $\mathcal{L}_\tau$ and the group multiplication table $f(\tau', \tau)$:

$$\mathcal{L}_{\tau'}\mathcal{L}_\tau = \mathcal{L}_{f(\tau',\tau)} \tag{13}$$

with $f(\tau', \tau) \neq \tau' + \tau$ in general. There exists a neutral element τ_0 for which we have $\mathcal{L}_{\tau_0} = \mathbf{1}$ and $f(\tau_0, \tau) = f(\tau, \tau_0) = \tau$. The generating operator $\hat{\mathcal{L}}$ is defined by

$$\hat{\mathcal{L}} := \left.\frac{\mathrm{d}\mathcal{L}_\tau}{\mathrm{d}\tau}\right|_{\tau_0}. \tag{14}$$

Here, $|_{\tau_0}$ means that the expression to the left is evaluated at τ_0. The action of the group on the coordinates and filters is (here in 2D) $x' = x'(x, y, \tau) = \mathcal{L}_\tau x$, $y' = y'(x, y, \tau) = \mathcal{L}_\tau y$, and $(\mathcal{L}_\tau F)(x, y) = F(x', y')$.

The representation of the group is considerably simplified if the *canonical parametrization* is introduced, denote by $\alpha = \alpha(\tau)$. It is defined by the following property of its group table:

$$\left.\frac{\partial f(\alpha, \alpha')}{\partial \alpha}\right|_{\alpha_0} = 1. \tag{15}$$

This is especially true for the canonical choice

$$f(\alpha, \alpha') = \alpha + \alpha' \quad \text{and} \quad \alpha_0 = 0. \tag{16}$$

A canonical parametrization is always possible (at least for all reasonable continuous deformations). For the canonical parametrization the following simple representations of the generating operator $\hat{\mathcal{L}}$ and the group elements $\mathcal{L}_\alpha$ hold:

$$\hat{\mathcal{L}} = \left.\frac{\mathrm{d}\mathcal{L}_\alpha x}{\mathrm{d}\alpha}\right|_{\alpha=0} \partial_x + \left.\frac{\mathrm{d}\mathcal{L}_\alpha y}{\mathrm{d}\alpha}\right|_{\alpha=0} \partial_y = \xi(x, y)\, \partial_x + \eta(x, y)\, \partial_y, \tag{17}$$

$$\mathcal{L}_\alpha = \mathrm{e}^{\alpha \hat{\mathcal{L}}}. \tag{18}$$

In addition, there always exist curvilinear *canonical coordinates* u, v that make the representation of the group especially simple.

$$u' = \mathcal{L}_\alpha u = u + \alpha, \qquad v' = \mathcal{L}_\alpha v = v, \qquad \hat{\mathcal{L}} = \partial_u. \tag{19}$$

In canonical coordinates and parametrization, the deformation is a translation. u and v are orthogonal coordinates, i.e. all lines u = const and v = const are locally orthogonal. The calculation of the canonical parameters and coordinates needs the solution of the partial differential equations that are derived from (15) and (19).

3.1. Steering general one-parameter deformations

For any one-parameter deformation, a steering equation can be obtained by transforming the problem to canonical parametrization and coordinates. Then, the formalism for steering translations is applied. The basis functions are derived from the eigenfunctions, $\mathrm{e}^{\mathrm{j}ku}$, of the generating operator: $A_k(u, v) = C_k(v)\, \mathrm{e}^{\mathrm{j}ku}$. The coefficients are $C_k(v) = \int F(u, v)\, \mathrm{e}^{-\mathrm{j}ku}\, m(u, v)\, \mathrm{d}u\, \mathrm{d}v$, where $m(u, v)\, \mathrm{d}u\, \mathrm{d}v = \mathrm{d}x\, \mathrm{d}y$ is the integration measure. The interpolation functions are $b_k(\alpha) = \mathrm{e}^{\mathrm{j}k\alpha}$ for all filters.

Depending on the definition of the deformation, the integration measure, and the periodicity or non-periodicity of the transformation, the details of the steering scheme differ slightly. We can most easily explain this for an example in Section 4.

