THE TWIST REPRESENTATION OF FREE-FORM OBJECTS *

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Abstract. We give a contribution to the representation problem of free-form curves and surfaces. Our proposal is an operational or kinematic approach based on the Lie group $SE(3)$. While in Euclidean space the modelling of shape as orbit of a point under the action of $SE(3)$ is limited, we are embedding our problem into the conformal geometric algebra $\mathbb{R}_{4,1}$ of the Euclidean space $\mathbb{R}^3$. This embedding results in a number of advantages which makes the proposed method a universal and flexible one with respect to applications. It makes possible the robust and fast estimation of the pose of 3D objects from incomplete and noisy image data. Especially advantagous is the equivalence of the proposed shape model to that of the Fourier representations.

Key words: shape representation, conformal geometric algebra, Lie algebra, Fourier transform, free-form curves, free-form surfaces, motor, twist

1. Introduction

Two objects can be said to have the same shape if they are similar in the sense of Euclidean geometry. By leaving out the property of scale invariance, we can define the shape of an object as that geometric concept that is invariant under the special Euclidean group. Furthermore, we allow our objects to change their shape in a well-defined manner under the action of some external forces.

* This work has been partially supported (G.S. and C.P.) by EC Grant IST-2001-3422 (VISATEC), by DFG Grant RO 2497/1-1 (B.R.), and by DFG Graduiertenkolleg No. 357 (B.R. and C.P.).

The literature on shape modelling and applications is vast. May it be visualization and animation in computer graphics or shape and motion recognition in computer vision. The central problem for the usefulness in either field is the chosen representation of shape.

Here we present a new approach to the modelling of free-form shape of curves and surfaces which has some features that make it especially attractive for computer vision and computer graphics. In our applications of pose estimation of 3D objects we could easily handle incomplete and noisy image data for numerically stable estimations with nearly video real-time capability.

That new representation results from the fusion of two concepts:

1) Free-form curves and surfaces are modelled as the orbit of a point under the action of the Lie group $SE(3)$, caused by a set of coupled infinitesimal generators of the group, called twists (Murray et al., 1994).
2) These object models are embedded in the conformal geometric algebra (CGA) of the Euclidean space $\mathbb{R}^3$ (Li et al., 2001), that is $\mathbb{R}_{4,1}$. Only in conformal geometry the above mentioned modelling of shape unfolds its rich set of useful features.

The concept of fusing a local with a global algebraic framework has been proposed already in (Sommer, 1997). But only the pioneering work in (Li et al., 2001) made it feasible to consider the Lie algebra $se(3)$, the space of tangents to an object, embedded in $\mathbb{R}_{4,1}$, as the source of our shape model instead of using $se(3)$ in $\mathbb{R}^3$.

The tight relations of geometry and kinematics are known to the mathematicians for centuries, see e.g. (Farouki, 2000). But in contrast to most applications in mechanical engineering we are not restricted in our approach by physically feasible motions nor will we get problems in generating spatial curves or surfaces.

By embedding our design method into CGA, both primitive geometric entities as points or objects on the one side and actions on the other side will have algebraic representations in one single framework. Furthermore, objects are defined by actions, and also actions can take on the role of operands.

Our proposed kinematic definition of shape uses infinitesimal actions to generate global patterns of low intrinsic dimension. This phenomen corresponds to the interpretation of the special Euclidean group in CGA, $SE(3)$, as a Lie group, where an element $g \in SE(3)$ performs a transformation of an entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$,

$$\underline{\boldsymbol{u}}' = \underline{\boldsymbol{u}}(\theta) = g\{\underline{\boldsymbol{u}}(0)\} \quad (1)$$

with respect to the parameter θ of g. Any special $g \in SE(3)$ that represents a general rotation in CGA corresponds to a Lie group operator $\boldsymbol{M} \in \mathbb{R}^+_{4,1}$ which is called a motor and which is applied by the bilinear spinor product

$$\underline{\boldsymbol{u}}' = \boldsymbol{M}\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}}, \tag{2}$$

where $\widetilde{\boldsymbol{M}}$ is the reverse of $\boldsymbol{M}$. This product indicates that $\boldsymbol{M}$ is an orthogonal operator. If g is an element of the Lie group $SE(3)$, than its infinitesimal generator, ξ, is defined in the corresponding Lie algebra, that is $\xi \in se(3)$. That Lie algebra element of the rigid body motion is geometrically interpreted as the rotation axis $\underline{\boldsymbol{l}}$ in conformal space. Then the motor $\boldsymbol{M}$ results from the exponential map of the generator $\underline{\boldsymbol{l}}$ of the group element, which is called a twist:

$$\boldsymbol{M} = \exp\left(-\frac{\theta}{2}\underline{\boldsymbol{l}}\right). \tag{3}$$

While θ is the rotation angle as the parameter of the motor, its generator is defined by the five degrees of freedom of a line $\underline{\boldsymbol{l}}$ in space.

In our approach, the motor $\boldsymbol{M}$ is the effective operator which causes arbitrarily complex object shape. This operator may result from the multiplicative coupling of a set of primitive motors $\{\boldsymbol{M}_i | i = n, ..., 1\}$,

$$\boldsymbol{M} = \boldsymbol{M}_n\boldsymbol{M}_{n-1}...\boldsymbol{M}_2\boldsymbol{M}_1. \tag{4}$$

Each of these motors $\boldsymbol{M}_i$ is representing a circular motion of a point around its own axis.

Based on that approach rather complex free-form objects can be designed which behave as algebraic entities. That means, they can be transformed by motors in a covariant and linear way. To handle complete objects in that way as unique entities makes sense from both a cognitive and a numeric point of view.

The conformal geometric algebra $\mathbb{R}_{4,1}$ makes this possible. This is caused by two essential facts. First, the representation of the special Euclidean group $SE(3)$ in $\mathbb{R}_{4,1}$ as a subgroup of the conformal group $C(3)$ is isomorphic to the special orthogonal group $SO^+(4,1)$. Hence, rigid body motion can be performed as rotation in CGA and therefore has a covariant representation. Second, the basic geometric entity of the conformal geometric algebra of the Euclidean space is the sphere. All geometric entities derived by incidence operations from the sphere can be transformed in CGA by an element $g \in SE(3)$, that is a motor $\boldsymbol{M} \in \mathbb{R}^+_{4,1}$, in the same linear way, just as a point in the homogeneous Euclidean space $\mathbb{R}^4$. Because there exists a dual representation of a sphere (and of all derived entities) in CGA, which considers points as the basic geometric entity of the Euclidean space in the conformal space, all the known concepts from Euclidean space can be transformed to the conformal one.

Finally, we can take advantage of the stratification of spaces by CGA. Since the seminal paper (Faugeras, 1995) the purposive use of stratified geometries became an important design principle of vision systems. This means that an observer in dependence of its possibilities and needs can have access to different geometries as projective, affine or metric ones. So far this could hardly be realized. In CGA we have quite another situation.

The CGA $\mathbb{R}_{4,1}$ is a linear space of dimension 32. This mighty space represents not only conformal geometry but also affine geometry. Note that the special Euclidean group is a special affine group. Because $\mathbb{R}_{4,1}$ is derived from the Euclidean space $\mathbb{R}^3$, it encloses also Euclidean geometry, which is represented by the geometric algebra $\mathbb{R}_{3,0}$. In addition, the projective geometric algebra $\mathbb{R}_{3,1}$ is enclosed in $\mathbb{R}_{4,1}$. Thus, we have the stratification of the geometric algebras $\mathbb{R}_{3,0} \subset \mathbb{R}_{3,1} \subset \mathbb{R}_{4,1}$. This enables to consider metric (Euclidean), projective and kinematic (affine) problems in one single algebraic framework.

