Estimation of Geometric Entities and Operators from Uncertain Data*

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Abstract. In this text we show how points, point pairs, lines, planes, circles, spheres, and rotation, translation and dilation operators and their uncertainty can be evaluated from uncertain data in a unified manner using the Geometric Algebra of conformal space. This extends previous work by Förstner et al. [3] from points, lines and planes to non-linear entities and operators, while keeping the linearity of the estimation method. We give a theoretical description of our approach and show the results of some synthetic experiments.

1 Introduction

In Computer Vision applications uncertain data occurs almost invariably. Appropriate methods to deal with this uncertainty do therefore play an important role. In this text we discuss the estimation of geometric entities and operators from uncertain data in a unified mathematical framework, namely Geometric Algebra. In particular, we will show that evaluating points, lines, planes, circle, spheres and their covariance matrices from a set of uncertain points can be done in much the same way as the evaluation of rotation, translation and dilation operators with corresponding covariance matrices. Using error propagation, further calculations can be performed with these uncertain entities, while keeping track of the uncertainty. This text builds on previous works by Förstner et al. [3] and Heuel [5] where uncertain points, lines and planes were treated in a unified manner. Perwass & Sommer previously discussed the linear estimation of rotation operators in Geometric Algebra [11], albeit without taking account of uncertainty. In [8] the description of uncertain circles and 2D-conics in Geometric Algebra was first discussed. The stratification of Euclidean, projective and affine spaces in Geometric Algebra, has been previously discussed in [12]. In this text, it is shown how the Geometric Algebra of the conformal space of 3D-Euclidean space can be used to deal with uncertain projective geometry and uncertain kinematics in a unified way. In particular, we will concentrate on the estimation of geometric entities and operators from uncertain data.

The structure of this text is as follows. First we give short introductions to Geometric Algebra, error propagation and the Gauss-Helmert model. Then we combine these methods to show how the various objects can be estimated. We conclude the text with some synthetic experiments and conclusions.

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2 Geometric Algebra

For a detailed introduction to Geometric Algebra see e.g. [10, 4]. Here we can only give a short overview. Geometric Algebra is an associative, graded algebra, whereby the algebra product is called *geometric product*. The Geometric Algebra over a n-dimensional Euclidean vector space $\mathbb{R}^n$ has dimension 2^n and is denoted by $\mathbb{G}(\mathbb{R}^n)$ or simply $\mathbb{G}_n$. Elements of different grade of the algebra can be constructed through the *outer product* of linearly independent vectors. For example, if $\{a_i\} \in \mathbb{R}^n$ are a set of k linearly independent vectors, then $A_{\langle k \rangle} := a_1 \wedge \ldots \wedge a_k$ is an element of $\mathbb{G}_n$ of grade k, which is called a *blade*, where $\wedge$ denotes the outer product. A general element of the algebra, called *multivector*, can always be expressed as a linear combination of blades of possibly different grades. Blades can be used to represent geometric entities. To combine projective geometry and kinematics we need to consider the Geometric Algebra of the (projective) *conformal space* of 3D-Euclidean space (cf. [10]). The embedding function $\mathcal{K}$ is defined as $\mathcal{K} : x \in \mathbb{R}^3 \mapsto x + \frac{1}{2} x^2 e_\infty + e_o \in \mathbb{R}^{4,1}$. The basis of $\mathbb{R}^{4,1}$ can be written as $\{e_1, e_2, e_3, e_\infty, e_o\}$. The various geometric entities that can be represented by blades in $\mathbb{G}_{4,1}$ are shown in table 1. In this table $X, Y, Z, U, V \in \mathbb{R}^{4,1}$ are embeddings of points $x, y, z, u, v \in \mathbb{R}^3$, respectively, and the $e_{ij} \equiv e_i \wedge e_j$ etc. denote the algebra basis elements of an entity.

