# Uncertain Geometry with Circles, Spheres and Conics

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2004/05/25

**Abstract.** Spatial reasoning is one of the central tasks in Computer Vision. It always has to deal with uncertain data. Projective geometry has become the working horse for modelling multiple view geometry, while modelling uncertainty with statistical tools has become a standard. Geometric reasoning in projective geometry with uncertain geometric elements has been advocated by Kanatani in the early 90's, and recently made transparent and generalized to basic entities in projective geometry including transformations by Förstner and Heuel, exploiting the multilinearity of nearly all relations, such as incidence and identity, which results from the underlying Grassmann-Cayley algebra (cf. [21, 8, 7]). This paper generalizes geometric reasoning under uncertainty towards circles, spheres and conics, which play a role in many computer vision applications. In particular it will be shown how within the Clifford algebra of conformal space, as introduced by Hestenes et al. [11, 16], circles can be constructed from three uncertain points in 3D-Euclidean space, while propagating the covariance matrices of the points. This then enables us to obtain and visualize the uncertainty of the resulting circle. We also introduce the Clifford algebra over the vector space of 2D-conics, which allows us to apply the same error propagation procedures as for the Clifford algebra of conformal space.

**Keywords:** Clifford Algebra, Error Propagation, Conformal Space, Conic Sections

## 1. Introduction

Spatial reasoning is one of the central tasks in Computer Vision. It always has to deal with uncertain data. Projective geometry has become the working horse for modelling multiple view geometry, while modelling uncertainty with statistical tools has become a standard. Geometric reasoning in projective geometry with uncertain geometric elements has been advocated by Kanatani in the early 90's, and recently made transparent and generalized to basic entities in projective geometry including transformations by Förstner and Heuel, exploiting the multilinearity of nearly all relations, such as incidence and identity, which results from the underlying Grassmann-Cayley algebra (cf. [21, 8, 7]).

This paper generalizes geometric reasoning under uncertainty towards circles and spheres, which play a role in many computer vision applications. The basic step is to embed all entities into a more general algebra, namely the Clifford algebra of conformal space as proposed by Hestenes et al. [11, 16]. The basic elements in conformal algebra are spheres in any dimension, including points, straight lines, planes but also point pairs, i. e. spheres in $\mathbb{E}^1$ and circles, i. e. spheres in $\mathbb{E}^2$. We also introduce the Clifford algebra over the vector space of 2D-conics, which, to the best of our knowledge, has not yet been discussed in the literature. This allows us to model 2D-conics and their intersections and thus also apply geometric reasoning under uncertainty to these entities. Clifford algebra is, for all intents and purposes, equivalent to Grassmann-Cayley algebra and can thus also cover projective geometry [13].

Modelling uncertainty of uncertain homogeneous entities is not straight forward (cf. [3]). In case of good relative accuracy, i. e. directional errors of less than 1 %, the representation with covariance matrices has been widely accepted (cf. e. g. [14, 4]). A direct integration into projective geometry has been proposed by Förstner [9]. For the simple case of the join $\mathbf{l} = \mathbf{x} \times \mathbf{y} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$, where $\mathsf{S}(\mathbf{x}) = [\mathbf{x}]_\times$ is the skew matrix induced by the 3-vector $\mathbf{x}$, we obtain:

$$\Sigma_{l,l} = \mathsf{S}(\mathbf{y})\Sigma_{x,x}\mathsf{S}^\mathsf{T}(\mathrm{y}) + \mathsf{S}(\mathbf{x})\Sigma_{y,y}\mathsf{S}^\mathsf{T}(\mathbf{x})$$

for independent 2D points with covariance matrices $\Sigma_{x,x}$ and $\Sigma_{y,y}$, e. g.

$$\Sigma_{x,x} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This type of uncertainty representation and propagation can be extended to all types of geoemtric entities and also transformations within projective geometry, in case the expressions are multilinear in the given entities.

The paper generalizes these developments towards circles, spheres and conic sections by embedding all entities in a more general algebra. The paper is organized as follows: Sect. 2 presents the basic concepts of Clifford algebra making the multilinearity of the expressions explicit. Sect. 3 describes the embedding of $n$-spheres into Clifford algebra via the special instance of conformal algebra and the versatility of the concept. Sect. 4 introduces the embedding of 2D-conics in a 6D-vector space and the Clifford algebra over this vector space. Based on the statistical error propagation in sect. 5 the uncertainty propagation in conformal algebra and the algebra of conics is demonstrated for 3D circles and 2D conics.

## 2. Clifford Algebra

Without explaining exactly what it is, we can define a Clifford algebra on $\mathbb{R}^n$, which is denoted by $\mathcal{C}\ell(\mathbb{R}^n)$ or simply $\mathcal{C}\ell_n$, if it is clear that we are forming the Clifford algebra over the reals. The latter will in fact be the case for the whole of this text. For more detailed introductions to Clifford algebra see e.g. [12, 20, 17, 6, 19]. A Clifford algebra $\mathcal{C}\ell_n$ over a vector space $\mathbb{R}^n$ has dimension $2^n$. An algebraic basis of $\mathcal{C}\ell_n$ may therefore be denoted by a set $\{E_i\}_{i=1}^{2^n}$ of so called *basis blades*. It may be shown that these basis blades satisfy a number of constraints with respect to the algebra product which is also called the *geometric* or *Clifford* product. This product will simply be denoted by juxtaposition, i.e. the geometric product of two elements $A, B \in \mathcal{C}\ell_n$ is written as $AB$. The basis blades of $\mathcal{C}\ell_n$ have the following properties:

$$\begin{array}{c} \exists E_1 \text{ such that } E_i E_1 = E_1 E_i = E_i,\ \forall i \in \{1, \ldots, 2^n\}, \\ E_i E_i = \lambda_i\, E_1,\ \lambda_i \in \{-1, 1\},\ \forall i \in \{1, \ldots, 2^n\}, \\ E_i E_j = \sum_{k=1}^{2^n} g^k{}_{ij}\, E_k,\ \forall i, j \in \{1, \ldots, 2^n\}. \end{array} \tag{1}$$

The last condition basically says that the geometric product of basis blades is invertible. For example, given indices $(i, j, k)$ such that $E_i E_j = E_k$, we find that

$$E_i E_j = E_k \iff E_i E_j E_j = E_k E_j \iff E_k E_j = \lambda_j\, E_i,$$

and thus $g^i{}_{kj} = \lambda_j$.