2. Rigid Body Motion in Conformal Geometric Algebra

After giving a bird's eye view on the construction of a geometric algebra and on the features of the conformal geometric algebra, we will present the possibilities of representing the rigid body motion in CGA.

2.1. SOME CONSTRUCTIVE PRINCIPLES OF A GEOMETRIC ALGEBRA

A geometric algebra (GA) $\mathbb{R}_{p,q,r}$ is a linear space of dimension 2^n, $n = p + q + r$, which results from a vector space $\mathbb{R}^{p,q,r}$. We call (p,q,r) the signature of the vector space of dimension n. This indicates that there are $p/q/r$ unit vectors $\boldsymbol{e}_i$ which square to $+1/-1/0$, respectively. While $n = p$ in case of the Euclidean space $\mathbb{R}^3$, $\mathbb{R}^{p,q,r}$ indicates a vector space with a metric different than the Euclidean one. In the case of $r \neq 0$ there is a degenerate metric. We will omit the signature indexes from right if the interpretation is unique, as in the case of $\mathbb{R}^3$.

The basic product of a GA is the geometric product, indicated by juxtaposition of the operands. This product is associative and anticommutative. There can be used a lot of other product forms in CA too, as the outer product ($\wedge$) and the inner product ($\cdot$).

The space $\mathbb{R}_{p,q,r}$ is spanned by a set of 2^n linear subspaces of different grade called blades. Giving the blades a geometric interpretation makes the difference of a GA from a Clifford algebra. A blade of grade k, a k-blade $\boldsymbol{B}_{\langle k \rangle}$, results from the outer product of k independent vectors $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_k\} \in$

$\mathbb{R}^{p,q,r} \equiv \langle \mathbb{R}_{p,q,r} \rangle_1$,

$$\boldsymbol{B}_{\langle k \rangle} = \boldsymbol{a}_1 \wedge ... \wedge \boldsymbol{a}_k = \langle \boldsymbol{a}_1 ... \boldsymbol{a}_k \rangle_k, \tag{5}$$

where $\langle \cdot \rangle$ is the grade operator. There are $l_k = \binom{n}{k}$ different blades of grade k, $\boldsymbol{B}_{\langle k \rangle j}, j = 1, ..., l_k$. If $\boldsymbol{e}_0 \in \mathbb{R}_{p,q,r}$, $\boldsymbol{e}_0 \equiv 1$, is the unit scalar element and $\boldsymbol{e}_{1...n} \in \mathbb{R}_{p,q,r}$, $\boldsymbol{e}_{1...n} \equiv \boldsymbol{e}_1 ... \boldsymbol{e}_n \equiv \boldsymbol{I}$, is the unit pseudoscalar element of the GA, then $\boldsymbol{B}_{\langle 0 \rangle}$ is the scalar blade and $\boldsymbol{B}_{\langle n \rangle} \equiv \boldsymbol{I}$ is the pseudoscalar blade. Hence, $\sum_{k=0}^{n} l_k = 2^n$ is the dimension of the GA. Blades are directed numbers, thus $\boldsymbol{I}_{\langle k \rangle} = \boldsymbol{e}_{i_1} \wedge ... \wedge \boldsymbol{e}_{i_k}$ gives the direction of a blade. Any linear combination

$$\boldsymbol{A}_k = \sum_{j=1}^{l^*} \alpha_j \boldsymbol{B}_{\langle k \rangle j} \qquad , \quad l^* \leq l_k \ , \ \alpha_j \in \mathbb{R} \tag{6}$$

is called a k-vector, $\boldsymbol{A}_k \in \langle \mathbb{R}_{p,q,r} \rangle_k$. This rich structure of a GA can be further increased by the linear combination of k-vectors,

$$\boldsymbol{A} = \sum_{k=k_*}^{k^*} \beta_k \boldsymbol{A}_k \qquad , \quad 0 \leq k_* < k^* \leq n \ , \ \beta_k \in \mathbb{R} \tag{7}$$

Here $\boldsymbol{A}$ is called a (general) multivector. It is composed of components of different grade. The multivector may result from the geometric product of an r-vector $\boldsymbol{A}_r$ with an s-vector $\boldsymbol{B}_s$,

$$\boldsymbol{A} = \boldsymbol{A}_r \boldsymbol{B}_s = \langle \boldsymbol{A}_r \boldsymbol{B}_r \rangle_{|r-s|} + \langle \boldsymbol{A}_r \boldsymbol{B}_s \rangle_{|r-s|+2} + ... + \langle \boldsymbol{A}_r \boldsymbol{B}_s \rangle_{r+s} \tag{8}$$

with the pure inner product

$$\boldsymbol{A}_r \cdot \boldsymbol{B}_s = \langle \boldsymbol{A}_r \boldsymbol{B}_s \rangle_{|r-s|} \tag{9}$$

and the pure outer product

$$\boldsymbol{A}_r \wedge \boldsymbol{B}_s = \langle \boldsymbol{A}_r \boldsymbol{B}_s \rangle_{r+s}. \tag{10}$$

All other components of $\boldsymbol{A}$ result from a mixture of inner and outer products. The product of two multivectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, can always be decomposed in the sum of an even and an odd component,

$$\boldsymbol{AB} = \frac{1}{2}(\boldsymbol{AB} + \boldsymbol{BA}) + \frac{1}{2}(\boldsymbol{AB} - \boldsymbol{BA}). \tag{11}$$

In the case of the product of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}, \boldsymbol{b} \in \langle \mathbb{R}_{p,q,r} \rangle_1$, we get

$$\boldsymbol{ab} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) + \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{12}$$

$$= \langle \boldsymbol{ab} \rangle_0 + \langle \boldsymbol{ab} \rangle_2 = \alpha + \boldsymbol{A}_2 \tag{13}$$

with $\alpha \in \langle \mathbb{R}_{p,q,r} \rangle_0$ and $\boldsymbol{A}_2 \in \langle \mathbb{R}_{p,q,r} \rangle_2$.

An important concept of a GA is that of duality. This means that it is possible to change the blade base of a multivector $\boldsymbol{A} \in \mathbb{R}_{p,q,r}$. Its dual is written as $\boldsymbol{A}^*$ and is defined as

$$\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{I}^{-1}, \tag{14}$$

where $\boldsymbol{I}$ is the unit pseudoscalar of $\mathbb{R}_{p,q,r}$. In the case where $\boldsymbol{A}_k \in \langle \mathbb{R}_{p,q,r} \rangle_k$ the dual is given by $\boldsymbol{A}_k^* = \boldsymbol{A}_{n-k} \in \langle \mathbb{R}_{p,q,r} \rangle_{n-k}$. The duality expresses the relations between the inner product null space, IPNS, and the outer product null space, OPNS, of a multivector, see (Perwass and Hildenbrand, 2003). The OPNS defines a collinear subspace of dimension k to a k-blade $\boldsymbol{B}_{\langle k \rangle} \subset \mathbb{R}_{p,q,r}$ which is given by all $\boldsymbol{x} \in \mathbb{R}^{p,q,r}$ so that

$$\boldsymbol{x} \wedge \boldsymbol{B}_{\langle k \rangle} = 0. \tag{15}$$

The IPNS defines a subspace of $\mathbb{R}_{p,q,r}$ which is orthogonal to a k-blade $\boldsymbol{B}_{\langle k \rangle} \subset \mathbb{R}_{p,q,r}$ and, hence

$$\boldsymbol{x} \cdot \boldsymbol{B}_{\langle k \rangle} = 0. \tag{16}$$

2.2. CGA OF THE EUCLIDEAN SPACE

The conformal geometry of Euclidean and non-Euclidean spaces is known for a long time (Yaglom, 1988) without giving strong impact on the modelling in engineering with the exception of electrical engineering. There are different representations of the conformal geometry. Most disseminated is a complex formulation (Needham, 1997). Based on an idea in (Hestenes, 1984), in (Li et al., 2001) and in two other papers of the same authors in (Sommer, 2001), the conformal geometries of the Euclidean, spherical and hyperbolic spaces have been worked out in the framework of GA.