3.2. Exact steerability

For approximations of the filter those basis functions with the largest norm are used (Section 2.3). We now calculate the L^2 approximation error. The Lie group basis functions are complete and orthogonal. Therefore, by using an infinite number of basis functions the reconstruction of the filter is exact: $F_\alpha = \sum_{k=-\infty}^{\infty} e^{jk\alpha} A_k$. For a finite number of basis functions the L^2 error is

$$||F_\alpha - \sum_{k=0}^{M} e^{jk\alpha} A_k||^2 = || \sum_{k=M+1}^{\infty} e^{jk\alpha} A_k||^2 = \sum_{k=M+1}^{\infty} ||A_k||^2. \tag{20}$$

The reconstruction error is obtained by adding the norm of the missing basis functions. It is independent of α. All deformed filters are reconstructed with the same error. The approximated filter itself, $F_\alpha^{\text{app}} := \sum_{k=0}^{M} e^{jk\alpha} A_k$, is exactly steerable. This is a special property of the Lie group basis functions.

4. Example: Steering scale

The example of steering the scale explains the application of the canonical Lie group formalism and some additional twists. Let $F(r, \varphi)$ be an N-dimensional (N-D) function in polar representation. The variable φ denotes the angular components that are omitted if no use is made of them. We define the scaling operator in N-D to be

$$\mathcal{L}_\alpha(F(r)) := e^{-\alpha N/2} F(e^{-\alpha} r) = F_\alpha(r). \tag{21}$$

The parameter α is the canonical scaling parameter: $\mathcal{L}_\alpha \mathcal{L}_{\alpha'} = \mathcal{L}_{\alpha+\alpha'}, \mathcal{L}_0 = \mathbf{1}$. We obtain the generating operator $\hat{\mathcal{L}}$ by a first-order Taylor approximation of $\mathcal{L}_\alpha F$:

$$e^{-\alpha N/2} F(e^{-\alpha} r) = F(r) - \alpha(\tfrac{1}{2}N + r\partial_r)F(r) + o(\alpha^2). \tag{22}$$

Hence, the generating operator for scaling in polar coordinates is:

$$-\hat{\mathcal{L}} = \tfrac{1}{2}N + r\partial_r = \tfrac{1}{2}N + \partial_t. \tag{23}$$

Comparing (23) and (19) we see that $t = \ln r$ is not the true canonical variable. The additional term $N/2$ stems from the normalization factor $e^{-\alpha N/2}$ in (21). However, it is convenient to use the coordinates (t, φ) in the following. In t-space the dilation/contraction becomes a simple shift (beside the normalization factor $e^{-\alpha N/2}$): $\mathcal{L}_\alpha F(r) = e^{-\alpha N/2} F(e^{-\alpha} r) = e^{-\alpha N/2} F(e^{t-\alpha})$.

The eigenvalue equation $(\frac{1}{2}N + r\partial_r)E_z(r) = zE_z(r)$ with the complex eigenvalue $z = a + jk$ is a simple differential equation with separated variables. The solution is $E_z(r) = C_z r^{z-N/2} = C_z r^{a-N/2} e^{jk \ln r}$ with C_z as an integration constant.

What is the appropriate choice for the parameters a and k, so that the set of eigenfunctions is complete and orthogonal. It turns out that this is the case for $a = 0$. A systematic derivation of this result may be found in (Michaelis, 1995). It is based on the property of hermitean operators to have complete and orthogonal eigenbases with *real* eigenvalues. The complete and orthogonal eigenfunctions are

$$E_k(r) = C_k r^{-N/2} e^{jk \ln r}. \tag{24}$$

An example of an eigenfunction for $N = 1$ and $k = 3$ is depicted in Fig. 2. The orthogonality (25) and completeness (26) of the eigenfunctions is stated by the following formulas (the angular integration merely gives a constant factor that is omitted):

$$\frac{1}{2\pi} \int_0^\infty r^{-N} e^{-jk \ln r} e^{+jk' \ln r} r^{N-1} \, dr = \delta(k - k'), \tag{25}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(rr')^{-N/2}\,\mathrm{e}^{-\mathrm{j}k\ln r'}\,\mathrm{e}^{+\mathrm{j}k\ln r}\,\mathrm{d}k = r^{-N+1}\delta(r-r') = \mathrm{e}^{-Nt}\,\delta(t-t'), \quad r,r'>0. \tag{26}$$

With the substitution $t = \ln r$ these equations are the ordinary orthogonality and completeness relations for complex exponentials. In (26) we applied the formula $\delta(\ln r - \ln r') = r'\delta(r-r')$. The factor r^{-N+1} in (26) compensates the factor r^{N-1} of the N-D integration measure. It should be emphasized that the optimality proof of Section 2.2 is not valid for dilations. In fact, significantly more basis functions are needed than for the SVD approach (Michaelis, 1995).