Entity	**Grade**	**No.**	**Basis Elements**
Point X	1	5	$e_1, e_2, e_3, e_\infty, e_o$
Point Pair $X \wedge Y$	2	10	$e_{23}, e_{31}, e_{12}, e_{1o}, e_{2o}, e_{3o}, e_{1\infty}, e_{2\infty}, e_{3\infty}, e_{o\infty}$
Line $X \wedge Y \wedge e_\infty$	3	6	$e_{23\infty}, e_{31\infty}, e_{12\infty}, e_{1o\infty}, e_{2o\infty}, e_{3o\infty}$
Circle $X \wedge Y \wedge Z$	3	10	$e_{23\infty}, e_{31\infty}, e_{12\infty}, e_{23\infty}, e_{31\infty}, e_{12\infty}, e_{1o\infty}, e_{2o\infty}, e_{3o\infty}, e_{123}$
Plane $X \wedge Y \wedge Z \wedge e_\infty$	4	4	$e_{123\infty}, e_{23o\infty}, e_{31o\infty}, e_{12o\infty}$
Sphere $X \wedge Y \wedge Z \wedge U$	4	5	$e_{123\infty}, e_{123o}, e_{23o\infty}, e_{31o\infty}, e_{12o\infty}$
Rotor R	0,2	4	$1, e_{23}, e_{31}, e_{12}$
Translator T	0,2	4	$1, e_{1\infty}, e_{2\infty}, e_{3\infty}$
Dilator D	0,2	2	$1, e_{o\infty}$
Motor RT	0,2,4	8	$1, e_{23}, e_{31}, e_{12}, e_{1\infty}, e_{2\infty}, e_{3\infty}, e_{123\infty}$
Gen. Rotor $TR\tilde{T}$	0,2	7	$1, e_{23}, e_{31}, e_{12}, e_{1\infty}, e_{2\infty}, e_{3\infty}$

Table 1. Entities and their algebra basis. Note that the operators are multivectors of mixed grade.

Apart from representing geometric entities by blades, it is also possible to define operators in Geometric Algebra. The class of operators we are particularly interested in are *versors*. A versor $V \in \mathbb{G}_n$ is a multivector that satisfies the following two conditions: $V\tilde{V} = 1$ and for any blade $A_{\langle k \rangle} \in \mathbb{G}_n$, $V\, A_{\langle k \rangle}\, \tilde{V}$ is also of grade k, i.e. a versor is *grade preserving*. The expression $\tilde{V}$ denotes the *reverse* of V. The reverse operation changes the sign of the constituent blade elements depending on their grade, which has an effect similar to complex conjugation in quaternions. The most interesting versors for our purposes in conformal space are rotation operators (rotors), translation operators (translators) and scaling operators (dilators).

If $\{E_i\}$ denotes the 2^n-dimensional algebra basis of $\mathbb{G}_n$, then a multivector $A \in \mathbb{G}_n$ can be written as $A = \mathsf{a}^i\, E_i$, where a^i denotes the i^{th} component of a vector $\mathsf{a} \in \mathbb{R}^{2^n}$ and a sum over the repeated index i is implied. We will use this Einstein summation convention also in the following. If $B = \mathsf{b}^i\, E_i$ and $C = \mathsf{c}^i\, E_i$, then the components of C in the algebra equation $C = A \circ B$ can be evaluated via $\mathsf{c}^k = \mathsf{a}^i\, \mathsf{b}^j\, g^k{}_{ij}$. Here $\circ$ is a placeholder for an algebra product and $g^k{}_{ij} \in \mathbb{R}^{2^n \times 2^n \times 2^n}$ is a tensor encoding this product.

If we define the matrices $\mathsf{U}, \mathsf{V} \in \mathbb{R}^{2^n \times 2^n}$ as $\mathsf{U}(\mathsf{a}) := \alpha^i\, g^k{}_{ij}$ and $\mathsf{V}(\mathsf{b}) := \beta^j\, g^k{}_{ij}$, then $\mathsf{c} = \mathsf{U}(\mathsf{a})\,\mathsf{b} = \mathsf{V}(\mathsf{b})\,\mathsf{a}$. Therefore, we can define an isomorphism Φ, such that for $A, B \in \mathbb{G}_n$, $\Phi(A) \in \mathbb{R}^{2^n}$ and $\Phi(A \circ B) = \mathsf{U}(\Phi(A))\,\Phi(B) = \mathsf{V}(\Phi(B))\,\Phi(A)$, where $\circ$ is a placeholder for an algebra product. This isomorphism allows us to apply standard numerical algorithms to Geometric Algebra equations. We can also reduce the complexity of the equations considerably by only mapping those components of multivectors that are actually needed. In the following we therefore assume that Φ maps to the minimum number of components necessary.