The basic approach is that a conformal geometric algebra (CGA) $\mathbb{R}_{p+1,q+1}$ is built from a pseudo-Euclidean space $\mathbb{R}^{p+1,q+1}$. If we start with an Euclidean space $\mathbb{R}^n$, the construction $\mathbb{R}^{n+1,1} = \mathbb{R}^n \oplus \mathbb{R}^{1,1}$, $\oplus$ being the direct sum, uses a plane with Minkowski signature for augmenting the basis of $\mathbb{R}^n$ by the additional basis vectors $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ with $\boldsymbol{e}_+^2 = 1$ and $\boldsymbol{e}_-^2 = -1$. Because that model can be interpreted as a homogeneous stereographic projection of all points $\boldsymbol{x} \in \mathbb{R}^n$ to points $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$, this space is called the homogeneous model of $\mathbb{R}^n$. Furthermore, by replacing the basis $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ with the basis $\{\boldsymbol{e}, \boldsymbol{e}_0\}$, the homogeneous stereographic representation will become a representation of null vectors. This is caused by the properties $\boldsymbol{e}^2 = \boldsymbol{e}_0^2 = 0$ and $\boldsymbol{e} \cdot \boldsymbol{e}_0 = -1$. The relation between the null basis $\{\boldsymbol{e}, \boldsymbol{e}_0\}$ and the basis $\{\boldsymbol{e}_+, \boldsymbol{e}_-\}$ is given by

$$\boldsymbol{e} := (\boldsymbol{e}_- + \boldsymbol{e}_+) \ \text{ and } \ \boldsymbol{e}_0 := \frac{1}{2}(\boldsymbol{e}_- - \boldsymbol{e}_+). \tag{17}$$

Any point $\boldsymbol{x} \in \mathbb{R}^n$ transforms to a point $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ according to

$$\underline{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^2\boldsymbol{e} + \boldsymbol{e}_0 \tag{18}$$

with $\underline{\boldsymbol{x}}^2 = 0$. In fact, any point $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ is lying on an n-dimensional subspace $N_e^n \subset \mathbb{R}^{n+1,1}$, called horosphere (Li et al., 2001). The horosphere is a non-Euclidean model of the Euclidean space $\mathbb{R}^n$.

It must be mentioned that the basis vectors $\boldsymbol{e}$ and $\boldsymbol{e}_0$ have a geometric interpretation. In fact, $\boldsymbol{e}$ corresponds the north pole and $\boldsymbol{e}_0$ corresponds the south pole of the hypersphere of the stereographic projection, embedded in $\mathbb{R}^{n+1,1}$. Thus, $\boldsymbol{e}$ is representing the points at infinity and $\boldsymbol{e}_0$ is representing the origin of $\mathbb{R}^n$ in the space $\mathbb{R}^{n+1,1}$.

By setting apart these two points from all others of the $\mathbb{R}^n$ makes $\mathbb{R}^{n+1,1}$ a homogeneous space in the sense that each $\underline{\boldsymbol{x}} \in \mathbb{R}^{n+1,1}$ is a homogeneous null vector without having reference to the origin. This enables coordinate-free computing to a large extent. Hence, $\underline{\boldsymbol{x}} \in N_e^n$ constitutes an equivalence class $\{\lambda\underline{\boldsymbol{x}}, \lambda \in \mathbb{R}\}$ on the horosphere. The reduction of that equivalence class to a unique entity with metrical equivalence to the point $\boldsymbol{x} \in \mathbb{R}^n$ needs a normalization.

The CGA $\mathbb{R}_{4,1}$, derived from the Euclidean space $\mathbb{R}^3$, offers 32 blades as basis of that linear space. This rich structure enables one to represent low order geometric entities in a hierarchy of grades. These entities can be derived as solutions of either the IPNS or the OPNS depending on what we assume as the basis geometric entity of the conformal space, see (Perwass and Hildenbrand, 2003). So far we only considered the mapping of an Euclidean point $\boldsymbol{x} \in \mathbb{R}^3$ to a point $\underline{\boldsymbol{x}} \in N_e^3 \subset \mathbb{R}^{4,1}$. But the null vectors on the horosphere are only a special subset of all the vectors of $\mathbb{R}^{4,1}$. All the vectors of $\mathbb{R}^{4,1}$ are representing spheres as the basic entities of the conformal space. A sphere $\underline{\boldsymbol{s}} \in \mathbb{R}^{4,1}$ is defined by its center position, $\boldsymbol{c} \in \mathbb{R}^3$, and its radius $\rho \in \mathbb{R}$ according to

$$\underline{\boldsymbol{s}} = \boldsymbol{c} + \frac{1}{2}(\boldsymbol{c} - \rho)^2\boldsymbol{e} + \boldsymbol{e}_0. \tag{19}$$

And because $\underline{\boldsymbol{s}}^2 = \rho^2 > 0$, it must be a non-null vector. A point $\underline{\boldsymbol{x}} \in N_e^3$ can be considered as a degenerate sphere of radius zero. Hence, spheres $\underline{\boldsymbol{s}}$ and points $\underline{\boldsymbol{x}}$ are entities of grade 1. By taking the outer product of spheres $\underline{\boldsymbol{s}}_i$, other entities of higher grade can be constructed. So we get a circle $\underline{\boldsymbol{z}}$ (grade 2), which exists outside the null cone in $\mathbb{R}^{4,1}$,

$$\underline{\boldsymbol{z}} \;=\; \underline{\boldsymbol{s}}_1 \wedge \underline{\boldsymbol{s}}_2 \tag{20}$$

as solution of the IPNS. If we consider the OPNS on the other hand, we are starting with points $\underline{\boldsymbol{x}}_i \in N_e^3$ and can proceed similarly to define a circle $\underline{\boldsymbol{Z}}$

and a sphere $\underline{\boldsymbol{S}}$ as entities of grade 3 and 4 derived from points $\underline{\boldsymbol{x}}_i$ on the null cone of $\mathbb{R}_{4,1}$ according to

$$\underline{\boldsymbol{Z}} = \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \wedge \underline{\boldsymbol{x}}_3 \tag{21}$$

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{x}}_1 \wedge \underline{\boldsymbol{x}}_2 \wedge \underline{\boldsymbol{x}}_3 \wedge \underline{\boldsymbol{x}}_4. \tag{22}$$

These sets of entities are obviously related by the duality $\underline{\boldsymbol{u}}^* = \underline{\boldsymbol{U}}$. Finally,

$$\underline{\boldsymbol{X}} = \boldsymbol{e} \wedge \underline{\boldsymbol{x}}$$

is called the affine representation of a point (Li et al., 2001). This representation of a point is used if the interplay of the projective with the conformal representation is of interest in applications as in (Rosenhahn, 2003). With respect to lines $\underline{\boldsymbol{l}}$ and planes $\underline{\boldsymbol{p}}$ or $\underline{\boldsymbol{L}}$ and $\underline{\boldsymbol{P}}$ we refer the reader to (Sommer et al., 2004).