A general element of $\mathcal{C}\ell_n$ is called *multivector*. In terms of basis blades a general multivector $A \in \mathcal{C}\ell_n$ may be given by $A = \sum_{i=1}^{2^n} \alpha^i\, E_i$. In the following we will use the Einstein summation convention, that a superscript index repeated within a product as a subscript index is implicitly summed over its range. That is, a multivector may be written as $A = \alpha^i\, E_i$, if it is clear that $i \in \{1, \ldots, 2^n\}$. The geometric product of two multivectors $A, B \in \mathcal{C}\ell_n$, with $A = \alpha^i E_i$ and $B = \beta^i E_i$, is then given by

$$AB = (\alpha^i\, E_i)\,(\beta^j\, E_j) = \alpha^i\, \beta^j\, E_i E_j = \alpha^i\, \beta^j\, g^k{}_{ij}\, E_k. \tag{2}$$

Writing the result multivector $M \in \mathcal{C}\ell_n$ of $M = AB$ as $M = \mu^i E_i$ then gives

$$M = AB \iff \mu^k E_k = \alpha^i\, \beta^j\, g^k{}_{ij}\, E_k \iff \mu^k = \alpha^i\, \beta^j\, g^k{}_{ij}\ \forall k. \tag{3}$$

This shows that if multivectors in $\mathcal{C}\ell_n$ are expressed as vectors in $\mathbb{R}^{2^n}$, the geometric product between them becomes a bilinear function. Therefore, if want to discuss error propagation in Clifford algebra, we can look at the error propagation of bilinear functions. Note that other products available in Clifford algebra like the inner and outer product, which will be discussed in the following, may also be expressed in this way.

As an example for an $\{E_i\}_{i=1}^{2^n}$ basis, consider projective space $\mathbb{PK}^3$ with orthonormal basis $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4\}$. A basis for the Clifford algebra $\mathcal{C}\ell(\mathbb{PE}^3)$ is then be given by

$$\begin{aligned} \{1,\, &\mathsf{e}_1,\, \mathsf{e}_2,\, \mathsf{e}_3,\, \mathsf{e}_2\mathsf{e}_3,\, \mathsf{e}_3\mathsf{e}_1,\, \mathsf{e}_1\mathsf{e}_2,\, \mathsf{e}_4\mathsf{e}_1,\, \mathsf{e}_4\mathsf{e}_2,\, \mathsf{e}_4\mathsf{e}_3, \\ &\mathsf{e}_2\mathsf{e}_3\mathsf{e}_4,\, \mathsf{e}_3\mathsf{e}_1\mathsf{e}_4,\, \mathsf{e}_1\mathsf{e}_2\mathsf{e}_4,\, \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3,\, \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3\mathsf{e}_4\} \end{aligned} \tag{4}$$

Each of the elements of this basis may now be denoted by one $E_i$. From the associativity of the algebra product $(\mathsf{e}_1(\mathsf{e}_2\mathsf{e}_4) = (\mathsf{e}_1\mathsf{e}_2)\mathsf{e}_3)$ and the signature of the vector space, in this case $\mathsf{e}_i\mathsf{e}_i = 1$, the particular values of the tensor $g^k{}_{ij}$ follow. Note that we can obtain an equivalent structure when using Grassmann-Cayley algebra.

The representation of algebra products in the form of equation (3) allows us to apply standard error propagation directly to Clifford algebra, as will be seen later on. However, this representation is not particularly enlightening when it comes to the description of geometry. Geometry is in fact represented through the null-spaces of algebraic entities with respect to particular algebra products. In Clifford algebra these are the inner and the outer product [12] and in Grassmann-Cayley algebra the meet and join [7].

The outer product is a special operation defined within Clifford algebra and is denoted by $\wedge$. It is, in fact, equivalent to the exterior product of Grassmann algebra. The outer product is associative and distributive. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{E}^n$ it is also anti-commutative, i. e. $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$. Another important property is that if $\mathbf{x} \wedge \mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. More generally, for a set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ of $k \leq n$ mutually linearly independent vectors, $(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k) \wedge \mathbf{y} = 0$ if and only if $\mathbf{y}$ is linearly dependent on $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. The outer product of a number of vectors is also called a *blade*. The *grade* of a blade is simply the number of vectors that "wedged" together give the blade. Hence, the outer product of $k$ linearly independent vectors gives a blade of grade $k$, a $k$-blade.

The null space of a $k$-blade in some $\mathcal{C}\ell(\mathbb{R}^n)$ with respect to the outer product, i. e. the *outer product null space* of a $k$-blade, is therefore a $k$-dimensional subspace of $\mathbb{R}^n$. Geometrically this means for Euclidean space $\mathbb{E}^3$ that a vector represents a line through the origin, a 2-blade (or *bivector*) a plane through the origin, and a 3-blade (or *trivector*) the whole $\mathbb{E}^3$. For more details see [13].

Instead of looking at the null space of algebraic entities with respect to the outer product, we can do the same for the *inner product* of

Clifford algebra. The inner product will be denoted by $\cdot$. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, their inner product is just the same as their scalar product denoted by $*$. That is, $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}*\mathbf{y} \ \in \mathbb{R}$. This may be called the "metric" property of the inner product, since the result of the scalar product of two vectors depends on the metric of the vector space they lie in. However, the inner product also has some purely algebraic properties for elements in $\mathcal{C}\ell(\mathbb{R}^n)$, which are independent of the metric of the vector space $\mathbb{R}^n$. For example, let $\mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then the inner product of $\mathbf{x}$ with $\mathbf{a} \wedge \mathbf{b}$ gives,

$$\mathbf{x} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{x} \cdot \mathbf{a})\,\mathbf{b} - (\mathbf{x} \cdot \mathbf{b})\,\mathbf{a}. \tag{5}$$

Since $(\mathbf{x} \cdot \mathbf{a})$ and $(\mathbf{x} \cdot \mathbf{b})$ are scalars, we see that the inner product of a vector with a bivector results in a vector. In terms of the null space of entities with respect to the inner product, this formula shows that vector $\mathbf{x}$ lies in the *inner product null space* of $\mathbf{a} \wedge \mathbf{b}$ if and only if $\mathbf{x}$ lies in the inner product null space of $\mathbf{a}$ *and* $\mathbf{b}$. That is, the inner product null space of $\mathbf{a}\wedge\mathbf{b}$ is the intersection of the inner product null spaces of $\mathbf{a}$ and $\mathbf{b}$. For example, in the Clifford algebra of projective space $\mathcal{C}\ell(\mathbb{PE}^3)$, vectors $\mathbf{a}$ and $\mathbf{b}$ may represent planes w.r.t. their inner product null space. Hence, the bivector $\mathbf{a} \wedge \mathbf{b}$ then represents the intersection line of the two planes.

## 3. Conformal Space

In the previous section it was shown how Clifford algebra can be used to represent geometric entities like lines and planes through the origin in $\mathcal{C}\ell(\mathbb{E}^3)$. Conformal space extends this idea by embedding a $n$-dimensional Euclidean space in a nonlinear manner in a $(n{+}2)$-dimensional space. Conformal space takes its name from the fact that certain types of reflections in conformal space represent inversion in Euclidean space and conformal transformations can be represented by combinations of inversions. See [18, 16] for more details. In this text we cannot go into all the details relating to conformal space and the Clifford algebra over this space. We can only state the important formulae and give a basic idea of how we can use conformal space to work with geometric entities.