3 Stochastic

In this section we give short descriptions of error propagation and the Gauss-Helmert model, which will be needed for the evaluation of multivectors from uncertain data. The *error propagation* we consider here is based on the assumption that the uncertainty of a (vector valued) measurement can be modeled by a Gaussian distribution. Hence, the probability density function of a random vector variable is fully described by a mean vector and a covariance matrix. Error propagation is a method to evaluate the mean and covariance of a function of random vector variables. In particular, this allows us to evaluate the mean and covariance of algebra products between multivector valued random variables. For a detailed introduction see [6, 7].

For example, we have to apply error propagation to the embedding of Euclidean vectors in conformal space. Let $\underline{a} \in \mathbb{R}^3$ be a Euclidean random vector variable with covariance matrix $\Sigma_{a,a}$, and $\underline{A} \in \mathbb{R}^{4,1}$ be defined by $\underline{A} := \mathcal{K}(\underline{a})$. It may then be shown that $\bar{A} = \mathcal{E}[\mathcal{K}(\underline{a})] = \bar{a} + \frac{1}{2}\,\bar{a}^2\, e_\infty + e_o + \frac{1}{2}\,\mathrm{tr}(\Sigma_{a,a})\, e_\infty$. Typically the trace of $\Sigma_{a,a}$ is negligible, which leaves us with $\bar{A} = \mathcal{K}(\bar{a})$. If we denote the Jacobi matrix of $\mathcal{K}$ evaluated at $\bar{a}$ by $\mathsf{J}_\mathcal{K}(\bar{a})$, then the error propagation equation for the covariance matrix can be written as $\Sigma_{A,A} = \mathsf{J}_\mathcal{K}(\bar{a})\ \Sigma_{a,a}\ \mathsf{J}^\mathsf{T}_\mathcal{K}(\bar{a})$.

The *Gauss-Helmert* model was introduced by Helmert in 1872 as the general case of least squares adjustment. It is also called the *mixed model* [6]. The Gauss-Helmert model is a linear, stochastic model. The idea is to find the smallest adjustment to the data points, such that a valid parameter vector exists.

Mathematically this is expressed as follows. Given is a set of M data vectors $\{\mathsf{b}_i\}$ with corresponding covariance matrices Σ_{b_i,b_i}. The goal is to find a parameter vector p, such that a given, vector valued constraint function g satisfies $g(\mathsf{b}_i, \mathsf{p}) = 0$ for all i. Furthermore, the set of valid parameter vectors is constraint by a function h, which has to satisfy $h(\mathsf{p}) = 0$. Since the Gauss-Helmert model is linear, the functions g and h have to be linearized. For this purpose it is assumed that the true data point b_i is given by the current estimate $\hat{\mathsf{b}}_i$ plus an adjustment $\Delta\mathsf{b}_i$, and similarly for the parameter vector. That is, $\mathsf{b}_i = \hat{\mathsf{b}}_i + \Delta\mathsf{b}$ and $\mathsf{p} = \hat{\mathsf{p}} + \Delta\mathsf{p}$, which implies that an initial estimate of the parameter vector has to be known, before the Gauss-Helmert method can be applied. Substituting these expressions for b_i and p in the constraint equation $g(\mathsf{b}_i, \mathsf{p}) = 0$ and considering only its Taylor expansion up to first order results in $\mathsf{U}_i\,\Delta\mathsf{p} + \mathsf{V}_i\,\Delta\mathsf{b}_i = \mathsf{c}_{g_i}$, where $\mathsf{U}_i := (\partial_{\mathsf{p}}\, g)(\hat{\mathsf{b}}_i, \hat{\mathsf{p}})$, $\mathsf{V}_i := (\partial_{\mathsf{b}_i}\, g)(\hat{\mathsf{b}}_i, \hat{\mathsf{p}})$ and $\mathsf{c}_{g_i} := -g(\hat{\mathsf{b}}_i, \hat{\mathsf{p}})$. The constraint function h is linearized in a similar way leading to the constraint equation $\mathsf{H}^{\mathsf{T}}\,\Delta\mathsf{p} = \mathsf{c}_h$, where $\mathsf{H}^{\mathsf{T}} := (\partial_{\mathsf{p}}\, h)(\hat{\mathsf{p}})$ and $\mathsf{c}_h := -h(\hat{\mathsf{p}})$.

