Projective Model for Central Catadioptric Cameras using Clifford Algebra*

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Abstract. A new method for describing the equivalence of catadioptric and stereographic projections is presented. This method produces a simple projection usable in all central catadioptric systems. A projective model for the sphere is constructed in such a way that it allows the effective use of Clifford algebra in the description of the geometrical entities on the spherical surface.

1 Introduction

Catadioptric cameras allow for a very large field of vision. This, in comparison to pinhole cameras, enables the system to perceive more visual information with one single image. The non-Euclidean geometry of the image enables more efficient self-calibration of the camera and reduces the complexity of algorithms needed to complete this task [5].

The mathematics used to model catadioptric cameras is slightly more complicated than for pinhole cameras. The main problem in the application of Clifford algebra to this modeling task is the local nature of the vector space structure on a curved manifold. This problem is solved in the following sections for central (single viewpoint) catadioptric systems, i.e. cameras with mirrors whose cross-sections are conic sections [1]. A projective model for parabolic, hyperbolic and elliptic mirrors is constructed taking the sphere as the unifying geometry. This model allows us to develop mathematical tools using Clifford algebra that are applicable to all these mirror geometries and works as a basis for our future research.

Clifford algebra has proven to be a powerfull tool in 2D-3D pose estimation (for example in [11],[12]). Using the model presented in this paper we hope these benefits gained in the Euclidean case of pinhole cameras will also be available in the omnidirectional vision using catadioptric cameras.

2 Unified mirror geometries

In [5] Geyer and Daniilidis present a unified model for single viewpoint catadioptric systems. In this model the world is first projected to the surface of a sphere

* This work has been supported by DFG grant So-320/2-3

with projective lines emerging from the center of the sphere. Stereographic projection from this spherical surface corresponds to the orthogonal projection from a parabolic mirror. Moving the projection point from the north pole of the sphere one may present perspective projections from the surfaces of elliptical and hyberbolical mirrors. Following the elegant description for the equivalence of the stereographic projection and orthogonal projection from a parabola by Penrose and Rindler [9], the unified model for single viewpoint catadioptric systems is reconstructed using a different mathematical method. This leads to simple projections for the different mirror geometries with a clear correspondence to the points on the sphere.

2.1 Modified Stereographic Projection

The stereographic projection is a one-to-one mapping between a sphere and a plane. Usually the sphere is defined along with the concept of ball:

Definition 1. *A n-ball of radius r centered at the origin is the set $B(0;r) = \{x \in \mathbb{R}^{n+1} \mid x^2 \leq r^2\}$.*

The surface $\mathbb{S}^2 = \{x \in \mathbb{R}^{n+1} \mid x^2 = 1\}$ of the unit 2-ball, is called the sphere.

Instead of using this more common concept of sphere as a subset of $\mathbb{R}^3$ the sphere is now formed in the 4-dimensional Minkowski space $\mathbb{R}^{3,1}$, i.e. vector space with the signature (-,+,+,+). This is done in order to stay consistent with the reference [9] and it offers the possibility to induce movement of points on the sphere by using Lorentz transformations which are known to be locally angle preserving.

The vectors $\mathbf{x} \in \mathbb{R}^{3,1}$ with $\mathbf{x}^2 = 0$ form a cone called the null cone. Let the vectors in $\mathbb{R}^{3,1}$ have the coordinates (t,x,y,z). The intersection of the null cone and the plane $t = 1$ forms a sphere. In stereographic projection a point $P(1, x, y, z)$ on this surface is projected to a plane T with $z = 0$ and $t = 1$ (see figure 1). The projective line is the line passing thru the north pole N and the point P. The intersection of this line and the plane T gives the coordinates of the projected point. To avoid inconsistencies in the projection of the point N the plane T has to be complex. This also enables the description of the projected point with just two parameters. Point A in figure 1 corresponds to the complex number $x + iy$. The x and y coordinates tell the position of the point P' in the complex plane and this is described by the complex number $\zeta = x' + iy'$. As the phase angle of the complex number $\zeta = x' + iy'$ is the azimuthal angle of the point (P' has the same direction from point C as point A) $P(1, x, y, z)$ on the sphere one has

$$A = hP' \quad \text{i.e} \quad x + iy = h\zeta, \tag{1}$$

where h is a real coefficient. The value of h is by geometric deduction (see figure 1)

$$h = \frac{CA}{CP'} = \frac{NP}{NP'} = \frac{NB}{NC} = 1 - z. \tag{2}$$

Using spherical coordinates ($0 \leq \phi \leq 2\pi, 0 < \theta, \pi$) to parameterise the sphere one gets