In the following we will denote vectors in a 3-dimensional Euclidean vector space $\mathbb{E}^3$ by small, bold faced letters as in $\mathbf{x}$. Note that even though we will work in the following with the conformal space of 3-dimensional Euclidean space, all formulae extent directly to $n$ dimensions. In order to obtain a conformal space, which we will denote by $\mathbb{PK}^n$, we extent the orthonormal basis $\{\mathsf{e}_i\}_{i=1}^n$ of $\mathbb{E}^n$ by two orthogonal basis vectors $\{\mathsf{e}_+, \mathsf{e}_-\}$ with $\mathsf{e}_+^2 = 1$ and $\mathsf{e}_-^2 = -1$. The embedding of a Euclidean vector $\mathbf{x}$ in conformal space is then given by

$$\mathbf{X} = \mathbf{x} + \tfrac{1}{2}\,\mathbf{x}^2\,\mathsf{e}_\infty + \mathsf{e}_o, \tag{6}$$

where $\mathsf{e}_\infty := \mathsf{e}_- + \mathsf{e}_+$ and $\mathsf{e}_o := \frac{1}{2}(\mathsf{e}_- - \mathsf{e}_+)$. The properties of $\mathsf{e}_\infty$ and $\mathsf{e}_o$ are therefore $\mathsf{e}_\infty^2 = \mathsf{e}_o^2 = 0$ and $\mathsf{e}_\infty \cdot \mathsf{e}_o = -1$. We use the null basis $\{\mathsf{e}_\infty, \mathsf{e}_o\}$ instead of the Minkowski basis $\{\mathsf{e}_+, \mathsf{e}_-\}$ since $\mathsf{e}_\infty$ and $\mathsf{e}_o$ have a clear semantic meaning as the point at infinity (there is only one) and the origin, respectively. We can now ask which Euclidean vectors $\mathbf{y} \in \mathbb{E}^3$ when embedded in conformal space, lie in the inner product null space of $\alpha\mathbf{X}$, with $\alpha \in \mathbb{R}$. Since we know that the embedding of $\mathbf{y}$ in conformal space is $\mathbf{Y} = \mathbf{y} + \frac{1}{2}\mathbf{y}^2\,\mathsf{e}_\infty + \mathsf{e}_o$, the question becomes for which $\mathbf{y}$ the inner product of $\mathbf{Y}$ and $\alpha\mathbf{X}$ becomes zero. We find that $\mathbf{Y} \cdot (\alpha\,\mathbf{X}) = \alpha\,(\mathbf{Y} \cdot \mathbf{X}) = \alpha\,(-\frac{1}{2}\,(\mathbf{y} - \mathbf{x})^2)$, which is clearly zero if and only if $\mathbf{y} = \mathbf{x}$. Similarly, we can ask what a vector of the form $\mathbf{S} = \mathbf{X} - \frac{1}{2}\,\rho^2\,\mathsf{e}_\infty$, with $\rho \in \mathbb{R}$, represents, where $\mathbf{X}$ is the same as above. We find that $\mathbf{Y} \cdot \mathbf{S} = -\frac{1}{2}\,(\mathbf{y} - \mathbf{x})^2 + \frac{1}{2}\,\rho^2$, which is zero if and only if $(\mathbf{x} - \mathbf{a})^2 = \rho^2$. That is, a vector of the form of $\mathbf{S}$ in $\mathbb{PK}^2$ represents a circle in $\mathbb{E}^2$ centered on $\mathbf{x}$ with radius $\rho$. In $\mathbb{PK}^3$, $\mathbf{S}$ represents a sphere centered on $\mathbf{x}$ with radius $\rho$ and in even higher dimensional spaces it would represent a hypersphere. This shows that it is possible to represent circles and spheres in a linear manner in conformal space, which is of course due to the non-linear embedding of Euclidean vectors.

Since equation (5) holds in any Clifford algebra, it is also valid for $\mathcal{C}\ell(\mathbb{PK}^3)$. Given two vectors $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{PK}^3$ both representing spheres in $\mathbb{E}^3$, their outer product $\mathbf{S}_1 \wedge \mathbf{S}_2$ represents the intersection circle of the spheres with respect to the inner product null space of the bivector. That is, we can also represent circles in $\mathbb{E}^3$ in a linear manner in conformal space $\mathbb{PK}^3$.

While a circle is represented in the inner product null space by the intersection of two spheres, it may be shown that in terms of the outer product null space a circle through three points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{E}^3$ can be represented by the outer product of the three corresponding conformal vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. Furthermore, four points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 \in \mathbb{PK}^2$ are co-circular if $\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 = 0$.

As it turns out, within the Clifford algebra over conformal space, the only geometric entity that can be represented is a sphere, albeit in any dimension and with any radius. For example, a sphere with infinite radius, i. e. a plane, can be represented with finite components. A point, on the other hand, is a sphere with zero radius and a sphere in $\mathbb{E}^1$ is a point pair. The following list shows the geometric entities in $\mathbb{E}^3$ represented by blades of different grades in $\mathcal{C}\ell(\mathbb{PK}^3)$, in terms of their outer product null space. The $\{\mathbf{X}_i\} \subset \mathbb{PK}^3$ are assumed to be the conformal embeddings of Euclidean vectors $\{\mathbf{x}_i\} \subset \mathbb{E}^3$.

$$\begin{array}{rl} \mathbf{X}_1 & : \text{ Point } \mathbf{x}_1 \\ \mathbf{X}_1 \wedge \mathbf{X}_2 & : \text{ Point pair } (\mathbf{x}_1,\, \mathbf{x}_2) \\ \mathbf{X}_1 \wedge \mathsf{e}_\infty & : \text{ Point pair } (\mathbf{x}_1,\, \infty) \\ \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 & : \text{ Circle through } \mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3 \\ \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathsf{e}_\infty & : \text{ Line through } \mathbf{x}_1,\, \mathbf{x}_2 \\ \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 & : \text{ Sphere through } \mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3, \mathbf{x}_4 \\ \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathsf{e}_\infty & : \text{ Plane through } \mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3 \\ \mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5 & : \text{ The whole space } \mathbb{E}^3. \end{array} \tag{7}$$

## 4. The Vector Space of Conic Sections

In conformal space we defined a particular embedding of Euclidean vectors in a higher dimensional vector space with particular properties. The Clifford algebra over this conformal vector space then allowed for the linear representation of circles, spheres, etc. A set of geometric entities of particular interest in computer vision are conic sections. It would therefore be advantageous to be able to form a Clifford algebra over a vector space such that conics and their intersections can be represented. This is indeed possible and may be done in the following way.

It is well known that given a symmetric $3 \times 3$ matrix $\mathsf{A}$, the set of vectors $\mathsf{x} = (x, y, 1)^\mathsf{T}$ that satisfy $\mathsf{x}^\mathsf{T}\, \mathsf{A}\, \mathsf{x} = 0$, lie on a conic. This can also be written using the scalar product of matrices, denoted here by $*$, as $(\mathsf{x}\mathsf{x}^\mathsf{T}) \,*\, \mathsf{A} = 0$. It makes therefore sense to define a vector space of symmetric matrices in the following way. If $a_{ij}$ denotes the component of matrix $\mathsf{A}$ at row $i$ and column $j$, we can define a transformation $\mathcal{T}$ that maps elements of $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^6$ as

$$\mathcal{T} \,:\, \mathsf{A} \in \mathbb{R}^{3\times 3} \mapsto \left(a_{13},\, a_{23},\, \tfrac{1}{\sqrt{2}}\, a_{33},\, \tfrac{1}{\sqrt{2}}\, a_{11},\, \tfrac{1}{\sqrt{2}}\, a_{22},\, a_{12}\right)^\mathsf{T} \in \mathbb{R}^6. \tag{8}$$

A vector $\mathsf{x} \in \mathbb{R}^3$ may now be embedded in the same six dimensional space via $\mathbf{x} := \mathcal{T}(\mathsf{x}\mathsf{x}^\mathsf{T})$. If we define $\mathbf{a} := \mathcal{T}(\mathsf{A})$, then $\mathsf{x}^\mathsf{T}\, \mathsf{A}\, \mathsf{x} = 0$ can be written as the scalar product

$$\mathbf{x} \cdot \mathbf{a} = 0 \iff x^2\, a_{11} + y^2\, a_{22} + 2xy\, a_{12} + 2x\, a_{13} + 2y\, a_{23} + a_{33} = 0. \tag{9}$$

Finding the vector $\mathbf{a}$ that best satisfies the above equation for a set of points is usually called the algebraic estimation of a conic, see e.g. [1].