We now try to solve for $\Delta\mathsf{b}_i$ and $\Delta\mathsf{p}$ such that $\Delta\mathsf{b}_i^{\mathsf{T}}\,\Sigma_{\mathsf{b}_i,\mathsf{b}_i}\,\Delta\mathsf{b}_i$ is minimized and the linearized constraint equations are satisfied for all i. This may be done using the method of Lagrange multipliers. This leads to the following equation system.

$$\begin{pmatrix} \mathsf{N} & \mathsf{H} \\ \mathsf{H}^{\mathsf{T}} & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathsf{p} \\ \mathsf{m} \end{pmatrix} = \begin{pmatrix} \mathsf{c}_n \\ \mathsf{c}_h \end{pmatrix}, \tag{1}$$

where m is a Lagrange multiplier vector, $\mathsf{N} := \sum_{i=1}^{M} \mathsf{U}_i^{\mathsf{T}}\,(\mathsf{V}_i\Sigma_{\mathsf{b}_i,\mathsf{b}_i}\mathsf{V}_i^{\mathsf{T}})^{+}\,\mathsf{U}_i$ and $\mathsf{c}_n := \sum_{i=1}^{M} \mathsf{U}_i^{\mathsf{T}}\,(\mathsf{V}_i\Sigma_{\mathsf{b}_i,\mathsf{b}_i}\mathsf{V}_i^{\mathsf{T}})^{+}\,\mathsf{c}_{g_i}$. The vector $\Delta\mathsf{p}$ can be evaluated directly equation (1), while $\Delta\mathsf{b}_i$ has to be evaluated by substituting $\Delta\mathsf{p}$ into the equation

$$\Delta\mathsf{b}_i = \Sigma_{\mathsf{b}_i,\mathsf{b}_i}\mathsf{V}_i^{\mathsf{T}}\,(\mathsf{V}_i\Sigma_{\mathsf{b}_i,\mathsf{b}_i}\mathsf{V}_i^{\mathsf{T}})^{+}\,(\mathsf{c}_{g_i} - \mathsf{U}_i\,\Delta\mathsf{p}). \tag{2}$$

The new estimates for b_i and p are then given by $\hat{\mathsf{p}}' = \hat{\mathsf{p}} + \Delta\mathsf{p}$ and $\hat{\mathsf{b}}_i' = \hat{\mathsf{b}}_i + \Delta\mathsf{b}_i$. If the constraint functions g and h are linear, then these new estimates are the best linear unbiased estimators for b_i and p, as is for example shown in [6, 7]. If the constraint functions are not linear, then this is a step in an iterative estimation procedure.

4 Estimation of Multivectors

In this section we show how Geometric Algebra offers a unified framework to derive the constraint equations for geometrical problems, so that the Gauss-Helmert method can be applied. Since the standard algebra operations between multivectors can be mapped to bilinear functions, the estimation of all algebra elements is basically the same. This means in particular that operators as well as geometric entities are represented by vectors and their estimation is therefore very similar. In order to apply the Gauss-Helmert estimation, we need to define a constraint function that relates the parameter vector and the data vectors, as well as a constraint function for the parameter vector alone. Furthermore, we need to obtain an initial estimate of the parameter vector. Part of the constraints is that we only use those multivector components that can be non-zero in the particular elements we consider. For example, table 1 shows that for a line we only need to consider a subset of those components necessary for a circle, even though both are blades of grade 3.

Let $P \in \mathbb{G}_{4,1}$ represent the geometric entity that is to be estimated and $B_n \in \mathbb{G}_{4,1}$ the n^{th} data point, then $g(B_n, P) = B_n \wedge P$, because $B_n \wedge P = 0$ if and only if B_n lies on P. Note that this constraint is only valid if B_n and/or P represents a point. For example, we could evaluate the best line (P) through a set of points (B_n), but also the best point (P) that lies on a set of lines (B_n). Mapping the g-constraint with Φ gives $\Phi(B_n \wedge P) = \mathsf{b}_n^i\, \mathsf{p}^j\, \mathsf{O}^k{}_{ij}$, where $\mathsf{O}^k{}_{ij}$ encodes the appropriate outer product.