$$\zeta = \frac{x+iy}{1-z} = e^{i\phi} \cot \frac{\theta}{2}. \tag{3}$$

As in the model by Geyer and Daniilidis the connection of different mirror geometries and the sphere is achieved by the movement of the projection point N. We start by moving the projection point N along the z direction which changes equation (2) to

$$h = \frac{CA}{CP'} = \frac{NP}{NP'} = \frac{NB}{NC} = \frac{\beta - z}{\beta} = 1 - \beta^{-1} z, \tag{4}$$

and equation (3) to

$$\zeta = \frac{x+iy}{1-\alpha z} = e^{i\phi} \frac{\sin\theta}{1-\alpha\cos\theta}, \text{ where } \alpha = \beta^{-1}. \tag{5}$$

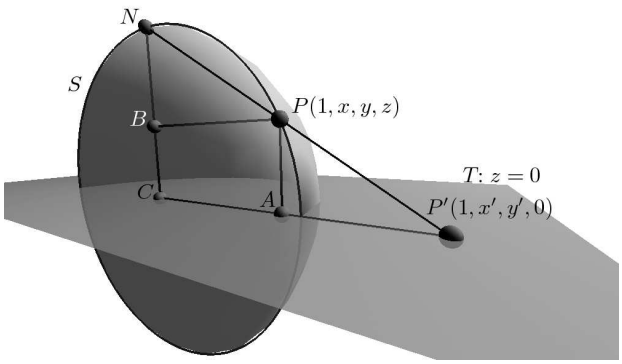

Fig. 1. Stereographic projection from sphere S to plane T. Only half of the Sphere S is drawn.

2.2 Connection to Conic Sections

This movement of the projection point is related to different conic sections in the following way. Let a null cone in $\mathbb{R}^{3,1}$ be intersected by the plane $t - z = 1$. This intersection forms a parabola. Let Q be a point of intersection of that plane and a line from the vertex of the cone to the point P given by $\mathbf{q} = u\mathbf{p}$, where $0 \leq u \leq 1$, $\mathbf{q}$ is the vector pointing at the point Q and $\mathbf{p}$ is the vector pointing at P (this is illustrated in the right part of figure 2). Solving the intersection of

the line defined by $\mathbf{p}$ and the plane $t - z = 1$ gives $u = \frac{1}{1-z}$. Thus point Q has the coordinates

$$Q = \left(\frac{1}{1-z}, \frac{x}{1-z}, \frac{y}{1-z}, \frac{z}{1-z} \right), \tag{6}$$

from which the coordinates in the $x - y$-plane given by orthogonal projection are

$$P'(X', Y') = \left(\frac{x}{1-z}, \frac{y}{1-z} \right). \tag{7}$$

Labeling the points in the $(x - y)$-plane with complex numbers the point Q is projected to a point $\zeta = \frac{x+iy}{1-z}$ as in (3). This equivalency of the stereographic projection from a sphere and the orthogonal projection from a parabola can be shown by intersecting planes. Let plane $t = 1$ intersect the null cone with vertex O. This intersection is the spherical surface $\mathbb{S}^2$. Let the north pole N of the sphere be at $(1, 0, 0, 1)$ and point Q be the intersection of the null line from O to P and the plane $t - z = 1$. The points O, Q, P, P' and N are coplanar and the points P, P' and N are collinear [9]. Thus the point P' is also the stereographic projection from the sphere $\mathcal{S}$ to the $(x - y)$-plane (see figure 2 representing the situation in one dimensional case).

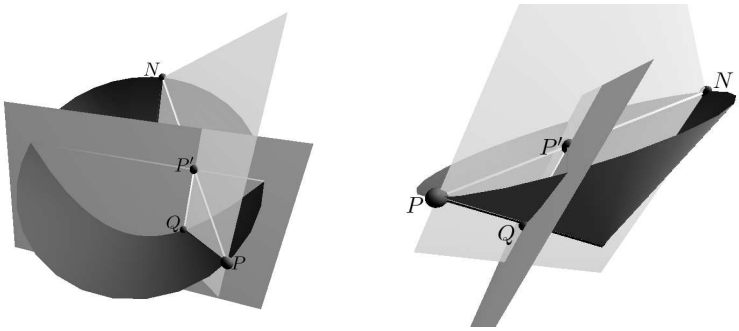

Fig. 2. Orthogonal projection from parabola and stereographic projection from circle. The parabola is formed by the intersection of the cone and the (non-transparent) $t - z = 1$ plane.