We will denote the 6D-vector space in which 2D-conics may be represented by $\mathbb{D}^2 \equiv \mathbb{R}^6$. A 2D-vector $(x, y) \in \mathbb{R}^2$ is transformed to $\mathbb{D}^2$ by the function

$$\mathcal{D}\ : \quad (x, y) \in \mathbb{R}^2 \ \mapsto\ (x,\, y,\, \tfrac{1}{\sqrt{2}},\, \tfrac{1}{\sqrt{2}}\, x^2,\, \tfrac{1}{\sqrt{2}}\, y^2,\, xy) \in \mathbb{D}^2. \tag{10}$$

The Clifford Algebra $\mathcal{C}\ell(\mathbb{D}^2)$ has (algebra) dimension $2^6 = 64$. The inner product null space of a vector $\mathbf{A} \in \mathbb{D}^2$ is the set of all those vectors $\mathbf{X} \in \mathbb{D}^2$ that satisfy $\mathbf{X} \cdot \mathbf{A} = 0$. As was shown before, this null space is a (possibly degenerate) conic. Furthermore, the inner product null space of the outer product of two vectors $\mathbf{A}, \mathbf{B} \in \mathbb{D}^2$, $\mathbf{A} \wedge \mathbf{B}$, now has to represent the intersection of the conics represented by $\mathbf{A}$ and $\mathbf{B}$. Let $\mathbf{x}_i \in \mathbb{R}^2$ and let $\mathbf{X}_i \in \mathbb{D}^2$ be defined by $\mathbf{X}_i = \mathcal{D}(\mathbf{x}_i)\, \forall\, i$. Then the outer product null space of blades in $\mathcal{C}\ell(\mathbb{D}^2)$ may be shown to represent the following objects.

$$\begin{aligned}
\mathbf{X}_1 &: \text{ Point } \mathbf{x}_1 \\
\mathbf{X}_1 \wedge \mathbf{X}_2 &: \text{ Point pair } (\mathbf{x}_1,\, \mathbf{x}_2) \\
\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 &: \text{ Point triplet } (\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3) \\
\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 &: \text{ Point quadruplet } (\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3, \mathbf{x}_4) \\
\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5 &: \text{ The conic through } \mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \mathbf{x}_4,\, \mathbf{x}_5.
\end{aligned} \tag{11}$$

In particular, it can be shown that the outer product null space of $\mathbf{X}_1 \wedge \mathbf{X}_2 \wedge \mathbf{X}_3 \wedge \mathbf{X}_4 \wedge \mathbf{X}_5$ is the same as the inner product null space of its dual, which is a vector. Hence, this is also a simple way to construct the symmetric matrix that represents a conic through five points. Note that to the best of our knowledge the Clifford algebra $\mathcal{C}\ell(\mathbb{D}^2)$ has not yet been discussed in the literature. We believe that it offers an intuitive way to deal with 2D-conics and warrants further investigation.

## 5. Error Propagation in Clifford Algebra

It was shown previously that operations like the geometric, inner and outer product in Clifford algebra are basically bilinear functions. This implies that standard error propagation methods (cf. e. g. [15]) can be applied in the evaluation of these products. Therefore, we can, for example, evaluate the mean circle through three points, given the three points with corresponding covariance and cross-covariance matrices in conformal space. The same could be done, given two spheres with appropriate covariance and cross-covariance matrices. Before the details of such calculations are presented, error propagation in Clifford algebra is introduced from a somewhat more general point of view.

Let $\{E_i\}_{i=1}^{2^n}$ denote again the algebra basis of $\mathcal{C}\ell(\mathbb{R}^n)$. Given three multivectors $A, B, M \in \mathcal{C}\ell(\mathbb{R}^n)$, with $A = \alpha^i E_i$, $B = \beta^i E_i$ and $M = \mu^i E_i$, we may regard them as vectors in some $\mathbb{R}^m$, with orthonormal basis $\{\mathsf{e}_i\}_{i=1}^{m}$, where $m = 2^n$. In this vector space the multivectors may be written as column vectors $\mathsf{a} = [\alpha^1, \ldots, \alpha^m]^\mathsf{T}$, $\mathsf{b} = [\beta^1, \ldots, \beta^m]^\mathsf{T}$ and $\mathsf{m} = [\mu^1, \ldots, \mu^m]^\mathsf{T}$, respectively. We use here sans serif letters to denote vectors in $\mathbb{R}^m$ in order to distinguish them from (multi-)vectors in $\mathcal{C}\ell(\mathbb{R}^n)$. The relation between multivectors in $\mathcal{C}\ell(\mathbb{R}^n)$ and their representation in $\mathbb{R}^m$ may be regarded as an isomorphism $\Phi$ between these two spaces, whereby $\Phi(A \in \mathcal{C}\ell_n) = \mathsf{a} \in \mathbb{R}^m$ and $\Phi^{-1}(\mathsf{a}) = A$. This isomorphism also transforms Clifford algebra products to matrix products with special matrices. For example, if $M = A \wedge B$ then

$$\mathsf{m} = \Phi(M) = \Phi(A \wedge B) = \mathsf{U}(\Phi(A))\, \Phi(B) = \mathsf{U}(\mathsf{a})\, \mathsf{b},$$

where $\mathsf{U}(\mathsf{a})$ is a matrix whose entries depend on $\mathsf{a}$. In the following all matrices will be written as capital sans-serif letters. The form of matrix $\mathsf{U}$ is derived through the following considerations. A product in $\mathcal{C}\ell(\mathbb{R}^n)$ between two multivectors can be expressed as a bilinear function $\mathsf{g}$ which is a map $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and may be written as $\mathsf{g}(\mathsf{a}, \mathsf{b}) := \alpha^i\, \beta^j\, g^k{}_{ij}\; \mathsf{e}_k$, where again we have implicit sums over $i, j$ and $k$. The object $g^k{}_{ij}$ is again the 3-valence tensor from equation (1). It encodes the relation between the basis blades of $\mathcal{C}\ell_n$ for a particular product. For example, if $g^k{}_{ij}$ encodes the outer product, then the equation $M = A \wedge B$ may be written in $\mathbb{R}^m$ as

$$\mathsf{m} = \mathsf{g}(\mathsf{a}, \mathsf{b}) \iff \mu^k = \alpha^i \beta^j\, g^k{}_{ij}\ \forall k. \tag{12}$$

If we now denote the matrix of derivatives of $\mathsf{g}(\mathsf{a}, \mathsf{b})$ with respect to the $\{\beta^j\}$ as $\mathsf{U}(\mathsf{a})$, and with respect to the $\{\alpha^i\}$ as $\mathsf{V}(\mathsf{b})$, we can write $M = A \wedge B$ equivalently in $\mathbb{R}^m$ as

$$\mathsf{m} = \mathsf{U}(\mathsf{a})\, \mathsf{b} = \mathsf{V}(\mathsf{b})\, \mathsf{a}. \tag{13}$$

Note that $\mathsf{U}$ and $\mathsf{V}$ are basically the Jacobi matrices of $\mathsf{g}$.