4.1. Steering equation and treating the singularity

According to the steering scheme from Section 3.1, the radial part of the basis functions as well as the interpolation functions are the same for all filters. We only have to calculate the angular part $C_k(\varphi)$ of the polar separable basis functions to obtain the steering equation. It is calculated by projecting the filter F to the orthogonal eigenfunctions.

We restrict the projection integrals to finite intervals $[t_{\min}, t_{\max}[$ or $[r_{\min}, r_{\max}[$, with $r_{\min} := \mathrm{e}^{t_{\min}}$ and $r_{\max} := \mathrm{e}^{t_{\max}}$. In Section 2.1 we pointed out that in practice the filter and the deformation are restricted to finite intervals to avoid a continuum of basis functions.

$$C_k(\varphi) := \int_{r_{\min}}^{r_{\max}} F(r,\varphi)\,r^{-N/2}\,\mathrm{e}^{-\mathrm{j}2\pi k(\ln r - t_{\min})/\Delta t}\,r^{N-1}\,\mathrm{d}r = \int_{t_{\min}}^{t_{\max}} \widetilde{F}(t,\varphi)\,\mathrm{e}^{-\mathrm{j}2\pi k(t-t_{\min})/\Delta t}\,\mathrm{d}t \tag{27}$$

The function $\widetilde{F}$ is defined as $\widetilde{F}(t,\varphi) := F(\mathrm{e}^t,\varphi)\,\mathrm{e}^{Nt/2}$. A scaling of $F(r)$ is a translation of $\widetilde{F}(t)$: $\widetilde{F}_\alpha(t) = \widetilde{F}(t-\alpha)$. (27) is an ordinary Fourier transform of $\widetilde{F}$. Although the eigenfunctions are singular for $r \to 0$ ($t \to -\infty$), the projection integral (27) is not. This is due to the exponential decay of $\widetilde{F}(t)$ for $t \to -\infty$ that stems from the integration measure. Therefore, we avoid the arbitrary modifications of the logarithm near the origin that are used in (Simoncelli et al., 1992) to regularize the singularity. *The singularity of the eigenfunctions is not worse than the singularity of ordinary complex exponentials.*

The filter is made periodic in t-space with the period $\Delta a = \Delta t := t_{\max} - t_{\min}$: $\sum_{n=-\infty}^{\infty} F_{\alpha+n\Delta a}$ (Fig. 1). Δt is chosen so that it covers the filter and all scaled versions (shifts in t-space) within the desired range of scales. The length of the interval is not critical but from Fourier theory it is clear that more basis functions are needed