Let us come back to the stratification of spaces mentioned in Section 1. Let be $\boldsymbol{x} \in \mathbb{R}^n$ a point of the Euclidean space, $\boldsymbol{X} \in \mathbb{R}^{n,1}$ a point of the projective space and $\underline{\boldsymbol{X}} \in \mathbb{R}^{n+1,1}$ a point of the conformal space. Then the operations which transform the representation between the spaces are for $\mathbb{R}_3 \longrightarrow \mathbb{R}_{3,1} \longrightarrow \mathbb{R}_{4,1}$

$$\underline{\boldsymbol{X}} = \boldsymbol{e} \wedge \boldsymbol{X} = \boldsymbol{e} \wedge (\boldsymbol{x} + \boldsymbol{e}_-), \tag{23}$$

and for $\mathbb{R}_{4,1} \longrightarrow \mathbb{R}_{3,1} \longrightarrow \mathbb{R}_3$

$$\boldsymbol{x} = -\frac{\boldsymbol{X}}{\boldsymbol{X} \cdot \boldsymbol{e}_-} = \frac{((\boldsymbol{e}_+ \cdot \underline{\boldsymbol{X}}) \wedge \boldsymbol{e}_-) \cdot \boldsymbol{e}_-}{(\boldsymbol{e}_+ \cdot \underline{\boldsymbol{X}}) \cdot \boldsymbol{e}_-}. \tag{24}$$

2.3. THE SPECIAL EUCLIDEAN GROUP IN CGA

A geometry is defined by its basic entity, the geometric transformation group which is acting in a linear and covariant manner on all the entities which are constructed from the basic entity by incidence operations, and the resulting invariances with respect to that group. The search for such a geometry was motivated in Section 1. Next we want to specify the required features of the special Euclidean group in CGA.

To make a geometry a proper one, we have to require that any action $\mathcal{A}$ of that group on an entity, say $\boldsymbol{u}$, is grade preserving, or in other words structure preserving. This makes it necessary that the operator $\boldsymbol{A}$ applies as versor product (Perwass and Sommer, 2002)

$$\mathcal{A}\{\boldsymbol{u}\} = \boldsymbol{A}\boldsymbol{u}\boldsymbol{A}^{-1}. \tag{25}$$

This means that the entity $\boldsymbol{u}$ should transform covariantly (Dorst and Fontijne, 2004). If $\boldsymbol{u}$ is composed by e.g. two representants $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of the basis entities of the geometry, then $\boldsymbol{u}$ should transform according to

$$\mathcal{A}\{\boldsymbol{u}\} = \mathcal{A}\{\boldsymbol{u}_1 \circ \boldsymbol{u}_2\} = (\boldsymbol{A}\boldsymbol{u}_1\boldsymbol{A}^{-1}) \circ (\boldsymbol{A}\boldsymbol{u}_2\boldsymbol{A}^{-1}) = \boldsymbol{A}\boldsymbol{u}\boldsymbol{A}^{-1}. \tag{26}$$

The invariants of the conformal group $C(3)$ in $\mathbb{R}^3$ are angles. The conformal group $C(3)$ is mighty (Needham, 1997), but other than (25) and (26) it is nonlinear and transforms not covariantly in $\mathbb{R}^3$. Besides, in $\mathbb{R}^3$ there exist no entities other than points which could be transformed.

As we have shown in Section 2.2, in $\mathbb{R}_{4,1}$ the situation is quite different because all the geometric entities derived there can be seen also as algebraic entities in the sense of Section 1. Not only the elements of the null cone transform covariantly but also those of the dual space of $\mathbb{R}_{4,1}$. Furthermore, the representation of the conformal group $C(3)$ in $\mathbb{R}_{4,1}$ has the required properties of (25) and (26), see (Li et al., 2001). All vectors with positive signature in $\mathbb{R}_{4,1}$, that is a sphere, a plane as well as the components inversion and reflection of $C(3)$ compose a multiplicative group. That is called the versor representation of $C(3)$. This group is isomorphic to the Lorentz group of $\mathbb{R}_{4,1}$. The subgroup, which is composed by products of an even number of these vectors, is the spin group $Spin^+(4,1)$, that is the spin representation of $O^+(4,1)$. To that group belong the subgroups of rotation, translation, dilatation, and transversion of $C(3)$. They are applied as a spinor $\boldsymbol{S}$, $\boldsymbol{S} \in \mathbb{R}^+_{4,1}$ and $\boldsymbol{S}\widetilde{\boldsymbol{S}} = |\boldsymbol{S}|^2$. A rotor $\boldsymbol{R}, \boldsymbol{R} \in \langle\mathbb{R}_{4,1}\rangle_2$ and $\boldsymbol{R}\boldsymbol{R}^2 = 1$, is a special spinor. Rotation and translation are represented in $\mathbb{R}_{4,1}$ as rotors.

The special Euclidean group $SE(3)$ is defined by $SE(3) = SO(3) \oplus \mathbb{R}^3$. Therefore, the rigid body motion $g = (R, \boldsymbol{t})$, $g \in SE(3)$ of a point $\boldsymbol{x} \in \mathbb{R}^3$ writes in Euclidean space

$$\boldsymbol{x}' = g\{\boldsymbol{x}\} = R\boldsymbol{x} + \boldsymbol{t}. \tag{27}$$

Here R is a rotation matrix and $\boldsymbol{t}$ is a translation vector. Because $SE(3) \subset C(3)$, in our choice of a special rigid body motion the representation of $SE(3)$ in CGA is isomorphic to the special orthogonal group, $SO^+(4,1)$. Hence, such $g \in SE(3)$ does not represent the full screw, but a general rotation in $\mathbb{R}_{4,1}$, that is the rotation axis in $\mathbb{R}^3$ is shifted out of the origin by the translation vector $\boldsymbol{t}$.

That transformation $g \in SE(3)$ is represented in CGA by a special rotor $\boldsymbol{M}$, called a motor, $\boldsymbol{M} \in \langle\mathbb{R}_{4,1}\rangle_2$. The motor may be written as in equation (3). To specify the line $\underline{\boldsymbol{l}} \in \langle\mathbb{R}_{4,1}\rangle_2$ by the rotation and translation in $\mathbb{R}^3$, the motor has to be decomposed into its rotation and translation components. The normal rotation in CGA is given by the rotor

$$\boldsymbol{R} = \exp\left(-\frac{\theta}{2}\boldsymbol{l}\right) \tag{28}$$

with $\boldsymbol{l} \in \langle \mathbb{R}_3 \rangle_2$ indicating the rotation plane which passes the origin. The translation in CGA is given by a special rotor, called a translator,

$$\boldsymbol{T} = \exp\left(\frac{\boldsymbol{et}}{2}\right) \tag{29}$$

with $\boldsymbol{t} \in \langle \mathbb{R}_3 \rangle_1$ as the translation vector. Rotors constitute a multiplicative group. If we interprete the rotor $\boldsymbol{R}$ as that entity of $\mathbb{R}_{4,1}$ which should be transformed by translation in a covariant manner, then

$$\boldsymbol{M} = \boldsymbol{T}\boldsymbol{R}\widetilde{\boldsymbol{T}}. \tag{30}$$

We call this special motor representation the twist representation. Its exponential form is given by

$$\boldsymbol{M} = \exp\left(\frac{1}{2}\boldsymbol{et}\right) \exp\left(-\frac{\theta}{2}\boldsymbol{l}\right) \exp\left(-\frac{1}{2}\boldsymbol{et}\right). \tag{31}$$

This equation expresses the shift of the rotation axis $\boldsymbol{l}^*$ in the plane $\boldsymbol{l}$ by the vector $\boldsymbol{t}$ to perform the normal rotation and finally shifting back the axis.