### 5.1. Error Propagation

Suppose now that multivectors $A$ and $B$ cannot be known exactly. Instead only their expectation value, covariance and cross-covariance matrices are known. The question is then how general Clifford algebra operations can be performed while propagating the covariances of the initial multivectors.

In the following we will denote random variables by underlining the variable name. That is, $\underline{A}$ and $\underline{B}$ denote two random multivector variables with an embedding in $\mathbb{R}^m$ as $\Phi(\underline{A}) = \underline{\mathsf{a}} = [\underline{\alpha}^i, \ldots, \underline{\alpha}^m]^\mathsf{T}$ and $\Phi(\underline{B}) = \underline{\mathsf{b}} = [\underline{\beta}^i, \ldots, \underline{\beta}^m]^\mathsf{T}$. The expectation value of a random variable will be denoted by overlining the variable name and the expectation value operator will be denoted by $\mathcal{E}$. The covariance matrix of $\mathsf{a}$ and $\mathsf{b}$ will be denoted by $\Sigma_{\mathsf{a},\mathsf{b}}$. Given the expectation values $\bar{\mathsf{a}}$, $\bar{\mathsf{b}}$, the covariance matrices $\Sigma_{\mathsf{a},\mathsf{a}}$, $\Sigma_{\mathsf{b},\mathsf{b}}$, and the cross-covariance $\Sigma_{\mathsf{a},\mathsf{b}}$ of $\underline{\mathsf{a}}$ and $\underline{\mathsf{b}}$, we ask what the expectation and covariance matrix of a bilinear function $\mathsf{g}(\underline{\mathsf{a}}, \underline{\mathsf{b}})$ as defined in equation (12) is. By expanding $\mathsf{g}(\underline{\mathsf{a}}, \underline{\mathsf{b}})$ with a Taylor expansion about the expectation values of $\underline{\mathsf{a}}$ and $\underline{\mathsf{b}}$, we find that

$$\bar{\mathsf{m}} = \mathcal{E}[\mathsf{g}(\underline{\mathsf{a}}, \underline{\mathsf{b}})] = \mathsf{U}(\bar{\mathsf{a}})\,\bar{\mathsf{b}} + \mathrm{tr}((\mathsf{H}^k)^\mathsf{T}\,\Sigma_{\mathsf{a},\mathsf{b}})\mathsf{e}_k, \tag{14}$$

where $\mathsf{H}^k$ is the Hesse matrix of the $k^{th}$ component of $\mathsf{g}$, and $\mathrm{tr}(\mathsf{U})$ denotes the trace of a matrix $\mathsf{U}$. In this case the Hesse matrix is simply $\mathsf{H}^k = {g^k}_{ij}$. Note that in most cases the term containing the Hesse matrix will be negligible. By using the same Taylor expansion of $\mathsf{g}$ as before, it may be shown that the covariance matrix of $\mathsf{g}(\underline{\mathsf{a}}, \underline{\mathsf{b}})$ is approximately given by

$$\begin{aligned}\Sigma_{\mathsf{m},\mathsf{m}} = \quad & \mathsf{V}(\bar{\mathsf{b}})\,\Sigma_{\mathsf{a},\mathsf{a},}\,\mathsf{V}^\mathsf{T}(\bar{\mathsf{b}}) + \mathsf{U}(\bar{\mathsf{a}})\,\Sigma_{\mathsf{b},\mathsf{b},}\,\mathsf{U}^\mathsf{T}(\bar{\mathsf{a}}) \\ + \; & \mathsf{V}(\bar{\mathsf{b}})\,\Sigma_{\mathsf{a},\mathsf{b},}\,\mathsf{U}^\mathsf{T}(\bar{\mathsf{a}}) + \mathsf{U}(\bar{\mathsf{a}})\,\Sigma_{\mathsf{b},\mathsf{a},}\,\mathsf{V}^\mathsf{T}(\bar{\mathsf{b}}),\end{aligned} \tag{15}$$

where we neglected an additional term $\mathrm{tr}((\mathsf{H}^r)^\mathsf{T}\,\Sigma_{\mathsf{a},\mathsf{b}})\,\mathrm{tr}((\mathsf{H}^s)^\mathsf{T}\,\Sigma_{\mathsf{a},\mathsf{b}})$ for each element $\Sigma^{rs}_{\mathsf{m},\mathsf{m}}$. For most applications it may be assumed that this is a good approximation. Furthermore, the cross-covariance matrix of $\mathsf{g}(\underline{\mathsf{a}}, \underline{\mathsf{b}})$ and another random multivector variable $\underline{\mathsf{c}} \in \mathbb{R}^m$ is given by

$$\Sigma_{\mathsf{m},\mathsf{c}} = \mathsf{U}(\bar{\mathsf{b}})\,\Sigma_{\mathsf{a},\mathsf{c}} + \mathsf{V}(\bar{\mathsf{a}})\,\Sigma_{\mathsf{b},\mathsf{c}}. \tag{16}$$

Note that in the previous two equations the matrices $\mathsf{U}$ and $\mathsf{V}$ are the Jacobean matrices of the bilinear function $\mathsf{g}$. Equations (14), (15) and (16) do in fact suffice to do error propagation for any combination of Clifford algebra operations.

### 5.2. Conformal Space

For any expression we want to obtain in the Clifford algebra of conformal space $\mathcal{C}\ell(\mathbb{PK}^3)$, we can now use the equations presented in the previous section to implement error propagation. Nevertheless, the initial expectation values and covariance matrices will typically only be given for vectors in Euclidean space $\mathbb{E}^3$ and not for the corresponding embedded vectors in $\mathbb{PK}^3$. We therefore first have to do the error propagation for the embedding of a Euclidean random vector variable $\underline{\mathbf{x}} \in \mathbb{E}^3$ into conformal space, where we will denote the corresponding conformal random vector variable by $\underline{\mathbf{X}} \in \mathbb{PK}^3$. Note that while $\underline{\mathbf{x}}$ is