Tilting the $t - z = 1$ plane to the plane $t - \alpha z = 1$ changes the coordinates of Q to

$$Q = \left(\frac{1}{1-\alpha z}, \frac{x}{1-\alpha z}, \frac{y}{1-\alpha z}, \frac{z}{1-\alpha z} \right), \tag{8}$$

and the coordinates of the projected point to

$$P'(X', Y') = \left(\frac{x}{1-\alpha z}, \frac{y}{1-\alpha z} \right), \tag{9}$$

where α is the eccentricity of the conic section. Exactly as in (1) the projected point has the coordinates

$$\zeta = \frac{x + iy}{1 - \alpha z}. \tag{10}$$

Moving the projection point N in the x direction in the stereographic projection corresponds to keeping the points O, P, P', Q and N coplanar. This is illustrated in figure 3.

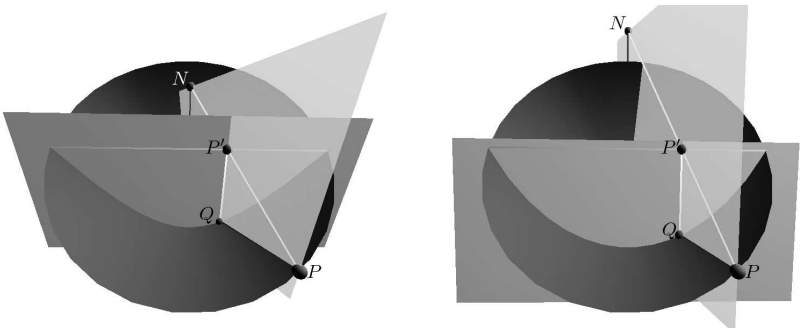

Fig. 3. Moving the point N keeps the points O, P, P', Q and N coplanar for different conic sections (center of the conic O not seen in image). The image on left shows the hyberbolic case and the image on right the elliptic case.

In order to use the equation (10) in elliptic and hyperbolic cases the orthogonal projection has to be changed to a perspective projection [2]. Let c be the distance between the foci and d the distance of the image plane from the second focal point. Then the point ζ will be projected to the point

$$\zeta' = -\frac{d}{c}\zeta \tag{11}$$

in the hyperbolic case and

$$\zeta' = \frac{d}{c}\zeta \tag{12}$$

in the elliptic case.

With this construction the projections from different conic sections have simple equations which are easy to implement in applications.

3 Spherical space and Clifford algebra

In this section a projective model for the sphere is constructed in such a way that it allows the description of geometrical entities on the sphere with simple algebraic expressions. In contrast to the previous section the sphere is now embedded to $\mathbb{R}^3$ as usual. This means that we consider only the subspace $(1, x, y, z)$ of $\mathbb{M}^4$

and this subspace has the same structure as $\mathbb{R}^3$. In this subspace the sphere can be described with the set of vectors $\mathbf{r}(\theta,\phi) = \sin(\theta)\cos(\phi)\mathbf{e}_1 + \sin(\theta)\sin(\phi)\mathbf{e}_2 + \cos(\theta)\mathbf{e}_3$.

3.1 Clifford algebra in parameter space

Let (V, g) be a vector space V equipped with a symmetric bilinear form (i.e. inner product) g. Algebra A over a ring R is compatible with the inner product space (V, g) if V is a subspace of A and for each $x \in V$, $x^2 = g(x, x)$. Clifford algebra $\mathbb{G}_{p,q,r}$ is the compatible algebra for $\mathbb{R}^{p,q,r}$ [8], where p, q, r are the numbers of unit vectors with positive, negative and null signature.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Then the Clifford algebra $\mathbb{G}_n$ has dimension 2^n and basis $\{e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n} | \epsilon_i = 0, 1\}$. For example the Clifford algebra $\mathbb{G}_3$ of $\mathbb{R}^3$ has the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}$

In practice it is useful to separate the *geometric product* of Clifford algebra in it's symmetric and antisymmetric parts: $xy = \frac{1}{2}(xy + yx) + \frac{1}{2}(xy - yx) = x{\cdot}y + x{\wedge}y$, where $(\cdot)$ is the inner product and $(\wedge)$ is the outer product ([6] contains a good introduction to the geometric product from a practical viewpoint).