Because $SE(3)$ is a Lie group, the line $\underline{\boldsymbol{l}} \in \langle \mathbb{R}_{4,1} \rangle_2$ is the representation of the infinitesimal generator of $\boldsymbol{M}$, $\xi \in se(3)$. We call the generator representation a twist because it represents rigid body motion as general rotation. It is parameterized by the position and orientation of $\underline{\boldsymbol{l}}$ which are the Plücker coordinates, represented by the rotation plane $\boldsymbol{l}$ and the inner product $(\boldsymbol{t} \cdot \boldsymbol{l})$, (Rosenhahn, 2003),

$$\underline{\boldsymbol{l}} = \boldsymbol{l} + \boldsymbol{e}(\boldsymbol{t} \cdot \boldsymbol{l}). \tag{32}$$

The twist model of the rigid body motion, equation (30), is that one we are using in that paper. The most general formulation of the rigid body motion is the screw motion (Rooney, 1978). But instead of presenting that in detail, we refer the reader to the report (Sommer et al., 2004).

A motor $\boldsymbol{M}$ transforms covariantly any entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$ according to

$$\underline{\boldsymbol{u}}' = \boldsymbol{M}\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}} \tag{33}$$

with $\underline{\boldsymbol{u}}' \in \mathbb{R}_{4,1}$. An equivalent equation is valid for the dual entity $\underline{\boldsymbol{U}} \in \mathbb{R}_{4,1}$. Because motors concatenate multiplicatively, a multiple-motor transformation of $\underline{\boldsymbol{u}}$ resolves recursively. Let be $\boldsymbol{M} = \boldsymbol{M}_2\boldsymbol{M}_1$, then

$$\underline{\boldsymbol{u}}'' = \boldsymbol{M}\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}} = \boldsymbol{M}_2\boldsymbol{M}_1\underline{\boldsymbol{u}}\widetilde{\boldsymbol{M}}_1\widetilde{\boldsymbol{M}}_2 = \boldsymbol{M}_2\underline{\boldsymbol{u}}'\widetilde{\boldsymbol{M}}_2. \tag{34}$$

It is a feature of any GA that also composed entities, which are built by the outer product of other ones, transform covariantly by a linear transformation. This is called outermorphism (Hestenes, 1991) and it means the preservation of the outer product under linear transformations. Following Section 1, this is an important feature of the chosen algebraic embedding that will be demonstrated in Section 3.

3. Shape Models from Coupled Twists

In this section we will approach step by step the kinematic design of algebraic and transcendental curves and surfaces by coupling a certain set of twists as generators of a multiple-parameter Lie group action.

3.1. THE KINEMATIC CHAIN AS MODEL OF CONSTRAINED MOTION

In the preceding section we argued that each entity $\underline{\boldsymbol{u}}_i$ contributing to the rigid model of another entity $\underline{\boldsymbol{u}}$ is performing the same transformation, represented by the motor $\boldsymbol{M}$. Now we assume an ordered set of non-rigidly coupled rigid components of an object. Such model is called a kinematic chain (Murray et al., 1994). In a kinematic chain the task is to formulate the net movement of the end-effector at the n-th joint by movements of the j-th joints, $j = 1, ..., n-1$, if the 0-th joint is fixed coupled with a world coordinate system. These movements are discribed by the motors $\boldsymbol{M}_j$. Let $\mathcal{T}_j$ be the transformation of an attached joint j with respect to the base coordinate system, then for $j = 1, ..., n$ the point $\underline{\boldsymbol{x}}_{j,i_j}, i_j = 1, ..., m_j$, transforms according to

$$\mathcal{T}_j(\underline{\boldsymbol{x}}_{j,i_j}, \boldsymbol{M}_j) = \boldsymbol{M}_1 ... \boldsymbol{M}_j \underline{\boldsymbol{x}}_{j,i_j} \widetilde{\boldsymbol{M}}_j ... \widetilde{\boldsymbol{M}}_1 \tag{35}$$

and

$$\mathcal{T}_0(\underline{\boldsymbol{x}}_{0,i_0}) = \underline{\boldsymbol{x}}_{0,i_0}. \tag{36}$$

The motors $\boldsymbol{M}_j$ are representing the flexible geometry of the kinematic chain very efficiently. This results in an object model $\mathcal{O}$ defined by a kinematic chain with n segments and described by any geometric entity $\underline{\boldsymbol{u}}_{j,i_j} \in \mathbb{R}_{4,1}$ attached to the j-th segment,

$$\mathcal{O} = \left\{ \mathcal{T}_0(\underline{\boldsymbol{u}}_{0,i_0}), \mathcal{T}_1(\underline{\boldsymbol{u}}_{1,i_1}, \boldsymbol{M}_1), ..., \mathcal{T}_n(\underline{\boldsymbol{u}}_{n,i_n}, \boldsymbol{M}_n) | n, i_0, ..., i_n \in \mathbb{N} \right\}. \tag{37}$$

If $\underline{\boldsymbol{u}}_{j,i_j}$ is performing a motion caused by the motor $\boldsymbol{M}$, then

$$\begin{aligned} \underline{\boldsymbol{u}}'_{j,i_j} &= \boldsymbol{M} \left(\mathcal{T}_j(\underline{\boldsymbol{u}}_{j,i_j}, \boldsymbol{M}_j) \right) \widetilde{\boldsymbol{M}} && (38) \\ &= \boldsymbol{M}(\boldsymbol{M}_1 ... \boldsymbol{M}_j \underline{\boldsymbol{u}}_{j,i_j} \widetilde{\boldsymbol{M}}_j ... \widetilde{\boldsymbol{M}}_1) \widetilde{\boldsymbol{M}}. && (39) \end{aligned}$$

3.2. THE OPERATIONAL MODEL OF SHAPE

We will now introduce another type of constrained motion, which can be realized by physical systems only in special cases but should be understood as a generalization of a kinematic chain. This is our proposed model of operational or kinematic shape (Rosenhahn, 2003). An operational shape means that a shape results from the net effect, that is the orbit, of a point under the action of a set of coupled operators. So the operators at the end are the representations of the shape. A kinematic shape means the shape for which these operators are the motors as representations of $SE(3)$ in $\mathbb{R}_{4,1}$. The principle is simple. It goes back to the interpretation of any $g \in SE(3)$ as a Lie group action (Murray et al., 1994), see equation (1). But only in $\mathbb{R}_{4,1}$ we can take advantage of its representation as rotation around the axis $\underline{\boldsymbol{l}}$, see equations (3), (30) and (31).

