Chapter 1

Numerical evaluation of Versors with Clifford Algebra

Christian B. U. Perwass, Gerald Sommer[1]

ABSTRACT This paper has two main parts. In the first part we discuss multivector null spaces with respect to the geometric product. In the second part we apply this analysis to the numerical evaluation of versors in conformal space. The main result of this paper is an algorithm that evaluates the best transformation between two sets of 3D-points. This transformation may be pure translation or rotation, or any combination of them. This is, of course, also possible using matrix methods. However, constraining the resultant transformation matrix to a particular transformation is not always easy. Using Clifford algebra it is straight forward to stay within the space of the transformation we are looking for.

1.1 Introduction

Clifford algebra has enjoyed an increasing popularity over the past years in the fields of computer vision and robotics. It is a particularly useful tool in these fields, since geometric objects like points, lines and planes can be expressed directly as algebraic entities. Furthermore, reflections, rotations and translations can be expressed by versors, which may act on any algebraic and hence geometric object. That is, if we denote a general versor by V, it acts on a geometric entity X via $VX\tilde{V}$, where V can be any combination of reflection, rotation and translation.

There exist many purely mathematical books on Clifford algebra, which, building on two simple axioms, analyze the complex structure of general Clifford algebras [1, 2, 3, 4]. The most useful Clifford algebras for our purpose are universal Clifford algebras. A Clifford algebra $\mathcal{C}l_n(\mathcal{V}^n)$ is called universal if it is build on an n-dimensional, non-degenerate vector space $\mathcal{V}^n$ and has dimension 2^n. A direct effect of this is, that a change of basis of $\mathcal{V}^n$ does not change $\mathcal{C}l_n$ up to an isomorphism. We will call a universal Clifford Algebra also *geometric algebra*. This term is preferred by many authors in the computer vision and robotics community [5, 6, 7, 8].

---

[1]Institut für Informatik, Preusserstr. 1-9, 24105 Kiel, Germany,
christian@perwass.com, gs@ks.informatik.uni-kiel.de.

A problem that turns up in robotics research is that given two sets of 3D-vectors $\{\mathbf{x}_i\}$, $\{\mathbf{y}_i\}$, related by a general rotation, we want to find that general rotation. If the given data is noisy, or if there are outliers, we want to find the best general rotation that relates the two 3D-point sets. Note that by general rotation, we mean a rotation about an arbitrary axis. This may also be represented by a rotation followed by a translation, or a twist. By a pure rotation we mean a rotation about an axis that passes through the origin of the space we are working in. We will refer to a pure rotation also simply by *rotation*.

1.2 Theory

Independent of the particular geometric algebra we are in, the versor equation $VA\tilde{V} = B$ may also be written as $VA - BV = 0$. We will show how this equation can be solved for V using a singular value decomposition (SVD). Before we describe the numerical algorithm, we should understand what the solution space of V is. We will start by analyzing the simpler equation $AB = 0$, where $A, B \in \mathcal{C}l_n$.

In order to perform our analysis it is convenient to look at Clifford algebra from a slightly different angle. We will assume here that you are familiar with the basic concepts of Clifford algebra. A standard way to construct a Clifford algebra $\mathcal{C}l_n(\mathcal{V}^n)$ is to take an orthonormal basis of $\mathcal{V}^n$ and then to combine these with the geometric product to obtain a 2^n dimensional basis of the Clifford algebra. It will be helpful to look at the properties of such a basis without referring to the underlying vector space basis.