Clifford algebra has proven to be a helpful tool in many applications with strong relation to geometry. Geometric transformations can be presented with simple geometric products and the inner and outer product null spaces are a simple way to present geometric entities of any dimension [10].

With the usual definition 1 of the sphere these benefits are lost as the inner and outer product null spaces describe the geometrical entities of the embedding space instead of the sphere itself. For example a line in $\mathbb{R}^n$ has at most two points common with the sphere. A conformal model for spherical geometry applying this kind of embedding can be found in [7]. Another possibility would be to use the Clifford algebra in the tangent spaces of $\mathbb{S}^2$, which is rather useless because it can only describe infinitesimal entities on the manifold.

A sphere can be parameterized in a number of ways. Parameterization with the least amount of ambiguities is the stereographic projection to the complex plane described in the previous section. In this projection the geodesic curves are mapped to curves in the complex plane, a fact which complicates their description with Clifford algebra. Instead, using the parameterization with azimuthal and polar angles $(\phi, \theta), 0 \leq \phi < 2\pi, 0 < \theta < \pi$ the geodesic lines have a simple description. Rectangular objects in projection on to the sphere can be described with lines in the (ϕ, θ) space and thus retain the 'rectangularity'. Figure 4 shows how the image captured with a parabolic mirror is transformed to the $V(\theta, \phi)$ space using (3).

To remove the periodicity in ϕ and θ on the image the following scaling is used:

$$\phi' = \frac{\phi}{2\pi - \phi} \text{ and } \theta' = \frac{\theta}{\pi - \theta}. \tag{13}$$

The vector space $V(\theta', \phi')$ equipped with the Euclidean inner product is clearly isomorphic to $\mathbb{R}^2$. Using the Euclidean inner product in $V(\theta', \phi')$ areas calculated in parameter space differ from areas on the sphere. When needed a scaling

Fig. 4. Image captured with a parabolical mirror and its mapping to $V(\phi, \theta)$.

between these areas can be calculated. Frequently used angular size $\Delta\alpha$ of an object is, for example, given by $\Delta\alpha = \sqrt{(\phi_2 - \phi_1)^2 + (\theta_2 - \theta_1)^2}$. Instead of using just the parameter space $V(\phi', \theta')$ a projective model is defined.

3.2 Clifford Algebra in the Projective Model

Definition 2. *The projective model of the sphere is the space* $\mathcal{S}_P = V(\phi', \theta') \times \{\mathbb{R}\backslash 0\}$ *equipped with the Euclidean inner product. The basis of* $\mathcal{S}_P$ *is* $\{\mathbf{e}_{\phi'}, \mathbf{e}_{\theta'}, \mathbf{e}_p\}$.

The corresponding Clifford algebra $\mathbb{G}(\mathcal{S}_P) \cong \mathbb{G}(\mathbb{R}^3)$ has the basis

$$\{1, \mathbf{e}_{\phi'}, \mathbf{e}_{\theta'}, \mathbf{e}_p, \mathbf{e}_{\phi'}\mathbf{e}_{\theta'}, \mathbf{e}_{\phi'}\mathbf{e}_p, \mathbf{e}_{\theta'}\mathbf{e}_p, \mathbf{e}_{\phi'}\mathbf{e}_{\theta'}\mathbf{e}_p\}. \tag{14}$$

A vector in $x \in V(\phi', \theta')$ is embedded in $\mathcal{S}_P$ with the mapping

$$\mathcal{P} : \mathbf{x} \in V(\phi', \theta') \mapsto \mathbf{x} + \mathbf{e}_p \in \mathcal{S}_P. \tag{15}$$

The inverse of $\mathcal{P}$ is

$$\mathcal{P}^{-1} : A \in \mathcal{S}_P \mapsto \frac{1}{A \cdot \mathbf{e}_p} \left[(A \cdot \mathbf{e}_{\phi'})\, \mathbf{e}_{\phi'} + (A \cdot \mathbf{e}_{\theta'})\, \mathbf{e}_{\theta'} \right] \tag{16}$$

In this projective model Euclidean inner and outer product null spaces, $\mathbb{NI}_E$ and $\mathbb{N}\rtimes_E$, give a simple description for points, lines and planes on the parameter space $V(\phi', \theta')$. As an example let $A, B, C \in \mathcal{S}_P$. Now

$$\mathbb{NO}(A \wedge B) = \{C \in \mathcal{S}_P \mid A \wedge B \wedge C = 0\}. \tag{17}$$