3-dimensional, $\underline{\mathbf{X}}$ is $(3+2)$-dimensional, and therefore also the corresponding covariance matrices will be of different dimensions. We find that the expectation value of $\underline{\mathbf{X}}$ is given by

$$\bar{\mathbf{X}} = \mathcal{E}[\mathcal{K}(\underline{\mathbf{x}})] = \bar{\mathbf{x}} + \tfrac{1}{2}\,\bar{\mathbf{x}}^2\,\mathsf{e}_\infty + \mathsf{e}_o \; + \tfrac{1}{2}\,\mathrm{tr}(\Sigma_{\mathbf{x},\mathbf{x}})\,\mathsf{e}_\infty, \tag{17}$$

where $\mathcal{K}$ is the function describing the embedding of a Euclidean vector in conformal space. The term $\mathrm{tr}(\Sigma_{\mathbf{x},\mathbf{x}})$ is typically very small and may be neglected. If we denote by $\mathsf{J}_\mathcal{K}(\bar{\mathbf{x}})$ the Jacobi matrix of $\mathcal{K}$ evaluated at $\bar{\mathbf{x}}$, then the covariance matrix $\Sigma_{\mathbf{X},\mathbf{X}}$ of $\underline{\mathbf{X}}$ is given in terms of the covariance matrix $\Sigma_{\mathbf{x},\mathbf{x}}$ of $\underline{\mathbf{x}}$ as

$$\Sigma_{\mathbf{X},\mathbf{X}} = \mathsf{J}_\mathcal{K}(\bar{\mathbf{x}})\,\Sigma_{\mathbf{x},\mathbf{x}}\,\mathsf{J}_\mathcal{K}^\mathsf{T}(\bar{\mathbf{x}}). \tag{18}$$

Denoting the components of $\bar{\mathbf{x}}$ by $\{\bar{\xi}^i\}$, the Jacobi matrix is in fact given by

$$\mathsf{J}_\mathcal{K}(\bar{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 & \bar{\xi}^1 & 0 \\ 0 & 1 & 0 & \bar{\xi}^2 & 0 \\ 0 & 0 & 1 & \bar{\xi}^3 & 0 \end{bmatrix}^\mathsf{T}. \tag{19}$$

The cross-covariance $\Sigma_{\mathbf{X},\mathbf{Y}}$ is simply given in terms of $\Sigma_{\mathbf{x},\mathbf{y}}$ as

$$\Sigma_{\mathbf{X},\mathbf{Y}} = \mathsf{J}_\mathcal{K}(\bar{\mathbf{x}})\,\Sigma_{\mathbf{x},\mathbf{y}}\,\mathsf{J}_\mathcal{K}^\mathsf{T}(\bar{\mathbf{y}}). \tag{20}$$

### 5.3. Evaluation of Circles

We mentioned earlier that in conformal space a circle may be represented by the outer product of three points, where a point is represented by a vector as given in equation (6). The problem we now want to discuss is, given three points in Euclidean space with associated covariance and cross-covariance matrices, what is the expected circle through these three points and what is its covariance matrix.

Let $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}} \in \mathbb{E}^3$ denote the expectation of three Euclidean vectors. Their corresponding covariance and cross-covariance matrices are $\Sigma_{\mathbf{x},\mathbf{x}}$, $\Sigma_{\mathbf{y},\mathbf{y}}$, $\Sigma_{\mathbf{z},\mathbf{z}}$, and $\Sigma_{\mathbf{x},\mathbf{y}}$, $\Sigma_{\mathbf{y},\mathbf{z}}$, $\Sigma_{\mathbf{z},\mathbf{x}}$. In section 5.2 we have shown how these three Euclidean vectors together with their covariance and cross-covariance matrices may be embedded in conformal space. The corresponding conformal vectors will be denoted by $\bar{\mathbf{X}}$, $\bar{\mathbf{Y}}$, $\bar{\mathbf{Z}}$ and the corresponding covariance and cross-covariance matrices likewise. Once this is done, we can use equations (14) and (15) to first evaluate $\bar{\mathbf{P}} = \mathcal{E}[\bar{\mathbf{X}} \wedge \bar{\mathbf{Y}}]$ and the corresponding $\Sigma_{\mathbf{P},\mathbf{P}}$. Then we use equation (16) to evaluate $\Sigma_{\mathbf{P},\mathbf{Z}}$. This then enables us to calculate $\bar{\mathbf{C}} = \mathcal{E}[\bar{\mathbf{P}} \wedge \bar{\mathbf{Z}}]$ and $\Sigma_{\mathbf{C},\mathbf{C}}$. We could, of course, also have evaluated the Jacobians directly for the trilinear product $\bar{\mathbf{X}} \wedge \bar{\mathbf{Y}} \wedge \bar{\mathbf{Z}}$ and then found the expectation and

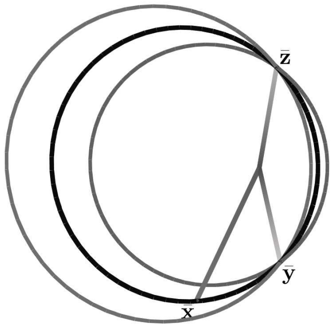


*Figure 1.* Standard deviation circles if vector $\bar{\mathbf{x}}$ has variance in only one component.

covariance. Note that the statistical relation between the components of the trivector $\bar{\mathbf{C}}$ are linear, while the relation between the actual radius, center and normal of the circle need not be. That is, due to the conformal embedding not only the representation of a circle is linearized but also the statistical relationship between its embedded components.

This allows us to very easily evaluate the standard deviation circles of the mean circle $\bar{\mathbf{C}}$. We do this by evaluating a singular value decomposition (SVD) on $\Sigma_{\mathbf{C},\mathbf{C}}$. The singular vectors that correspond to non-zero singular values give the principal components of $\Sigma_{\mathbf{C},\mathbf{C}}$, while the singular values give the variances along them. If $\bar{\mathbf{C}}$ were a point, the principal components would give the axes of an ellipsoid which represents the surface of standard deviation about this point. In the present case, where $\bar{\mathbf{C}}$ represents a circle, we have to draw for each point on the ellipsoid a circle. Hence, if $\Sigma_{\mathbf{C},\mathbf{C}}$ only has one principal component, we obtain two standard deviation circles as shown in figure 1. Here points $\bar{\mathbf{y}}$ and $\bar{\mathbf{z}}$ were held fixed and only point $\bar{\mathbf{x}}$ was taken to have a variance along one dimension. The central black circle is the mean circle which goes through all three points. The two gray circles are the ones that will occur with a likelihood of $\exp(-\frac{1}{2})$, i. e. they give the standard deviation from the mean.

If we now only hold point $\bar{\mathbf{z}}$ fixed and assume that $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ each have a variance in one dimension, then $\Sigma_{\mathbf{C},\mathbf{C}}$ has two principal components that give the axes of an ellipse. If we draw for each point on the ellipse one circle, we obtain the surface shown in figure 2. That is, each circle on the surface has a probability of $\exp(-\frac{1}{2})$ to occur. How the actual circle parameter may be extracted from a trivector $\mathbf{C} \in \mathcal{C}\ell(\mathbb{PK}^3)$ that represents it, may be found in some detail, for example, in [16]. Only a short overview will be given here.

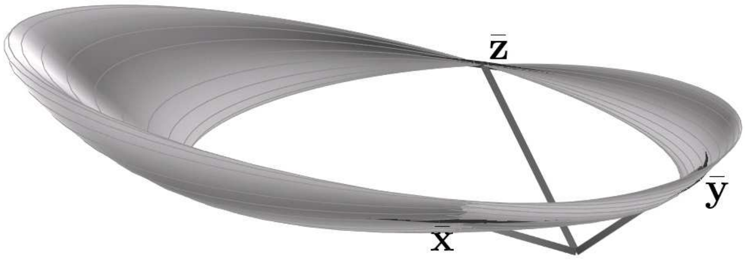


*Figure 2.* Standard deviation surface if vectors $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ have each a variance in only one component.