Let $\mathcal{E}_n$ be a basis of some universal Clifford Algebra $\mathcal{C}l_n$. Note that at this point we are not interested in the concept of "grade" of the basis elements. Let $\mathcal{U}_n := \{1, 2, \ldots, 2^n\}$, $i, j, k \in \mathcal{U}_n$, ${g_{ij}}^k \in \{-1, 0, +1\}$ and let $\{E_i\}$ denote the elements of $\mathcal{E}_n$. The elements of $\mathcal{E}_n$ have the following properties. 1) The 2^n elements of $\mathcal{E}_n \subset \mathcal{C}l_n$ are linearly independent; 2) there exists an identity element which we choose to denote by E_1; 3) $E_i E_i = {g_{ii}}^1 E_1$; 4) $E_i E_j = {g_{ij}}^k E_k$, with $i \neq j$. The tensor ${g_{ij}}^k$ is the metric of $\mathcal{C}l_n$. If the Clifford algebra is universal then, if two indices of ${g_{ij}}^k$ are held fixed, there is only exactly one value for the third index such that ${g_{ij}}^k \neq 0$. Furthermore, ${g_{ii}}^1 \neq 0$ for all i. Therefore, each element of $\{E_i\}$ has a unique inverse with respect to the geometric product (property 3). The tensor ${g_{ij}}^k$ also encodes whether two elements of $\mathcal{E}_n$ commute or anti-commute. From property 4 we find $E_i E_j = \lambda_{ij}\, E_j E_i$, with $\lambda_{ij} \equiv {g_{ij}}^k / {g_{ji}}^k$. We will use λ_{ij} as a short hand for ${g_{ij}}^k / {g_{ji}}^k$. Note that $\lambda_{ij} = \lambda_{ji}$. If $\{E_i\}$ was constructed from a set of anti-commuting elements, as for example the orthonormal basis of some $\mathcal{V}^n$, we could evaluate ${g_{ij}}^k$. Here we want to assume that the ${g_{ij}}^k$ tensor is known for a given $\mathcal{C}l_n$.

We can now write a general multivector $A \in \mathcal{C}l_n$ as $A = \alpha^i E_i$, with

$\{\alpha^i\} \subset \mathcal{R}$. We use here the Einstein summation convention: a superscript index repeated as a subscript, or vice versa, within a product, implies a summation over the index. That is, $\alpha^i E_i \equiv \sum_i \alpha^i E_i$. The product of two multivectors $A, B \in \mathcal{C}l_n$ where $A = \alpha^i E_i$, $B = \beta^i E_i$, can thus be written as $AB = \alpha^i \beta^j \, g_{ij}{}^k E_k$.

The concept of *duality* will play an important role later on. Usually, the dual of a multivector is defined as its product with the inverse pseudoscalar. For our purposes it will be convenient to introduce a more general concept of duality. It may be shown that for some $E_p \in \mathcal{E}_n$ there exists a subset $\mathcal{D} \subset \mathcal{E}_n$ with $E_1 \in \mathcal{D}$ and $E_p \notin \mathcal{D}$, such that the intersection of $\mathcal{D}$ with the coset $\mathcal{D}E_p$ is the empty set and their union gives $\mathcal{E}_n$. In fact, $\mathcal{D}$ forms a basis of some $\mathcal{C}l_{(n-1)}$. The coset $\mathcal{D}E_p$ may be regarded as dual to $\mathcal{D}$ with respect to E_p.

After these preliminaries we will now return to our initial problem. Let $A, B \in \mathcal{C}l_n$ be two multivectors such that $AB = 0$. What are the properties of A and B? First of all note that neither A nor B can have an inverse.

Lemma 1.1. *Let $A, B \in \mathcal{C}l_n$, $A \neq 0$, $B \neq 0$, satisfy the equation $AB = 0$. Then neither A nor B have an inverse. Also, if some $A \in \mathcal{C}l_n$ does have an inverse, then there exists no non-zero $X \in \mathcal{C}l_n$ such that $AX = 0$ or $XA = 0$.*

Proof. Suppose A had an inverse denoted by A^{-1}. Then

$$AB = 0 \iff A^{-1}AB = 0 \iff E_1 B = 0 \iff B = 0,$$

where E_1 denotes the identity element of $\mathcal{C}l_n$ as before. This contradicts the assumption that $B \neq 0$. Hence, A does not have a left inverse. Furthermore, if two elements $X, Y \in \mathcal{C}l_n$ satisfy $XY = 1$, then all their components have to mutually anti-commute. This anti-commutativity is independent of whether we write XY or YX. Therefore, it is clear that every left inverse is also a right inverse and vice versa. Thus A does neither have a left nor a right inverse element in $\mathcal{C}l_n$. It also follows that if A does have an inverse, the equation $AB = 0$ is only satisfied for $B = 0$. All this may be shown in a similar way for B. ∎

The next thing we can observe is that if there exist two multivectors $A, B \in \mathcal{C}l_n$ that satisfy the equation $AB = 0$, then for every $X \in \mathcal{C}l_n$ the equation $(AB)X = 0$ is also satisfied. Due to the associativity of the geometric product we can write this equation also as $A(BX) = 0$. Hence, there exists a whole set of multivectors that right multiplied with A give zero.