The magnitude of $B_n \wedge P$ is only proportional to the Euclidean distance between a point B_n and the element represented by P, if P represents a point, line or plane. If P represents a point pair, circle or sphere, this is not the case, in general. However, the closer points lie to these entities, the better proportionality is satisfied. In 2D-Euclidean space the fitting of a circle P to a set of points $\{B_n\}$ with the constraint $B_n \wedge P = 0$, is equivalent to the well known algebraic fitting of circles [2]. However, $B_n \wedge P = 0$ is valid independent of the embedding dimension, which allows us to readily extract the algebraic constraint equations for circles in 3D-Euclidean space.

The constraints on P alone depend on the grade of P. If P is of grade 1 or 4, i.e. it represents a point, a plane or a sphere, then the only constraint is that the scale of P is fixed. This is needed, since we are working in a projective space and thus all scaled, non-zero versions of P represent the same geometric entity. If $\mathsf{p} = \Phi(P)$, then this constraint can be written as $h_1(\mathsf{p}) = \mathsf{p}^\mathsf{T}\, \mathsf{p} - 1$, such that $h_1(\mathsf{p}) = 0$ if $\|\mathsf{p}\| = 1$.

If P represents a point pair (grade 2), a line (grade 3) or a circle (grade 3), then there is an additional constraint that ensures that P is in fact a blade. Recall that a blade of grade k is the outer product of k vectors. However, if $k = 2$ or $k = 3$, *not* all linear combinations of the respective algebra basis elements form a blade. The constraints that ensure that P is a blade, are the Plücker constraints. In Geometric Algebra these constraints can be expressed by the equation $P \wedge P = 0$ if P is of grade 2 and $P^* \wedge P^* = 0$ if P is of grade 3, where P^* denotes the dual of P. The dual operation in Geometric Algebra is the geometric product with a constant element of the algebra (cf. [10]). Mapped with Φ we thus obtain the constraint equation $h_2(\mathsf{p}) = \Phi(P^* \wedge P^*) = \mathsf{p}^p\, \mathsf{p}^q\, \mathsf{D}^i{}_p\, \mathsf{D}^j{}_q\, \mathsf{O}^k{}_{ij}$, if P is of grade 3. Here $\mathsf{D}^i{}_p$ encodes the dual operation.

For versors $V \in \mathbb{G}_{4,1}$ the constraint functions are different. Suppose $A_n, B_n \in \mathbb{G}_{4,1}$ represent pairs of geometric entities of the same type. The problem now is to find the V that best satisfies $B_n = V A_n \tilde{V}$. Since $V\tilde{V} = 1$, this can also be written as $V A_n - B_n V = 0$. Hence, $g(B_n, V) = V A_n - B_n V$. The constraint function on V alone is $h(V) = V\tilde{V} - 1$, which is zero, if V is a versor. Mapping the latter constraint with Φ gives $g(\mathsf{b}_n, \mathsf{v}) = \Phi(V A_n - B_n V) = \mathsf{v}^i\, (\mathsf{a}_n^j\, \mathsf{G}^k{}_{ij} - \mathsf{b}_n^j\, \mathsf{G}^k{}_{ji})$, where $\mathsf{G}^k{}_{ij}$ encodes the appropriate geometric product. The h-constraint becomes $h(\mathsf{v}) = \Phi(V\tilde{V} - 1) = \mathsf{v}^i\, \mathsf{v}^q\, \mathsf{R}^j{}_q\, \mathsf{G}^k{}_{ij} - \mathsf{w}^k$, where $\mathsf{R}^j{}_p$ encodes the reverse operation and w^k is zero everywhere apart from the entry representing the scalar component, which is unity.

Table 2 summarizes the Jacobi matrices of the constraint equations for the various entities, as needed in equation 1 in the Gauss-Helmert estimation. Matrices H_1^T and H_2^T have to be combined column-wise to result in H^T. These Jacobi matrices have to be evaluated for current estimates of the parameter and data vectors as described in section 3. The contractions of the algebra product tensors ${\mathsf{O}^k}_{ij}$ and ${\mathsf{G}^k}_{ij}$ with vectors can, for example, be evaluated with the software *CLUCalc* [9]. An initial estimate of the parameter vector, i.e. p or v, is given by the right null space that the respective set of U_n matrices have in common. This can be evaluated by finding the right null space of $\mathsf{U} := \sum_n \mathsf{U}_n^\mathsf{T}\mathsf{U}_n$ using, for example, a singular value decomposition (SVD). Note that the matrices U_n and V for points, lines and planes as given in table 2 are equivalent to the matrices S, Π and Γ as defined by Förstner et al. in [3].