In Section 2.2 we introduced the sphere and the circle from IPNS and OPNS, respectively. We call these definitions the canonical ones. On the other hand, a circle has an operational definition which is given by the following. Let $\underline{\boldsymbol{x}}_\phi$ be a point which is a mapping of another point $\underline{\boldsymbol{x}}_0$ by $g \in SE(3)$ in $\mathbb{R}_{4,1}$. This may be written as

$$\underline{\boldsymbol{x}}_\phi = \boldsymbol{M}_\phi \underline{\boldsymbol{x}}_0 \widetilde{\boldsymbol{M}}_\phi \tag{40}$$

with $\boldsymbol{M}_\phi$ being the motor which rotates $\underline{\boldsymbol{x}}_0$ by an angle ϕ,

$$\boldsymbol{M}_\phi = \exp\left(-\frac{\phi}{2}\Psi\right). \tag{41}$$

Here again is Ψ the twist as a generator of the rotation around the axis $\underline{\boldsymbol{l}}$, see equation (3). Note that $\Psi = \alpha \boldsymbol{l}, \alpha \in \mathbb{R}$. If ϕ covers densely the whole span $[0, ..., 2\pi]$, then the generated set of points $\{\underline{\boldsymbol{x}}_\phi\}$ is also dense. The infinite set $\{\underline{\boldsymbol{x}}_\phi\}$ is the orbit of a rotation caused by the infinite set $\{\boldsymbol{M}_\phi\}$, which has the shape of a circle in $\mathbb{R}^3$. The set $\{\boldsymbol{x}_\phi\}$ represents the well-known subset concept in a vector space of geometric objects in analytic geometry. In fact, that circle is on the horosphere N_e^3 because it is composed only by points. We will write for the circle $\underline{\boldsymbol{z}}_{\{1\}}$ instead of $\{\underline{\boldsymbol{x}}_\phi\}$ to indicate the different nature of that circle in comparison to either $\underline{\boldsymbol{z}}$ or $\underline{\boldsymbol{Z}}$ of Section 2.2. The index $\{1\}$ means that the circle is generated by one twist from a continuous argument ϕ. So the circle, embedded in $\mathbb{R}_{4,1}$, is defined by

$$\underline{\boldsymbol{z}}_{\{1\}} = \left\{\underline{\boldsymbol{x}}_\phi | \text{ for all } \phi \in [0, ..., 2\pi]\right\}. \tag{42}$$

Its radius is given by the distance of the chosen point $\underline{\boldsymbol{x}}_0$ to the axis $\underline{\boldsymbol{l}}$ whose orientation and position in space depends on the parameterization of $\underline{\boldsymbol{l}}$. That $\underline{\boldsymbol{z}}_{\{1\}}$ is defined by an infinite set of arguments is no real problem in the case of computational geometry or applications where only discretized shape is of interest. More interesting is the fact that in the canonical definitions of Section 2.2 the geometric entities are all derived from either spheres or points. In the case of the operational definition of shape, the circle is the basic geometric entity instead, respectively rotation is the basic operation.

A sphere results from the coupling of two motors, $\boldsymbol{M}_{\phi_1}$ and $\boldsymbol{M}_{\phi_2}$, whose twist axes meet at the center of the sphere and which are perpendicularly arranged.

The resulting constrained motion of a point $\underline{\boldsymbol{x}}_{0,0}$ performs a rotation on a sphere given by $\phi_1 \in [0, ..., 2\pi]$ and $\phi_2 \in [0, ..., \pi]$,

$$\underline{\boldsymbol{x}}_{\phi_1,\phi_2} = \boldsymbol{M}_{\phi_2}\boldsymbol{M}_{\phi_1}\underline{\boldsymbol{x}}_{0,0}\widetilde{\boldsymbol{M}}_{\phi_1}\widetilde{\boldsymbol{M}}_{\phi_2}. \tag{43}$$

The complete orbit of a sphere is given by

$$\underline{\boldsymbol{s}}_{\{2\}} = \left\{\underline{\boldsymbol{x}}_{\phi_1,\phi_2} \middle| \text{ for all } \phi_1 \in [0, ..., 2\pi], \phi_2 \in [0, ..., \pi]\right\}. \tag{44}$$

Let us come back to the point of generalization of the well-known kinematic chains. These models of linked bar mechanisms have to be physically feasible. Instead, our model of coupled twists is not limited by that constraint. Therefore, the sphere expresses a virtual coupling of twists. This includes both location and orientation in space, and the possibility of fixating several twists at the same location, for any dimension of the space $\mathbb{R}^n$. There are several extensions of the introduced kinematic model which are only possible in CGA.

First, while the group $SE(3)$ can only act on points, its representation in $\mathbb{R}_{4,1}$ may act in the same way on any entity $\underline{\boldsymbol{u}} \in \mathbb{R}_{4,1}$ derived from either points or spheres. This results in high complex free-form shapes caused from the motion of relatively simple generating entities and low order sets of coupled twists.

Second, only by coupling a certain set of twists, high complex free-form shapes may be generated from a complex enough constrained motion of a point.

Let $\underline{\boldsymbol{u}}_{\{n\}}$ be the shape generated by n motors $\boldsymbol{M}_{\phi_1}, ..., \boldsymbol{M}_{\phi_n}$. We call it the n-twist model,

$$\underline{\boldsymbol{u}}_{\{n\}} = \left\{\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} \middle| \text{ for all } \phi_1, ..., \phi_n \in [0, ..., 2\pi]\right\} \tag{45}$$

with

$$\underline{\boldsymbol{x}}_{\phi_1,...,\phi_n} = \boldsymbol{M}_{\phi_n}...\boldsymbol{M}_{\phi_1}\underline{\boldsymbol{x}}_{0,...,0}\widetilde{\boldsymbol{M}}_{\phi_1}...\widetilde{\boldsymbol{M}}_{\phi_n}. \tag{46}$$

3.3. FREE-FORM OBJECTS

There are a lot of more degrees of freedom to design free-form objects embedded in $\mathbb{R}_{4,1}$ by the motion of a point caused by coupled twists.
While a single rotation-like motor generates a circle, a single translation-like motor generates a line as a root of non-curved objects. Of course, several of both variants can be mixed. Other degrees of freedom of the design result from the following extensions:

- Introducing an individual angular frequency λ_i to the motor $\boldsymbol{M}_{\phi_i}$ also influences the synchronization of the rotation angles ϕ_i.
- Rotation within limited angular segments $\phi_i \in [\alpha_{i_1}, ..., \alpha_{i_2}]$ with $0 \leq \alpha_{i_1} < \alpha_{i_2} \leq 2\pi$ is possible.

Let us consider the simple example of a 2-twist model of shape,

$$\underline{\boldsymbol{u}}_{\{2\}} = \left\{ \underline{\boldsymbol{x}}_{\phi_1,\phi_2} | \text{ for all } \phi_1, \phi_2 \in [0, ..., 2\pi] \right\} \tag{47}$$

with

$$\underline{\boldsymbol{x}}_{\phi_1,\phi_2} = \boldsymbol{M}_{\lambda_2\phi_2} \boldsymbol{M}_{\lambda_1\phi_1} \underline{\boldsymbol{x}}_0 \widetilde{\boldsymbol{M}}_{\lambda_1\phi_1} \widetilde{\boldsymbol{M}}_{\lambda_2\phi_2}, \tag{48}$$

$\lambda_1, \lambda_2 \in \mathbb{R}$ and $\phi_1 = \phi_2 = \phi \in [0, ..., 2\pi]$.

That model can generate not only a sphere, but an ellipse ($\lambda_1 = -2, \lambda_2 = 1$), several well-known algebraic curves (in space), see (Rosenhahn, 2003), such as cardioid, nephroid or deltoid, transcendental curves like a spiral, or surfaces. For the list of examples see Table I.

Interestingly, the order of nonlinearity of algebraic curves grows faster than the number of the generating motors.

3.4. EXTENSIONS OF THE CONCEPTS

By replacing the initial point $\underline{\boldsymbol{x}}_0$ by any other geometric entity, $\underline{\boldsymbol{u}}_0$, built from either points or spheres by applying the outer product, the concepts remain the same. This makes the kinematic object model in conformal space a recursive one.