Definition 1.1. *Denote the set of multivectors $X \in \mathcal{C}l_n$ that satisfy the equation $AX = 0$ for some $A \in \mathcal{C}l_n$ by $\mathcal{N}_A$. Formally $\mathcal{N}_A$ is defined as*
$\mathcal{N}_A := \{X \; : \; AX = 0 \, , \; X \in \mathcal{C}l_n\}$.

Lemma 1.2. *Some properties of $\mathcal{N}_A$. Let $A \in \mathcal{C}l_n$ and let $\mathcal{N}_A$ denote its set*

of right null-multivectors. $\mathcal{N}_A$ has the following properties.
a) If $A \neq 0$ then $\mathcal{N}_A$ does not contain the identity element of $\mathcal{C}l_n$;
b) any linear combination of elements of $\mathcal{N}_A$ is an element of $\mathcal{N}_A$;
c) for any $X \in \mathcal{N}_A$ and $M \in \mathcal{C}l_n$, $(XM) \in \mathcal{N}_A$.

Proof. Let $X, Y \in \mathcal{N}_A$ and $\alpha, \beta \in \mathcal{R}$.
a) If the identity element E_1 of $\mathcal{C}l_n$ was an element of $\mathcal{N}_A$ then $AE_1 = 0$. This is only possible if $A = 0$. Therefore, if $A \neq 0$, $E_1 \notin \mathcal{N}_A$.
b) From the distributivity of the geometric product it follows that $A(\alpha X + \beta Y) = \alpha(AX) + \beta(AY) = 0$. Hence, $(\alpha X + \beta Y) \in \mathcal{N}_A$.
c) Since the geometric product is associative $A(XM) = (AX)M = 0M = 0$ and thus $(XM) \in \mathcal{N}_A$. ■

From this lemma it follows that $\mathcal{N}_A$ is a subspace of $\mathcal{C}l_n$, albeit not a subalgebra due to its lack of the identity element. The question now is, what dimension $\mathcal{N}_A$ has. If the dimension of $\mathcal{N}_A$ was the same as that of $\mathcal{C}l_n$, i.e. 2^n, then E_1 would have to be an element of $\mathcal{N}_A$. Since this is not the case by lemma 1.2, $\dim \mathcal{N}_A < 2^n$.

Consider the set $\mathcal{M}_{AB} := \{BE_i : B \in \mathcal{N}_A,\ B \neq 0,\ \forall\ E_i \in \mathcal{E}_n\}$. That is, we take the product of all elements in $\mathcal{E}_n$ with some $B \in \mathcal{N}_A$ not equal to zero. Also note, that since we can write $B = \beta^i E_i$, $BE_j \neq 0$ for all $E_j \in \mathcal{E}_n$. Hence, $\mathcal{M}_{AB}$ has 2^n non-zero elements, from which we can build the whole space $\mathcal{N}_A$. Nevertheless, $\mathcal{M}_{AB}$ is not a basis of $\mathcal{N}_A$, since $\dim \mathcal{N}_A < 2^n$. This means that there has to be at least one element of $\mathcal{M}_{AB}$ that is linearly dependent on the others. Therefore, there exists a set $\{\alpha^i\} \subset \mathcal{R}$ such that $\alpha^i\,(BE_i) = (BE_k)$, with $i \in \mathcal{U}_n \backslash k$, where $i \in \mathcal{U}_n \backslash k$ stands for the set of integers from 1 to 2^n without k. Note that this does *not* imply that $\alpha^i E_i = E_k$, since B has no inverse. Right-multiplying this equation with ${g_{kk}}^1 E_k$ gives ${g_{kk}}^1 \alpha^i\, B(E_i E_k) = BE_1$. That is, there also exists a set $\{\beta^i\} \subset \mathcal{R}$ such that $\beta^i\,(BE_i) = (BE_1)$, with $i \in \mathcal{U}_n \backslash 1$. In general we will call any element $J \in \mathcal{C}l_n$ that satisfies $BJ = \lambda B$, an *eigen-multivector* of B with eigenvalue λ. Clearly, E_1 is an eigen-multivector of any multivector in $\mathcal{C}l_n$.