First of all, evaluate $\mathbf{L} = \mathbf{C} \cdot \mathsf{e}_\infty$ and $\mathbf{A} = \mathbf{C} \wedge \mathsf{e}_\infty$. It turns out that $\mathbf{L}$ represents w.r.t. the inner product null space, the line through the center of the circle with direction perpendicular to the plane the circle lies in. $\mathbf{A}$ represents w.r.t. the outer product null space, the plane the circle lies in. The intersection of $\mathbf{A}$ and $\mathbf{L}$ may simply be evaluated by $\mathbf{P} = \mathbf{L} \cdot \mathbf{A}$, whence $\mathbf{P}$ is of the form $\mathbf{P} = \mathbf{X} \wedge \mathsf{e}_\infty$, if $\mathbf{X}$ gives the center of the circle. The normal of the plane the circle lies in is given by $\mathbf{N} = \mathbf{L} \cdot (\mathsf{e}_3 \wedge \mathsf{e}_2 \wedge \mathsf{e}_1)$. However, $\mathbf{N}$ still has to be normalized, since its magnitude is related to the radius of the circle. The radius $r$ can simply be evaluated by $r^2 = -(\mathbf{C} \cdot \mathbf{C})/(\mathbf{A} \cdot \mathbf{A})$. In fact, $\mathbf{S} = \mathbf{C}/\mathbf{A}$ results in a vector of the same form as the vector representing a sphere in section 3. The center and radius of $\mathbf{S}$ are then the same as those of the circle $\mathbf{C}$. Note that error propagation can be applied to all of the above calculations, such that expectation values and covariance matrices are available for all of these properties.

### 5.4. Evaluation of Conics

Constructing a conic from five uncertain points in $\mathbb{D}^2$ is very similar to constructing a circle from three uncertain points in conformal space $\mathbb{PK}^3$. We assume that we are given five points in $\mathbb{R}^2$, each with an associated covariance matrix. These are embedded in $\mathbb{D}^2$ using standard error propagation.

Let $\mathcal{D}$ again denote the function embedding vectors from $\mathbb{R}^2$ in $\mathbb{D}^2$. A random vector variable $\underline{\mathbf{x}} \in \mathbb{R}^2$ is embedded in $\mathbb{D}^2$ via $\underline{\mathbf{X}} = \mathcal{D}(\underline{\mathbf{x}})$. The expectation value of $\underline{\mathbf{X}}$ is then given by $\bar{\mathbf{X}} = \mathcal{E}[\mathcal{D}(\underline{\mathbf{x}})] \approx \mathcal{D}(\bar{\mathbf{x}})$. If we denote by $\mathsf{J}_\mathcal{D}(\bar{\mathbf{x}})$ the Jacobi matrix of $\mathcal{D}$ evaluated at $\bar{\mathbf{x}}$, then the covariance matrix $\Sigma_{\mathbf{X},\mathbf{X}}$ of $\underline{\mathbf{X}}$ is given in terms of the covariance matrix $\Sigma_{\mathbf{x},\mathbf{x}}$ of $\underline{\mathbf{x}}$ as

$$\Sigma_{\mathbf{X},\mathbf{X}} = \mathsf{J}_\mathcal{D}(\bar{\mathbf{x}})\, \Sigma_{\mathbf{x},\mathbf{x}}\, \mathsf{J}_\mathcal{D}^\mathsf{T}(\bar{\mathbf{x}}). \tag{21}$$

Denoting the components of $\bar{\mathbf{x}}$ by $\{\bar{\xi}^i\}$, the Jacobi matrix is given by

$$\mathsf{J}_{\mathcal{D}}(\bar{\mathbf{x}}) = \begin{bmatrix} 1 & 0 & 0 & \sqrt{2}\bar{\xi}^1 & 0 & \bar{\xi}^2 \\ 0 & 1 & 0 & 0 & \sqrt{2}\bar{\xi}^2 & \bar{\xi}^1 \end{bmatrix}^{\mathsf{T}} . \tag{22}$$

The cross-covariance $\Sigma_{\mathbf{X},\mathbf{Y}}$ is simply given in terms of $\Sigma_{\mathbf{x},\mathbf{y}}$ as

$$\Sigma_{\mathbf{X},\mathbf{Y}} = \mathsf{J}_{\mathcal{D}}(\bar{\mathbf{x}})\,\Sigma_{\mathbf{x},\mathbf{y}}\,\mathsf{J}_{\mathcal{D}}^{\mathsf{T}}(\bar{\mathbf{y}}). \tag{23}$$

Figure 3 shows an example for such a construction. Given are five points, of which two have a non-zero covariance matrix indicated by small black bars. Taking the outer product of these five points after having them embedded in $\mathbb{D}^2$, we can evaluate the mean conic, represented as black conic, and also the covariance matrix of the conic. In this case the covariance matrix is of rank 2, which generates a whole set of conics that have probability $\exp(-\frac{1}{2})$ of a occurring, represented by the gray conics. It can be seen that the area swept by this set of "standard deviation conics" has a highly non-linear shape. Nevertheless, this surface is represented by the covariance matrix of the conic in $\mathbb{D}^2$.

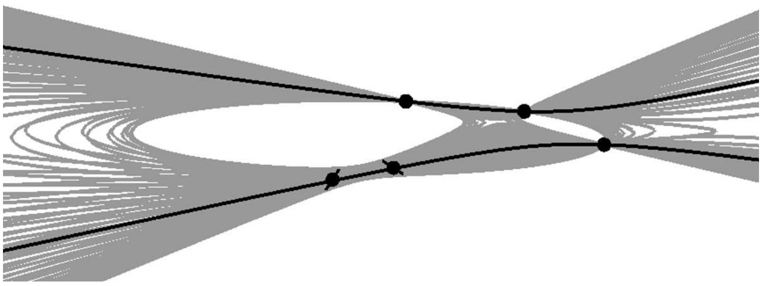

*Figure 3.* Standard deviation conics if two of the five points have rank 1 covariance matrices (indicated by small black bars).

## 6. Fitting of Circles and Conics to Data

There is also a linear solution to find the best circle that passes through a set of points in $\mathbb{E}^3$, or the best conic that passes through a set of points in $\mathbb{E}^2$, in a least squares sense. This follows directly from equation (13). In both cases the entities we would like to evaluate can be calculated from a set of linear constraint equations. In conformal space we can in this way extend the method given, for example, in [5, 1] for fitting circles in 2D-Euclidean space, to 3D-Euclidean space.

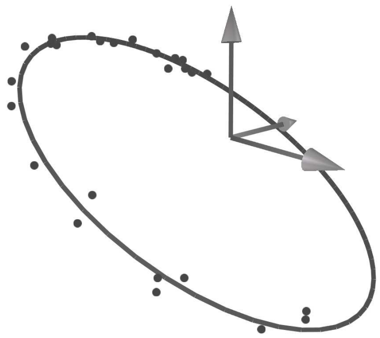

*Figure 4.* Result of linear fitting of a circle to a set of slightly scattered points in $\mathbb{E}^3$.