As only the perpendicular component of C contributes to (17) one gets

$$\mathbb{NO}(A \wedge B) = \{C \in \mathcal{S}_P \mid A \wedge B \wedge C_\perp = 0\}, \tag{18}$$

i.e. C lies in the plane spanned by A and B. The corresponding euclidian outer product null space is given by the projection of the plane $A \wedge B$ to $V(\phi', \theta')$:

$$\begin{aligned} &\mathbb{NO}_E\,(A \wedge B) = \mathcal{P}^{-1}\,(\mathbb{NO}(A \wedge B)) = \\ &\mathcal{P}^{-1}\ (\alpha A + \beta B) = \mathcal{P}^{-1}\,(\alpha A - \alpha B + \alpha B + \beta B) = \\ &\mathcal{P}^{-1}\ [\alpha(A - B) + (\alpha + \beta)B] = \mathbf{b} + \frac{\alpha}{\alpha + \beta}(\mathbf{a} - \mathbf{b}) \\ &= \quad \mathbf{b} + t(\mathbf{a} - \mathbf{b}), t \in \mathbb{R}, \end{aligned} \tag{19}$$

which is a line in through points $\mathbf{a}$ and $\mathbf{b}$ in $V(\phi', \theta')$. In a similar manner $\mathbb{NO}_E(A) = \mathbf{a}$.

In order to consider also the radial position of objects in the enviroment of the camera one has to add also the radial dimension $\mathbf{e}_r$ to the model. This addition does not have any other effect on the model than the addition of one extra dimension.

4 Conclusion

In this paper a simple method for unifying central catadioptric systems was presented. Using Clifford algebra on the parameter space of the sphere allows an efficient method for describing rectangular objects that are also mapped to rectangular objects in the parameter space. This has not been possible in the previous models using Clifford algebra [3],[4].

Using the parameter space of the sphere the distance $\Delta\phi$ between points on the geodesics of the sphere have the simple form $\Delta\phi = \frac{2\pi\phi_2'}{\phi_2'+1} - \frac{2\pi\phi_1'}{\phi_1'+1} = \phi_2 - \phi_1$. One can also calculcate the distances between points on a line in the parameter space using basic algebra instead of using line integrals on the surface of the sphere (which lead in many cases to incomplete elliptic integrals).

In the parameter space the rotation of the sphere is achieved with the translation operator $T(\mathbf{x}) = \mathbf{x} + \mathbf{t}$. In order to linearise the translation operator the parameter space has to be embedded to a confromal space. This conformal model is included in our ongoing research as is the movement of the sensor in the enviroment. Using Clifford algebra on suitably embedded parameter space allows the description of the geometric entities and the camera movement.

References

1. Baker, S., Nayar, S. A theory of catadioptric image formation. Proc. Int. Conf on Computer Vision (1998) 35–42
2. Baker, S., Nayar, S. A theory of single view-point catadioptric image formation. International Journal of Computer Vision **35** (1999) 175–196
3. Bayro-Corrochano, E., López-Franco, C.: Omnidirectional vision: unified model using conformal geometry. ECCV 2004, Proceedings, Part I (2004) 536–548
4. Cameron, J.I.: Applications of Geometric Algebra, first year report. Signal Processing Laboratory, Department of Engineering, University of Cambridge, 2004.
5. Geyer, C., Daniilidis, K.: Catadioptric projective geometry. International Journal of Computer Vision **45** (2001) 223–243
6. Doran, C. Lasenby, A.: Geometric Algebra for Physicists. Cambridge University Press (2003)
7. Li, H., Hestenes, D., Rockwood, A.: Spherical conformal geometry with Geometric algebra. In Sommer, G. (Ed.): Geometric Computing with Clifford Algebras. Springer Verlag Berlin Heidelberg (2001) 61–76
8. McDonald, B.R: Geometric Algebra over Local Rings. Marcel Dekker, Inc. (1976)
9. Penrose, R., Rindler, W.: Spinors & Space-Time 1. Cambridge University Press (1984)

10. Perwass, C., Hildenbrand, D.: Aspects of Geometric Algebra in Euclidean, Projective and Conformal Space. Technical Report Nr.0308, University of Kiel (2002)
11. Rosenhahn, B., Sommer, G.: Pose estimation in conformal geometric algebra, Part II: Real-time pose estimation using extended feature concepts. Journal of Mathematical Imaging and Vision **22** (2005) 49–70
12. Rosenhahn, B., Perwass, C., Sommer, G. : Pose Estimation of free-form contours. Int. Journal of Computer Vision **62** (2005) 267–289