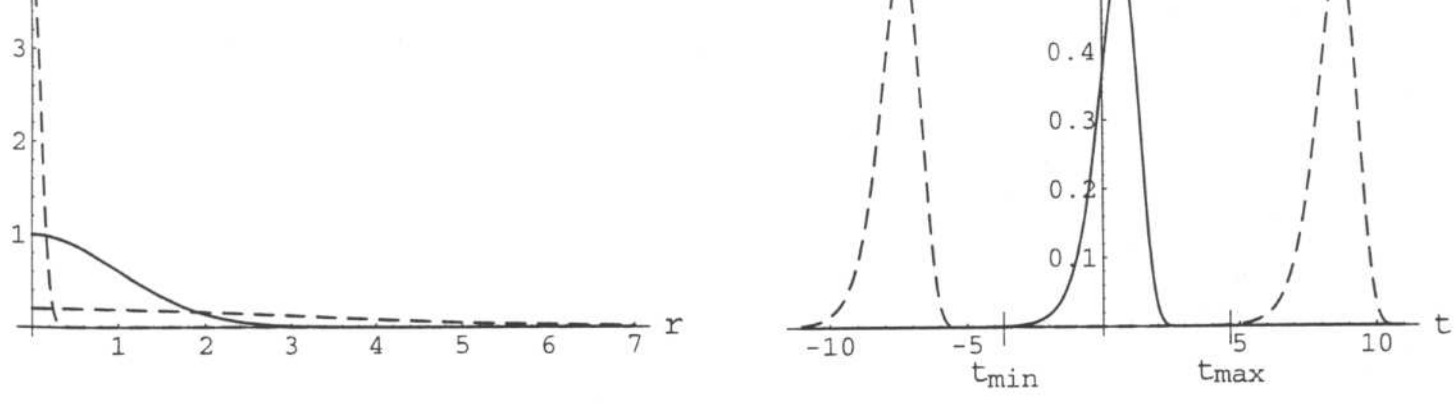

Fig. 1. Periodic filter in r-space (left) and t-space (right). In r-space it is a Gaussian ($F(r) = \mathrm{e}^{-r^2}$), in t-space it is the corresponding filter $\widetilde{F}(t) = \exp(\mathrm{e}^{-2t} - t)$. The solid line is the original filter, the broken lines are the periods that are added to avoid a continuum of basis functions.

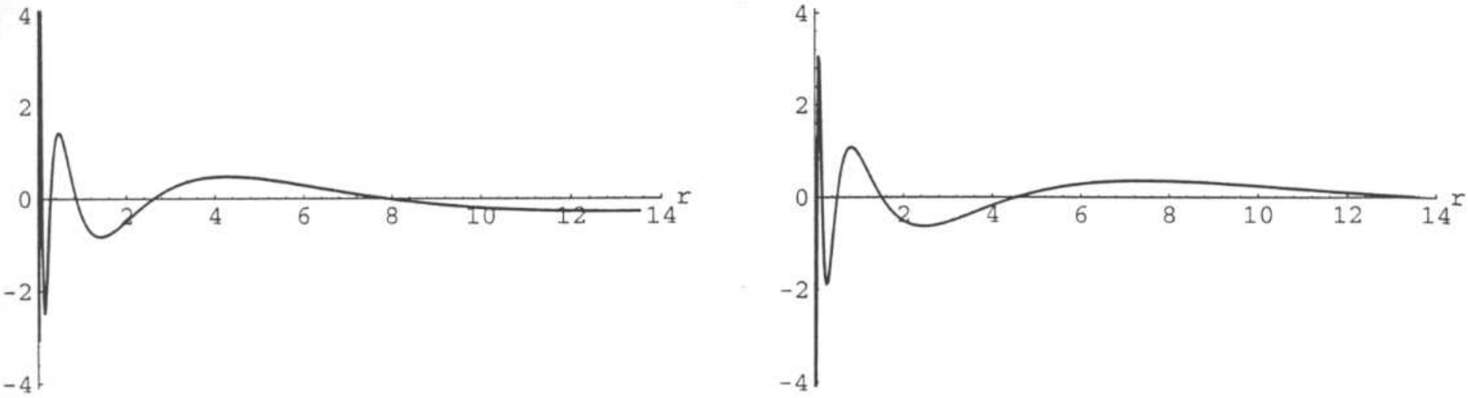

Fig. 2. Depicted is a 1-D basis functions for steering the scale (real left, imaginary right), with $k = 3$, $t_{\min} = -5$, $t_{\max} = 2.6$.

for larger intervals, e.g. if a larger range of scales is steered. The amplitude of F_α decreases fast with increasing scale because the filter is L^2-normalized for all scales. Hence, if Δa is chosen large enough the filter F_α is not changed significantly by the next period $F_{\alpha+\Delta a}$. The next smaller period $F_{\alpha-\Delta a}$ has no influence either in the discrete case because it vanishes between the origin and the first pixel.