The infinite set of arguments ϕ_i of the motor $\boldsymbol{M}_{\phi_i}$ to generate the entity $\underline{\boldsymbol{u}}_{\{n\}}$ will in practice reduce to a finite one, which results in a discrete entity $\underline{\boldsymbol{u}}_{[n]}$. The index $[n]$ indicates that n twists are used with a finite set of arguments $\{\phi_{i,j_i} | j_i \in \{0, ..., m_i\}\}$.

The previous formulations of free-form shape did assume a rigid model. As in the case of the kinematic chain, the model can be made flexible. This happens by encapsulating the entity $\underline{\boldsymbol{u}}_{[n]}$ into a set of motors $\left\{\boldsymbol{M}_j^d | j = J, ..., 1\right\}$, which results in a deformation of the object.

$$\underline{\boldsymbol{u}}_{[n]}^d = \boldsymbol{M}_J^d ... \boldsymbol{M}_1^d \underline{\boldsymbol{u}}_{[n]} \widetilde{\boldsymbol{M}}_1^d ... \widetilde{\boldsymbol{M}}_J^d \tag{49}$$

TABLE I. Simple geometric entities generated from up to three twists

Entity	Generation	Class
point	twist axis intersected with a point	0twist curve
circle	twist axis non-collinear with a point	1twist curve
line	twist axis is at infinity	1twist curve
conic	2 parallel non-collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = -2$
line segment	2 twists, building a degenerate conic	2twist curve $\lambda_1 = 1, \lambda_2 = -2$
cardioid	2 parallel non-collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = 1$
nephroid	2 parallel non-collinear twists	2twist curve $\lambda_1 = 1, \lambda_2 = 2$
rose	2 parallel non-collinear twists, j loops	2twist curve $\lambda_1 = 1, \lambda_2 = -j$
spiral	1 finite and 1 infinite twist	2twist curve $\lambda_1 = 1, \lambda_2 = 1$
sphere	2 perpendicular twists	2twist surface $\lambda_1 = 1, \lambda_2 = 1$
plane	2 parallel twists at infinity	2twist surface
cylinder	2 twists, one at infinity	2twist surface
cone	2 twists, one at infinity	2twist surface
quadric	a conic rotated with a third twist	3twist surface

Finally, the entity $\underline{\boldsymbol{u}}^d_{[n]}$ may perform a motion under the action of a motor $\boldsymbol{M}$, which itself may be composed by a set of motors $\{\boldsymbol{M}_i | i = I, ..., 1\}$ according to equation (4),

$$\underline{\boldsymbol{u}}^{d'}_{[n]} = \boldsymbol{M} u^d_{[n]} \widetilde{\boldsymbol{M}}. \tag{50}$$

But a twist is not only an operator but it may play in CGA also the role of an operand,

$$\Psi' = \boldsymbol{M} \Psi \boldsymbol{M}. \tag{51}$$

This causes a dynamic shape model as an alternative to (49).

So far, the entity $\boldsymbol{u}_{\{n\}}$ was embedded in the Euclidean space. Lifting up the entity to the conformal space, $\underline{\boldsymbol{u}}_{\{n\}} \in \mathbb{R}_{4,1}$, is simply done by

$$\underline{\boldsymbol{u}}_{\{n\}} = \boldsymbol{e} \wedge \left(\boldsymbol{u}_{\{n\}} + \boldsymbol{e}_-\right) = \boldsymbol{e} \wedge \boldsymbol{U}_{\{n\}} \tag{52}$$

with $\boldsymbol{U}_{\{n\}}$ being the shape in the projective space $\mathbb{R}_{3,1}$.

4. Twist Models and Fourier Representations

The message of the last subsection is the following. A finite set of coupled twist (or nested motors) performs a constrained motion of any set of geometric entities, whose orbit uniquely represents either a curve, a surface or a volume of arbitrary complexity. This needs a parameterized model of the generators of the shape. In some applications the reverse problem may be of interest. That is to find a parameterized twist model for a given shape. That task can be solved: Any curve, surface or volume of arbitrary complexity can be mapped to a finite set of coupled twists, but in a non-unique manner. That means, that there are different models which generate the same shape.

We will show here that there is a direct and intuitive relation between the twist model of shape and the Fourier representations. The Fourier series decomposition and the Fourier transforms in their different representations are well-known techniques of signal analysis and image processing. The interesting fact that this equivalence of representations results in a fusion of concepts from geometry, kinematics, and signal theory is of great importance in engineering. Furthermore, because the presented modelling of shape is embedded in a conformal space, there is also a single access for embedding the Fourier representations in either conformal or projective geometry. This is quite different from the recent publication (Turski, 2004).

4.1. THE CASE OF A CLOSED PLANAR CURVE

Let us consider a closed curve $\boldsymbol{c} \in \mathbb{R}^2$ in a parametric representation with $t \in \mathbb{R}$. Then its Fourier series representation is given by

$$\boldsymbol{c}(t) = \sum_{\nu=-\infty}^{\infty} \gamma_\nu \exp\left(\frac{j2\pi\nu t}{T}\right) \tag{53}$$

with the Fourier coefficients γ_ν, $\nu \in \mathbb{Z}$ as frequency and j, $j^2 = -1$, as the imaginary unit and T as the curve length.

This model of a curve has been used for a long time in image processing for shape analysis by Fourier descriptors (these are the Fourier coefficients) (Zahn and Roskies, 1972).

We will translate this spectral representation into the model of an infinite number of coupled twists by following the method presented in (Rosenhahn et al., 2004). Because equation (53) is valid in an Euclidean space, the twist model has to be reformulated accordingly. This will be shown for the case of a 2-twist curve $\underline{\boldsymbol{c}}_{\{2\}}$ based on equation (27). Then equation (48) can be written in $\mathbb{R}_3$ for $\phi_1 = \phi_2 = \phi$ as

$$\boldsymbol{x}_\phi = \boldsymbol{R}_{\lambda_2\phi}\left((\boldsymbol{R}_{\lambda_1\phi}(\boldsymbol{x}_0 - \boldsymbol{t}_1)\widetilde{\boldsymbol{R}}_{\lambda_1\phi} + \boldsymbol{t}_1) - \boldsymbol{t}_2\right)\widetilde{\boldsymbol{R}}_{\lambda_2\phi} + \boldsymbol{t}_2 \tag{54}$$

$$= \boldsymbol{p}_0 + \boldsymbol{V}_{1,\phi}\boldsymbol{p}_1\widetilde{\boldsymbol{V}}_{1,\phi} + \boldsymbol{V}_{2,\phi}p_2\widetilde{\boldsymbol{V}}_{2,\phi}. \tag{55}$$

Here the translation vectors have been absorbed by the vectors $\boldsymbol{p}_i$ and the $\boldsymbol{V}_i$ are built by certain products of the rotors $\boldsymbol{R}_{\lambda_i\phi}$. We call the $\boldsymbol{p}_i$ the phase vectors. Next, for the aim of interpreting that equation as a Fourier series expansion, we rewrite the Fourier basis functions as rotors of an angular frequency $i \in \mathbb{Z}$, in the plane $\boldsymbol{l} \in \mathbb{R}_2$, $\boldsymbol{l}^2 = -1$,

$$\boldsymbol{R}_{\lambda_i\phi} = \exp\left(-\frac{\lambda_i\phi}{2}\boldsymbol{l}\right) = \exp\left(-\frac{\pi i\phi}{T}\boldsymbol{l}\right). \tag{56}$$