Suppose there exists only exactly one element $J = \beta^i E_i$ with $i \in \mathcal{U}_n \backslash 1$, such that $BJ = B$. We would like to show that $JJ = E_1$. This is not quite as trivial as it might seem, since $BJJ = B$ does not necessarily imply $JJ = E_1$ because B has no inverse. Nevertheless, we know that (JJ) is also an eigen-multivector of B. Since we assumed the only eigen-multivectors of B to be E_1 and J, $(JJ) \in \{E_1, J\}$. We therefore have to show that J cannot square to itself. We do this by first observing that J has no scalar component (no E_1 component), and then proving that a multivector without a scalar component cannot be idempotent (square to itself). Unfortunately, due to space constraints we cannot give this proof here (but see `www.perwass.de`). In any case, J cannot be idempotent, since it has no E_1 component. Thus, J has to square to E_1. We can write $J = \alpha^i E_i$ with $i \in \mathcal{U}_J \subset \mathcal{U}_n \backslash 1$. Since $JJ = E_1$, all components of J must be mutually anti-commuting.

A standard construction of a universal Clifford algebra $\mathcal{C}l_n$ shows that it can be constructed from n mutually anti-commuting elements that square to the identity [9, 10]. Such a set may be regarded as the basis of an n-dimensional vector space $\mathcal{V}^n$. J can therefore be seen as a basis vector of some $\mathcal{V}'^{\,n}$, which is the result of a change of basis of $\mathcal{V}^n$. Since we assumed $\mathcal{C}l_n$ to be universal, it may also be constructed from $\mathcal{V}'^{\,n}$.

We can therefore construct a new basis $\mathcal{E}'_n \equiv \{E'_i\}$ of $\mathcal{C}l_n$, with $E'_1 = E_1$ and $E'_2 = J$. As we mentioned earlier, we can now construct a subset $\mathcal{D}' \subset \mathcal{E}'_n$ with $E'_1 \in \mathcal{D}'$ and $E'_2 \notin \mathcal{D}'$ such that $\mathcal{D}' \cap (\mathcal{D}' E'_2) = \emptyset$ and $\mathcal{D}' \cup (\mathcal{D}' E'_2) = \mathcal{E}'_n$. Our goal is now to see of what form B is. If we denote the elements of $\mathcal{D}'$ by $\{D'_i\}$ we can parameterize B as $B = \beta^i_1 D'_i + \beta^j_2 D'_j E'_2$. It then follows that

$$BE'_1 = BE'_2 \iff \beta^i_1\, D'_i + \beta^j_2\, D'_j E'_2 = \beta^i_1\, D'_i E'_2 + \beta^j_2\, D'_j \iff \beta^i_1 = \beta^i_2. \tag{1.1}$$

Therefore, B can be written as $B = \beta^i D'_i\,(E'_1 + E'_2)$. To summarize, we started with the equation $AB = 0$ and then showed that B has to have at least one right eigen-multivector. Assuming that B has exactly one right eigen-multivector, we could show that B is of the form given above. We still have to show under what circumstances B is a possible solution to the equation $AB = 0$. If we write $A = \alpha^i_1 D'_i + \alpha^i_2 D'_i E'_2$ with $\{\alpha^i\} \subset \mathcal{R}$ we find that $AB = 0$ if either of the following two conditions is satisfied.