Altogether, the N-D basis functions for steering scale are

$$A_k(r,\varphi) = C_k(\varphi) r^{-N/2} e^{j2\pi k(t-t_{\min})/\Delta t}, \quad t = \ln r \tag{28}$$

According to the above scheme, the origin plays a special role: For $r = 0$ all basis functions are defined to be zero, except one that is defined by $A_k(0,\varphi) = e^{-\alpha N/2} F(0,\varphi)$. This basis function can be chosen arbitrarily but it is most convenient to choose the first (most important) basis function.

The completeness of the eigenfunctions guarantees that the filter is reconstructed by summing up all A_k. Together with the trivial steerability of the A_k we obtain the steering equation:

$$F_\alpha(x) = \sum_k e^{jk\alpha} A_k(x). \tag{29}$$

Examples for the basis functions are shown in Fig. 2 for 1-D and in Fig. 3 for 2-D. The depicted basis functions for the 2-D case do not look polar separable because (28) is complex polar separable. Therefore, the real and imaginary part are sums of two real polar separable functions. However, the real polar separable basis functions could be used just as well. Even though the basis functions are infinitely supported, they can be truncated to the size of the largest steered filter in practice.

5. Steering multiple-parameter deformations

The canonical basis functions for steering the scale (28) are polar separable. The variable φ is the canonical coordinate for steering the orientation. Therefore, it is easy to steer scale and orientation simultaneously by the one-parameter formalism. The basis functions are $A_{kl}(r,\varphi) = C_{kl}\, e^{jl\varphi}\, r^{-N/2}\, e^{jkt}$. The coefficients C_{kl} are calculated by $C_{kl} = \int\int \widetilde{F}(t,\varphi)\, e^{-jkt}\, e^{-jl\varphi}\, dt\, d\varphi$ (see (28) and (27) for details). The interpolation functions are $b_{kl}(\theta,\alpha) = e^{j(l\theta+k\alpha)}$. This holds in general for a set of mutually commuting deformations because the generating operators then have a common set of eigenfunctions.

Perona (1992) also steers scale and orientation. He first steers the orientation by a Fourier decomposition, subsequently the scale of all basis functions is steered. With the presented formalism we achieve a symmetry between scale and orientation in the steering formalism.

In general, the generating operators of Lie groups do not commute. This is the case e.g. for the 3-D rotation group $SO(3)$ or for rotations and translation. The latter example demonstrates another twist. New deformations