Entity	U_n	V	H_1^T	H_2^T
Point Pair	$\mathsf{b}_n^i\, {\mathsf{O}^k}_{ij}$	$\mathsf{p}^j\, {\mathsf{O}^k}_{ij}$	$2\,\mathsf{p}^j$	$2\,\mathsf{p}^j\, {\mathsf{O}^k}_{ij}$
Line, Circle				$2\,\mathsf{p}^q\, {\mathsf{D}^i}_p\, {\mathsf{D}^j}_q\, {\mathsf{O}^k}_{ij}$
Point, Plane, Sphere				n/a
Versor	$\mathsf{a}_n^j\, {\mathsf{G}^k}_{ij} - \mathsf{b}_n^j\, {\mathsf{G}^k}_{ji}$	$-\mathsf{v}^i\, {\mathsf{G}^k}_{ji}$	$2\,\mathsf{v}^q\, {\mathsf{R}^j}_q\, {\mathsf{G}^k}_{ij}$	n/a

Table 2. Jacobi matrices used in Gauss-Helmert estimation for different entities. A repeated index in a product implies summation over its range. The first index of a tensor denotes the row in matrix representation.

5 Experiments & Conclusions

To show the quality of the proposed estimation method of geometric entities and operators, we present two synthetic experiments. In the first experiment we fit 3D-circles to uncertain data points and in the second experiment we estimate general rotations between two 3D-point clouds.

To generate the uncertain data to which a circle is to be fitted, we first create a "true" circle C of radius one, oriented arbitrarily in 3D-space. We then randomly select N points $\{a_n \in \mathbb{R}^3\}$ on the true circle within a given angle range. For each of these points a covariance matrix Σ_{a_n,a_n} is generated randomly, within a certain range. For each of the a_n, Σ_{a_n,a_n} is used to generate a Gaussian distributed random error vector r_n. The data points $\{b_n\}$ with corresponding covariance matrices Σ_{b_n,b_n} are then given by $b_n = a_n + r_n$ and $\Sigma_{b_n,b_n} = \Sigma_{a_n,a_n}$. The standard deviation of the set $\{\|r_n\|\}$ will be denoted by σ_r. For each angle range, 30 sets of true points $\{a_n\}$ and for each of these sets, 40 sets of data points $\{b_n\}$ were generated.

A circle is then fitted to each of the data point sets. We will denote a circle estimate by $\hat{C}$ and the shortest vector between a true point a_n and $\hat{C}$ by d_n. For each $\hat{C}$ we then evaluate two quality measures: the Euclidean RMS distance $\delta_E := \sqrt{\sum_n d_n^\mathsf{T}\, d_n / N}$ and the Mahalanobis RMS distance $\delta_\Sigma := \sqrt{\sum_n d_n^\mathsf{T}\, \Sigma_{a_n,a_n}^{-1}\, d_n \,/ N}$. The latter measure uses the covariance matrices as local metrics for the distance measure. δ_Σ is a unit-less value that is > 1, $= 1$ or < 1 if d_n lies outside, on or inside the standard deviation error ellipsoid represented by Σ_{a_n,a_n}. For each true point set, the mean and standard deviation of the δ_E and δ_Σ over all data point sets is denoted by Δ_E, σ_E and Δ_Σ, σ_Σ, respectively. Finally, we take the mean of the Δ_E, σ_E and Δ_Σ, σ_Σ over all true point sets, which

σ_r	Angle Range	$\bar{\Delta}_\Sigma$ ($\bar{\sigma}_\Sigma$)		$\bar{\Delta}_E$ ($\bar{\sigma}_E$)	
		SVD	GH	SVD	GH
0.07	10°	2.13 (0.90)	1.26 (0.52)	0.047 (0.015)	0.030 (0.009)
	60°	1.20 (0.44)	0.92 (0.31)	0.033 (0.010)	0.028 (0.009)
	180°	1.38 (0.56)	0.97 (0.36)	0.030 (0.009)	0.025 (0.008)
0.15	10°	2.17 (0.90)	1.15 (0.51)	0.100 (0.032)	0.057 (0.019)
	60°	1.91 (0.99)	1.35 (0.68)	0.083 (0.033)	0.069 (0.028)
	180°	1.21 (0.44)	0.90 (0.30)	0.070 (0.022)	0.058 (0.018)

Table 3. Results of circle estimation for SVD method (SVD) and Gauss-Helmert method (GH).