$$A = \alpha^i D'_i (1 - E'_2) \quad \text{and} \quad B = \beta^{j_c} D'_{j_c} (1 + E'_2), \tag{1.2}$$
$$A = \alpha^i D'_i (1 + E'_2) \quad \text{and} \quad B = \beta^{j_a} D'_{j_a} (1 + E'_2), \tag{1.3}$$

where $i \in \mathcal{U}_{(n-1)}$, $j_c \in \{j \in \mathcal{U}_{(n-1)} : D'_j E'_2 = E'_2 D'_j\}$ and $j_a \in \{j \in \mathcal{U}_{(n-1)} : D'_j E'_2 = -E'_2 D'_j\}$. We can now answer the question regarding the dimension of $\mathcal{N}_A$, the right null space of A. From $BD'_{j_c} E'_2 = BE'_2 D'_{j_c} = BD'_{j_c}$ and $BD'_{j_a} E'_2 = -BE'_2 D'_{j_a} = -BD'_{j_a}$ it follows that B maps half the basis of $\mathcal{C}l_n$ onto the other half. Hence, $\{BD_i : i \in \mathcal{U}_{(n-1)}\}$ is a basis of $\mathcal{N}_A$ and $\dim \mathcal{N}_A = 2^{(n-1)}$. We may conjecture that if B has k right eigen-multivectors then $\dim \mathcal{N}_A = 2^{(n-k)}$. A proof of this will not be given here.

We can draw some conclusions from this analysis. First of all, we saw that if a multivector has a right null space, the dimension of the null space is a power of 2. Furthermore, the existence of a right null space implies the existence of at least one right eigen-multivector and vice versa. That is, instead of looking for the null space of A, we could also try to evaluate its eigen-multivectors.

We will not discuss the versor equation $VA\tilde{V} = B$ in all its generality. Note that we can write this equation also as $VA = BV$. In this form it is similar to the eigen-multivector equation $VA = \lambda V$ where $\lambda \in \mathcal{R}$. The form we will consider the versor equation in is $VA - BV = 0$. If there exists a set of multivectors $\mathcal{J}_B := \{J \in \mathcal{C}l_n : JB = BJ\}$ then $(\mathcal{J}_B V)A - B(\mathcal{J}_B V) = 0$.

This shows that the solution space for V has dimension $\dim \mathcal{J}_B$. Note that if $J_1, J_2 \in \mathcal{J}_B$ then $(J_1 J_2) \in \mathcal{J}_B$. Therefore, the dimension of $\mathcal{J}_B$ is a power of 2.

Consider for example $\mathcal{C}l_3$. Let $\mathbf{a}, \mathbf{b} \in \mathcal{C}l_3$ be two unit vectors in a Euclidean 3D-space. We are looking for the rotor R such that $R\mathbf{a}\tilde{R} = \mathbf{b}$. We know that the appropriate rotor is $R_{ab} = e^{U_{ab}\theta/2}$, where U_{ab} is the unit bivector representing the rotation plane, and θ is the angle of rotation. However, if we were trying to solve for R in $R\mathbf{a} - \mathbf{b}R = 0$, we would find the solution set to be $\mathcal{J}_R := \{R_{ab}, \mathbf{b}R_{ab}, U_b R_{ab}, I R_{ab}\}$, where I is the pseudoscalar of $\mathcal{C}l_3$ and $U_b = \mathbf{b}I^{-1}$, the plane normal to $\mathbf{b}$. We cannot even lift this ambiguity completely by introducing more vector pairs that are related by the same rotor. If we introduce a second vector pair the solution space becomes $\mathcal{J}_R := \{R_{ab}, I R_{ab}\}$. That is, although the rotation is now uniquely defined, the rotor is only given "up to duality", since I commutes with all vectors. Therefore, we would have to enforce the solution space $\mathcal{J}_R$ to be an even subspace of $\mathcal{C}l_3$, in order to reduce the solution space to $\mathcal{J}_R := \{R_{ab}\}$. This is what we will do in our numerical algorithm.

1.3 Implementation

In order to solve Clifford equations like $AB = 0$ or $VA\tilde{V} = B$ in $\mathcal{C}l_n$ numerically, we regard multivectors as 2^n dimensional vectors with an appropriate metric. Then we can use standard methods for solving sets simultaneous linear equations, e.g. singular value decomposition (SVD). Nevertheless, we always stay in the particular Clifford algebra. This means, for example, that we can solve for a rotor linearly, while still remaining within the space of rotors.