All rotors of a planar curve lie in the same plane as the phase vectors $\boldsymbol{p}_i$. After some algebra, see (Rosenhahn et al., 2004), we get for the transformed point

$$\boldsymbol{x}_\phi = \sum_{i=0}^{2}\boldsymbol{p}_i \exp\left(\frac{2\pi i\phi}{T}\boldsymbol{l}\right) \tag{57}$$

and for the curve as subspace of $\mathbb{R}^3$ the infinite set of points

$$\boldsymbol{c}_{\{2\}} = \{\boldsymbol{x}_\phi | \text{ for all } \phi \in [0, ..., 2\pi] \text{ and for all } i \in \{0, 1, 2\}\}. \tag{58}$$

A general (planar) curve is given by

$$\boldsymbol{c}_{\{\infty\}} = \{\boldsymbol{x}_\phi | \text{ for all } \phi \in [0, ..., 2\pi] \text{ and for all } i \in \mathbb{Z}\}, \tag{59}$$

respectively as Fourier series expansion, written in the language of kinematics

$$\boldsymbol{c}_{\{\infty\}} = \left\{\lim_{n\longrightarrow\infty}\sum_{i=-n}^{n}\boldsymbol{p}_i \exp\left(\frac{2\pi i\phi}{T}\boldsymbol{l}\right)\right\} \tag{60}$$

$$= \left\{\lim_{n\longrightarrow\infty}\sum_{i=-n}^{n}\boldsymbol{R}_{\lambda_i\phi}\boldsymbol{p}_i\widetilde{\boldsymbol{R}}_{\lambda_i\phi}\right\}. \tag{61}$$

A discretized curve is called a contour. In that case equation (60) has to consider a finite model of n twists and the Fourier series expansion becomes the inverse discrete Fourier transform. Hence, a planar contour is given by the finite sequence $\boldsymbol{c}_{[n]}$ with the contour points $c_k, -n \leq k \leq n$, in parametric representation

$$c_k = \sum_{i=-n}^{n}\boldsymbol{p}_i \exp\left(\frac{2\pi ik}{2n+1}\boldsymbol{l}\right), \tag{62}$$

and the phase vectors are computed as a discrete Fourier transform of the contour

$$\boldsymbol{p}_i = \frac{1}{2n+1} \sum_{k=-n}^{n} c_k \exp\left(-\frac{2\pi i k}{2n+1}\boldsymbol{l}\right). \tag{63}$$

These equations imply that the angular argument ϕ_k is replaced by k.

4.2. EXTENSIONS OF THE CONCEPTS

The extension of the modelling of a planar curve, embedded in $\mathbb{R}^3$, to a 3D curve is easily done. This happens by taking its projections to either $\boldsymbol{e}_{12}$, $\boldsymbol{e}_{23}$, or $\boldsymbol{e}_{31}$ as periodic planar curves. Hence, we get the superposition of these three components. Let $\boldsymbol{c}^j_{[n]}$ be these components in the case of a 3D contour with the rotation axes $\boldsymbol{l}^*_j$ perpendicular to the rotation planes $\boldsymbol{l}_j$. Then

$$\boldsymbol{c}_{[n]} = \sum_{j=1}^{3} \boldsymbol{c}^j_{[n]} \tag{64}$$

with the contour points of the projections c^j_k, $j = 1, 2, 3$ and $-n \le k \le n$,

$$c^j_k = \sum_{i=-n}^{n} \boldsymbol{p}^j_i \exp\left(\frac{2\pi i k}{2n+1}\boldsymbol{l}_j\right). \tag{65}$$

Another useful extension is with respect to surface representations, see (Rosenhahn et al., 2004). If this surface is a 2D function orthogonal to a plane spanned by the bivectors $\boldsymbol{e}_{ij}$, then the twist model corresponds to the 2D inverse FT. In the case of an arbitrary orientation of the rotation planes $\boldsymbol{l}_j$ instead, or in the case of the surface of a 3D object, the procedure is comparable to that of equation (65). The surface is represented as a two-parametric surface $\boldsymbol{s}(t_1, t_2)$ as superposition of the three projections $\boldsymbol{s}^j(t_1, t_2)$.

In the case of a discrete surface in a two-parametric representation we have the finite surface representation $\boldsymbol{s}_{[n_1,n_2]}$,

$$\boldsymbol{s}_{[n_1,n_2]} = \sum_{j=1}^{3} \boldsymbol{s}^j_{[n_1,n_2]} \tag{66}$$

with the surface points of the projections $s^j_{k_1,k_2}$, $j = 1, 2, 3$ and $-n_1 \le k_1 \le n_1$, $-n_2 \le k_2 \le n_2$,

$$s^j_{k_1,k_2} = \sum_{i_1=-n_1}^{n_1} \sum_{i_2=-n_2}^{n_2} \boldsymbol{p}^j_{i_1,i_2} \exp\left(\frac{2\pi i_1 k_1}{2n_1+1}\boldsymbol{l}_j\right) \exp\left(\frac{2\pi i_2 k_2}{2n_2+1}\boldsymbol{l}_j\right) \tag{67}$$

and the phase vectors

$$\boldsymbol{p}^{j}_{i_1,i_2} = \frac{1}{2n_1+1}\frac{1}{2n_2+1}\boldsymbol{p}^{j'}_{i_1,i_2} \tag{68}$$

$$\boldsymbol{p}^{j'}_{i_1,i_2} = \sum_{k_1=-n_1}^{n_1}\sum_{k_2=-n_2}^{n_2} s^{j}_{k_1,k_2} \exp\left(-\frac{2\pi i_1 k_1}{2n_1+1}\boldsymbol{l}_j\right)\exp\left(-\frac{2\pi i_2 k_2}{2n_2+1}\boldsymbol{l}_j\right) \tag{69}$$

Finally, we will give the hint to an alternative model of a curve $\underline{\boldsymbol{c}} \in \mathbb{R}_{4,1}$, see (Rosenhahn, 2003). While equation (60) expresses the additive superposition of rotated phase vectors in Euclidean space, the multiplicative coupling of the twists directly in conformal space is possible.

The discussed equivalence of the twist model and the Fourier representation has several advantages in practical use of the model. The most important may be the applicability to low-frequency approximations of the shape. For instance in pose estimation (Rosenhahn, 2003) the estimations of the motion parameters of non-convex objects can be regularized efficiently in that way. Instead of estimating motors, the parameters of the twists are estimated because of numeric reasons.

5. Summary and Conclusions

We presented an operational or kinematic model of shape in $\mathbb{R}^3$. This model is based on the Lie group $SE(3)$, embedded in the conformal geometric algebra $\mathbb{R}_{4,1}$ of the Euclidean space. While the modelling of shape in $\mathbb{R}^3$ caused by actions of $SE(3)$ is limited, a lot of advantages result from the chosen algebraic embedding in real applications. As one of these the possibility of conformal (and projective) shape models should be mentioned. We did not discuss any applications in detail. Instead, we refer the reader to the website http://www.ks.informatik.uni-kiel.de with respect to the problem of pose estimation. In that work we could show that the pose estimation based on the presented shape model can cope with incomplete and noisy data. In addition to that robustness the pose estimation is numerically stable and fast.

Because the chosen twist model is equivalent to the Fourier representation (in some aspects it overcomes that), the proposed shape representation unifies geometry, kinematics, and signal theory. It can be expected that this will have a great impact on both theory and practice in computer vision, computer graphics and modelling of mechanisms.

An extended version of this paper can be found as report (Sommer et al., 2004).

6. Acknowledgment

The authors thank the anonymous reviewers for their valuable suggestions for improving the quality of this paper.

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Part I

Discrete Geometry

Part II

Approximation and Regularization