σ_r	$\bar{\Delta}_\Sigma$ ($\bar{\sigma}_\Sigma$)			$\bar{\Delta}_E$ ($\bar{\sigma}_E$)		
	Std	SVD	GH	Std	SVD	GH
0.09	1.44 (0.59)	1.47 (0.63)	0.68 (0.22)	0.037 (0.011)	0.037 (0.012)	0.024 (0.009)
0.18	1.47 (0.62)	1.53 (0.67)	0.72 (0.25)	0.078 (0.024)	0.079 (0.026)	0.052 (0.019)

Table 4. Result of general rotation estimation for standard method (Std), SVD method (SVD) and Gauss-Helmert method (GH).

are then denoted by $\bar{\Delta}_E$, $\bar{\sigma}_E$ and $\bar{\Delta}_\Sigma$, $\bar{\sigma}_\Sigma$. These quality measures are evaluated for the circle estimates by the SVD and the Gauss-Helmert (GH) method. In table 3 the results for different values of σ_r and different angle ranges is given. In all cases 10 data points are used.

It can be seen that for different levels of noise (σ_r) the Gauss-Helmert method always performs better in the mean quality and the mean standard deviation than the SVD method. It is also interesting to note that the Euclidean measure $\bar{\Delta}_E$ is approximately doubled when σ_r is doubled, while the "stochastic" measure $\bar{\Delta}_\Sigma$, only increases slightly. This is to be expected, since an increase in σ_r implies larger values in the Σ_{a_n,a_n}. Note that $\bar{\Delta}_\Sigma < 1$ implies that the estimated circle lies mostly inside the standard deviation ellipsoids of the true points.

For the evaluation of a general rotor, the "true" points $\{a_n\}$ are a cloud of Gaussian distributed points about the origin with standard deviation 0.8. These points are then transformed by a "true" general rotation R. Given the set $\{a'_n\}$ of rotated true points, noise is added to generate the data points $\{b_n\}$ in just the same way as for the circle. For each of 40 sets of true points, 40 data point sets are generated and a general rotor $\hat{R}$ is estimated. Using $\hat{R}$ the true points are rotated to give $\{\hat{a}'_n\}$. The distance vectors $\{d_n\}$ are then defined as $d_n := a'_n - \hat{a}'_n$. From the $\{d_n\}$ the same quality measures as for the circle are evaluated. In table 4 we compare the results of the Gauss-Helmert (GH) method with the initial SVD estimate and a standard approach (Std) described in [1]. Since the quality measures did not give significantly different results for rotation angles between 3 and 160 degrees, the mean of the respective values over all rotation angles are shown in the table. The rotation axis always points along the z-axis and is moved one unit away from the origin along the x-axis.

In all experiments 10 points are used. It can be seen that for different levels of noise (σ_r) the Gauss-Helmert method always performs significantly better in the mean quality and the mean standard deviation than the other two. Just as for the circle the Euclidean measure $\bar{\Delta}_E$ is approximately doubled when σ_r is doubled, while the "stochastic" measure $\bar{\Delta}_\Sigma$, only increases slightly. Note that $\bar{\Delta}_\Sigma < 1$ implies that the points $\{\hat{a}'_n\}$ lie mostly inside the standard deviation ellipsoids of the $\{a'_n\}$.

In conclusion it was shown by the synthetic experiments that accounting for the uncertainty in the data when estimating geometric and kinematic entities, does improve the results. Geometric Algebra offers a unifying framework where the constraints on geometric and kinematic entities can be expressed succinctly and dimension independently in such a way that linear estimation procedures may be applied. We believe that these properties can be of great value for many applications in Computer Vision.

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