As described in the previous section we write a multivector $A \in \mathcal{C}l_n$ as $A = \alpha^i E_i$, where the $\{E_i\} \equiv \mathcal{E}_n$ form a basis of $\mathcal{C}l_n$. We assume $\mathcal{C}l_n$ to be universal. Therefore, the metric ${g_{ij}}^k$ satisfies the properties specified at the beginning of the previous section. We assume that ${g_{ij}}^k$ is known for the particular $\mathcal{C}l_n$ we use.

Let $A, B, X \in \mathcal{C}l_n$ be defined as $A := \alpha^i E_i$, $B := \beta^i E_i$ and $X := \xi^i E_i$. To solve $AX = 0$ for X we write this equation as $\alpha^i \xi^j {g_{ij}}^k = \xi^j\, G^{A\,k}_j = 0$, where $G^{A\,k}_j \equiv \alpha^i {g_{ij}}^k$ is a matrix of dimensions $2^n \times 2^n$. Now we simply have to evaluate the null space of G^A to obtain the set of multivectors X that satisfy $AX = 0$. Similarly we can solve the equation $AX = B$ for X. In this case we write $\xi^j\, G^{A\,k}_j = \beta^k$, and then evaluate the inverse of G^A. If $B = 1$ we obtain the inverse of A in this way, provided A does have an inverse. The versor equation $VA\tilde{V} = B$ can be solved analogously. Let $V := \eta^i E_i$ and write the versor equation as $VA - BV = 0$. This becomes

$$\eta^i \alpha^j {g_{ij}}^k - \beta^j \eta^i {g_{ji}}^k = \eta^i \left(\alpha^j {g_{ij}}^k - \beta^j {g_{ji}}^k\right) = \eta^i\, G^{AB\,k}_i = 0, \tag{1.4}$$

where $G_i^{AB\,k} \equiv (\alpha^j g_{ij}{}^k - \beta^j g_{ji}{}^k)$. Note that the difference between left and right multiplication with V is expressed by the transposition of the indices i and j in $g_{ij}{}^k$. Again we only have to find the null space of G^{AB} in order to find V. If we know that V only has components $\eta^i E_i$ with $i \in \mathcal{U}_V \subset \mathcal{U}_n$, we simply take out the appropriate columns of G^{AB} to implement these constraints. This enables us, for example, to reduce the solution space of V to the even subalgebra of $\mathcal{C}l_n$, or even to a particular rotation plane. Furthermore, if we know that two factors η^i have to be equal we can simply add the appropriate columns of G^{AB}.

The general rotation between two sets of 3D-points may be evaluated in this way in a conformal space. The appropriate Clifford algebra for this space is $\mathcal{C}l_{4,1}(\mathcal{V}^{4,1})$, i.e. 4 basis vectors square to $+1$ and one to -1. We will only give a very short introduction to the properties of this algebra. See [11] for more details. Let the basis of $\mathcal{V}^{4,1}$ be given by $\{e_1, e_2, e_3, e_+, e_-\}$ where $e_i^2 = e_+^2 = +1$, $i \in \{1, 2, 3\}$ and $e_-^2 = -1$. We define two composite elements $e \equiv e_+ + e_-$ and $\bar{e} \equiv \frac{1}{2}(e_- - e_+)$, such that $e^2 = \bar{e}^2 = 0$ and $e \cdot \bar{e} = -1$. A vector in Euclidean space, $\mathbf{x} \in \mathcal{V}^3$, is represented in this conformal space as $X = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e + \bar{e}$, such that $X^2 = 0$. The advantage of working in this space is that there exist versors for rotation *and* translation. The versor for a pure rotation about the rotation axis given by the normalized vector $\hat{\mathbf{b}}$ and angle θ is $R_b = \cos\theta + \sin\theta\, \hat{U}_b$, where $\hat{U}_b \equiv \hat{\mathbf{b}} I_3^{-1}$ and $I_3^{-1} \equiv e_3 e_2 e_1$. The versor for a pure translation by a vector $\mathbf{a}$ is given by $T_a = 1 - \frac{1}{2}\mathbf{a}e$. Therefore, the general rotation about an axis with orientation $\hat{\mathbf{b}}$ and offset $\mathbf{a}$ is given by $W_{ab} \equiv T_a R_b \tilde{T}_a = \cos\theta + \sin\theta \hat{U}_b + \sin\theta(\hat{U}_b \cdot \mathbf{a})e$. Therefore, W_{ab} has 7 components: $W_{ab} = \eta^0 + \eta^1 e_2 e_3 + \eta^2 e_3 e_1 + \eta^3 e_1 e_2 + \eta_e^1 e_1 e + \eta_e^2 e_2 e + \eta_e^3 e_3 e$. However, recall that e is a composite element, which means that the three elements $\{\eta_e^i e_i e\}$ are really six elements $\{\eta_+^i\, e_i e_+,\, \eta_-^i\, e_i e_-\}$ when we solve for W_{ab} numerically. But as we mentioned above, the constraint $\eta_+^i = \eta_-^i$ can be implemented quite easily. A much more serious problem is that the seven parameters of W_{ab} are not independent. This is because the $\{\eta_e^i\}$ not only depend on $\mathbf{a}$ but also on $\hat{U}_b$, the rotation plane. In fact, $\mathbf{a}' \equiv \hat{U}_b \cdot \mathbf{a}$ is the orthogonal complement of the projection of $\mathbf{a}$ onto $\hat{U}_b$ [7]. When we evaluate W_{ab} numerically as described above, i.e. regarding all parameters as independent, there is no guarantee that the vector $\mathbf{a}'$ does indeed lie on $\hat{U}_b$. To obtain a proper versor we therefore adjust W_{ab} after evaluation such that this condition is satisfied. The adjusted $\mathbf{a}'$ is then given by $\mathbf{a}'_{adj} = (\mathbf{a}' \cdot \hat{U}_b^{-1}) \cdot \hat{U}_b$. This is of course quite an arbitrary adjustment and may introduce an error. Nevertheless, we will investigate this method of evaluation of W_{ab} in the following.