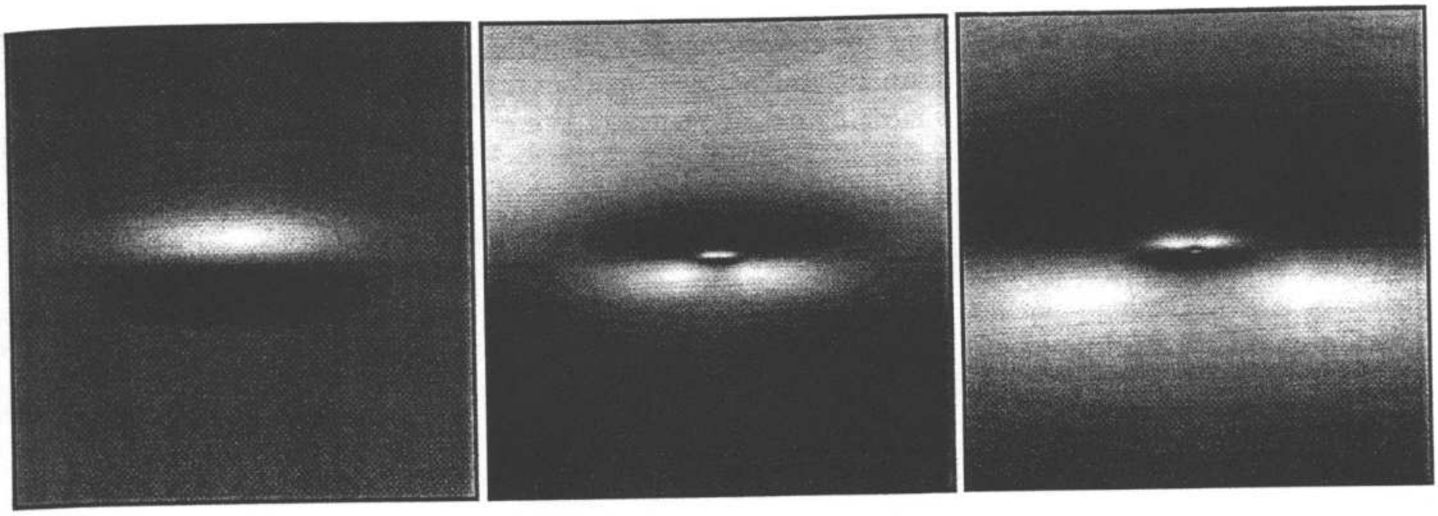

Fig. 3. Example of a 2-D basis functions for steering the scale (real middle, imaginary right, $k = 2$). The filter (left) is a first derivative of Gaussian times Gaussian with an aspect ratio of 3. The basis functions are displayed without the strong radial decay.

have to be added to make the group of deformations closed. The two generators for rotations and x-translations for example do not commute: $[\hat{\mathcal{L}}^x, \hat{\mathcal{L}}^r] = \hat{\mathcal{L}}^y$. The y-translations must be added what is evident because rotations mix x- and y-translations.

For those general closed groups of deformations we cannot apply the full one-parameter scheme. However, the following remains the same: A characterization of the exactly steerable filters and a steering equation is obtained by the canonical basis functions of the irreducible representations of the group. In case of $SO(3)$ these are the spherical harmonics Y_l^m, which are used by Freeman and Adelson (1991) to steer 3-D functions in orientation. The spherical harmonics and the canonical basis functions in general (for unitary representations) are orthogonal and hence it is easy to calculate the interpolation functions. If A_k are the orthogonal basis functions the interpolation functions b_k are calculated by $b_k(\alpha) = \langle F_\alpha \mid A_k \rangle$.

For the approximate steerability with a limited number of basis functions, the number of basis functions is not increased one by one. Complete irreducible subspaces are added instead. In case of one-parameter deformations, the irreducible subspaces are 1D. In general, the irreducible subspaces are multi dimensional or even infinite dimensional. In the latter case, no filter is exactly steerable by a finite number of basis functions for these deformations. For these cases, the SVD approach to steerability (Perona, 1992; Michaelis, 1995) is definitely more appropriate.

6. Discussion

We applied the Lie group formalism to steerable filters. This approach provides a theoretical basis and a generalization of the steering scheme by Freeman and Adelson (1991) and Simoncelli et al. (1992). Our generalization allows one formalism to be applied to all one-parameter continuous deformations as well as those multiple-parameter deformations for which the generating operators commute. A special property of the Lie group approach is the exact steerability of the approximated filters. The method does not provide the most parsimonious set of basis functions, except for (periodic) translations and rotations.

The canonical Lie group basis functions are not deformed copies of the filter. However, steerability is a property of the span of the basis functions rather than of the functions themselves. All linear independent functions within this span, including deformed filters, can serve as basis functions as well. We usually prefer the canonical Lie group basis functions. They are orthogonal and they reconstruct all deformed filters simultaneously with the same error (Section 3.2). Nevertheless, Simoncelli et al. (1992) use deformed filters. They argue that the projections coefficients of the basis functions have to support some degree of interpretability and invariance. For the same reason they argue for an over-complete sampling of the deformation parameter. These requirements hamper the choice of the most convenient basis functions for a given task. From our point of view the basis functions themselves have no meaning or interpretation. Interpretation and invariance are delayed to the reconstruction level of the filter responses.

The problem of steerable filters received explicit attention only for the last few years. Nevertheless, the mathematical principles of steerability are well known from several other methods. Steerability has many features in common with sampling and interpolation, wavelets, invariants, and principal component analysis (Michaelis, 1995). The log-polar canonical coordinates for steering scale and orientation, are well known from the Mellin transform and from conformal mappings. The Lie group formalism has been applied also by Lenz (1990) who focused on invariant pattern recognition but did not explicitly address the problem of steerability. Ferraro and Caelli (1988) applied Lie group theory to invariant integral transform representations of images. Their transform kernels are identical to our basis functions. However, they pursue a different goal. In invariance theory, the projection coefficients of the kernels are of interest for themselves as invariant representations. For steerability, these coefficients are merely a tool to calculate the filter responses. However, both applications are strongly related and may benefit from another.

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