In the space of conics $\mathbb{D}^2$, it is not only possible to fit conics but also the intersection of conics to data by solving a linear system of equations. For example, if the data consists of four clusters of points, then fitting the intersection of two conics to the data will return a point quadruplet whose points specify the centers of the clusters.

### 6.1. Fitting in Conformal Space

Here is a short description of how a circle may be fitted to a set of 3D-points using the Clifford algebra of conformal space. It was mentioned earlier that if a point $\mathbf{X}$ lies on a circle $\mathbf{C}$, then $\mathbf{X} \wedge \mathbf{C} = 0$. If we write $\mathsf{c} = \Phi(\mathbf{C})$ and $\mathsf{x} = \Phi(\mathbf{X})$, then this condition can be written as $\mathsf{U}(\mathsf{x})\,\mathsf{c} = \mathbf{0}$. That is, $\mathsf{c}$ lies in the null space of the matrix $\mathsf{U}(\mathsf{x})$. Given a set of points $\{\mathsf{x}_1, \ldots, \mathsf{x}_k\}$ that all have to lie on a circle, we can define a matrix $\mathsf{W}$ that contains the set of matrices $\{\mathsf{U}(\mathsf{x}_1), \ldots, \mathsf{U}(\mathsf{x}_k)\}$ stacked on top of each other. The condition a circle passing through all these points then has to satisfy becomes $\mathsf{W}\mathsf{c} = \mathbf{0}$. We could now simply find the null space of $\mathsf{W}$ using a SVD. However, this would give the subspace of multivectors and not trivectors that satisfy the constraint. Therefore, we first remove those columns from $\mathsf{W}$ that are not related to trivector components and only then find the null space. Since a SVD gives the best solution in a least squares sense, we should obtain a fairly good solution for the best circle fit, even though we have not taken the covariance matrix of the $\{\mathsf{x}_i\}$ into account. As discussed in [2], this simple method is therefore only likely to supply a good initial guess for an iterative algorithm [10]. Figure 4 shows an example of a circle fitted to a set of artificially generated noisy 3D-points using this method. Of course, any linear regression method, as for example the Gauss-Helmert model may be applied here.

### 6.2. Fitting in the Space of Conics

In order to fit a conic to a set of points, the same method as above can be used, only this time in $\mathcal{C}\ell(\mathbb{D}^2)$. A conic $\mathbf{C} \in \mathcal{C}\ell(\mathbb{D}^2)$ is represented by the outer product of five points, and any point represented by a vector $\mathbf{X} \in \mathbb{D}^2$ that lies on the conic satisfies $\mathbf{X} \wedge \mathbf{C} = 0$. The dual representation of a conic in $\mathcal{C}\ell(\mathbb{D}^2)$ is a vector. This vector can be evaluated by the dual operation in Clifford algebra. Writing the dual of $\mathbf{C}$ as $\mathbf{C}^*$, the constraint a point satisfies when it lies on the conic is $\mathbf{X} \cdot \mathbf{C}^* = 0$. Writing this constraint again as $\mathsf{U}(\Phi(\mathbf{X}))\,\Phi(\mathbf{C}) = 0$ allows us to apply the same method we used for circles to evaluate the conic.

As mentioned before, this way of fitting a conic to data is well known. However, using the Clifford algebra representation, we can use the same linear approach to fit any entity that can be represented in the algebra to any other representable entity. For example, the outer product of two vectors $\mathbf{X}, \mathbf{Y} \in \mathbb{D}^2$ representing points in $\mathbb{R}^2$, represents this pair of points. Hence, we can also fit a conic to point pairs. Maybe more interestingly, we can also fit point pairs to a set of data points. Since the outer product of four vectors in $\mathcal{C}\ell(\mathbb{D}^2)$ represents a point quadruplet, it is also possible to fit a point quadruplet to a set of points. An example of this is shown in figure 5. In $\mathcal{C}\ell(\mathbb{D}^2)$ a point quadruplet can also be regarded as the intersection of two conics, since for every point quadruplet there exists a whole pencil of conics who all intersect in the same four points. For better visualization two conics of such a pencil are drawn in figure 5. As can be seen, the two conics intersect more or less in the centers of the four clusters. Hence, $\mathcal{C}\ell(\mathbb{D}^2)$ offers a simple, linear method to find the centers of up to four clusters in a set of data points.

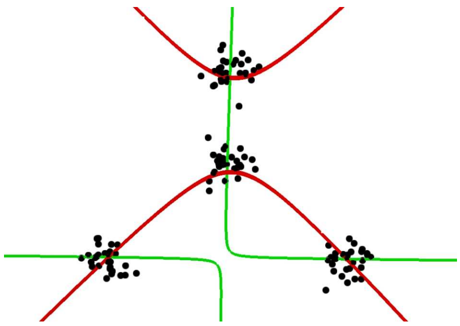

*Figure 5.* Result of linear fitting of a point quadruplet represented as the intersection of two conics to a set of scattered points in $\mathbb{E}^2$.

Figure 6 shows the result of fitting point quadruplets to line segment structures, which are of particular interest in computer vision problems.

It can be seen that the two conics drawn in each example intersect on the line structures in such a way that the structures may be further analyzed. Potentially this offers a method to distinguish between junctions and corners in images and also to evaluate the opening angle and orientation of corners. This will be investigated further in future research.

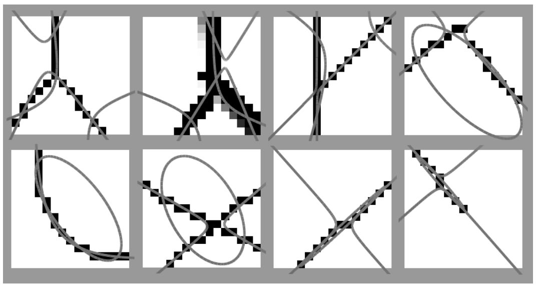

*Figure 6.* Fitting point quadruplets (represented as intersections of two conics) to line segment structures.

## 7. Conclusions

In this text we presented a method of constructing circles in 3D-Euclidean space and conics in 2D-Euclidean space from a number of uncertain points using error propagation methods. The main advantage of representing circles in $\mathbb{E}^3$ and conics in $\mathbb{E}^2$ through elements of a Clifford algebra, is that this representation is (multi-)linear. This allows us to employ standard error propagation methods to find the mean circle through three points or the mean conic through five points and also their respective covariance matrices. These covariance matrices may then also be used to visualize the standard deviation of the circle and conics, respectively. In this setting it is also possible to extend the well known linear model of fitting circles in 2D-Euclidean space, as presented, for example, in [5, 1], to 3D-Euclidean space. Furthermore, it is possible to fit the intersection of conics, which may be point quadruplets, triplets, doublets or single points, to sets of data vectors. In future work we will investigate the application of standard statistical estimation models to this problem. Another topic of investigation is to develop statistical methods in Clifford algebra to test, for example, whether a line intersects a circle (conic), or whether a point lies on a circle (conic), etc. Note that a software tool called CLUCalc is available from `www.clucalc.info`, for investigating and visualizing the Clifford algebra expressions and their error propagation as presented here.

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