FIGURE 1.1.

1.4 Experiments

We performed two synthetic experiments, to test the evaluation quality of the above described method for evaluating general rotations. In the first experiment we evaluated a pure rotation and in the second experiment a general rotation. In both experiments we used the following setup. A random distribution of $N = 100$ points was created within a sphere of radius 10 about the origin. Then this set of points was rotated by an angle of $\pi/4$ radians about the rotation axis given by the vector $(e_1 + 2e_2 + 3e_3)$. For the general rotation experiment the rotation axis was also translated by the vector $(e_1 - e_2 + 2e_3)$. Let the initial and transformed sets of points be denoted by $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$, respectively. Then Gaussian noise was induced into both sets to produce data sets $\{\mathbf{a}'_i\}$ and $\{\mathbf{b}'_i\}$, which in turn were used to evaluate the appropriate versor (V^{eval}). Even though we added Gaussian noise evenly to all points, it is clear that this has a stronger effect for points close to the rotation axis, than for points further away from it. To obtain a statistically meaningful error measure we therefore evaluated $M = 100$ versors $\{V_i^{eval}\}$ from different random sets of 3D-points at each noise level. Two error measures were calculated at each noise level.

$$\Delta := \frac{1}{M}\sum_i^M \sqrt{\sum_j^N \frac{(\mathbf{b}_j - \mathbf{b}_{ij}^{eval})^2}{N}}, \; \Delta' := \frac{1}{M}\sum_i^M \sqrt{\sum_j^N \frac{(\mathbf{b}'_j - \mathbf{b}_{ij}^{eval})^2}{N}}, \tag{1.5}$$

where $\mathbf{b}_{ij}^{eval} := V_i^{eval}\,\mathbf{a}_j\,\tilde{V}_i^{eval}$. Figure 1.1 shows the result of these evaluations. The units on the x-axis refer to the Gaussian mean deviation of the radii of the error vectors added to the point sets $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$ to give $\{\mathbf{a'}_i\}$ and $\{\mathbf{b'}_i\}$. In the figure the values for Δ are denoted by "Compared with True" and the values for Δ' with "Compared with Data". The series with pure rotation is indicated by "(R)" and the series where a general rotation was evaluated is indicated by "(R&T)". The line denoted "Expected" indicates the mean geometric error we might naively expect. That is, if we induce noise with a Gaussian mean deviation of δ, we might also expect the versor evaluated from this noisy data to produce a mean geometric error of δ. However, what we see is quite different.

Maybe the most surprising result is that if we have pure rotation of a rigid body about a known origin, the appropriate rotor can be evaluated with fairly high accuracy, even for large noise (*Compared with True (R)*). This shows that pure rotation is a very strong constraint if the origin is known. Nevertheless, the figure also shows that if the origin is not known (*Compared with True (R&T)*), i.e. we have to evaluate a general rotation, the mean geometric error grows very quickly. This large error cannot be due to our adjustment of the evaluated rotor, since we found that the geometric product of the rotor with its reverse deviates only slightly from unity.

The two series that show the values of Δ' indicate how close the evaluated versors reproduce the data they were calculated from. In the case of pure rotation the mean geometric error is as big as the induced noise. This shows that the method of evaluation can only reproduce the noisy data up to its noise level. However, when evaluating a general rotation, the evaluated versor cannot even reproduce the noisy data up to its noise level.

1.5 Conclusion

As with most mathematical work, this paper was actually developed from back to front. That is, after realizing how we could solve equations like $AB = 0$ or $AB = C$ numerically, while staying within the given Clifford algebra, we were somewhat surprised by the results. Especially the observation that the null space of a multivector always seemed to have the dimension of a power of 2, "cried out" for some analytical justification. The results of this analysis are presented in the first section of this paper. As you will have realized, in this first section we deviated quite a bit from our declared goal to investigate the numerical evaluation of versors. However, we found this analysis to be very interesting and informative, since it taught us a number of things about universal Clifford algebras that we had not seen discussed before. This analysis then helped us to understand our numerical algorithms better.

Clifford algebra seemed an attractive tool to evaluate the general rotation between two 3D-point sets, since rotation and translation can be expressed as linear operators. That is, we can evaluate a general rotation through a set of simultaneous linear equations, while always staying within the space of the appropriate operator. As we have seen, this is not quite true, since the versor for a general rotation does not consist of six independent parameters. Nevertheless, one could easily implement a minimization routine that evaluates such a versor from just six parameters, e.g. via a gradient descent method. In any case, for small noise levels the algorithm we presented here gives good results. It is also fairly fast, where its speed depends mainly on the implementation of the SVD algorithm. In our case the evaluation of a versor for a general rotation from 100 3D-point pairs took on average 0.08 seconds on an AMD-K6 III, 400 MHz running Windows Me. The algorithm was implemented with Christian Perwass' geometric algebra C++ library (CLU library) and is available for download as part of this library from `www.perwass.com`.

[1] J. E. Gilbert and M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis.* Cambridge University Press, 1991.

[2] I. R. Porteous, *Clifford Algebras and the Classical Groups.* Cambridge University Press, 1995.

[3] M. Riesz, *Clifford Numbers and Spinors.* Kluwer Academic Publishers, 1993.

[4] A. J. Hahn, *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups.* Springer–Verlag, 1994.

[5] D. Hestenes and R. Ziegler, "Projective Geometry with Clifford Algebra," *Acta Applicandae Mathematicae*, vol. 23, pp. 25–63, 1991.

[6] C. Perwass and J. Lasenby, "A Unified Description of Multiple View Geometry," in *Geometric Computing with Clifford Algebra* (G. Sommer, ed.), Springer-Verlag, 2001 to be published.

[7] L. Dorst, "Honing geometric algebra for its use in the computer sciences," in *Geometric Computing with Clifford Algebra* (G. Sommer, ed.), Springer-Verlag, 2000 to be published.

[8] J. Lasenby, W. J. Fitzgerald, A. Lasenby, and C. Doran, "New geometric methods for computer vision: An application to structure and motion estimation," *International Journal of Computer Vision*, vol. 3, no. 26, pp. 191–213, 1998.

[9] P. Lounesto, *Clifford Algebras and Spinors.* Cambridge University Press, 1997.

[10] C. Perwass, *Applications of Geometric Algebra in Computer Vision.* PhD thesis, Cambridge University, 2000.

[11] H. Li, D. Hestenes, and A. Rockwood, “Generalized homogeneous coordinates for computational geometry,” in *Geometric Computing with Clifford Algebra* (G. Sommer, ed.), Springer-Verlag, 2001 to be published.