Hilbert Spaces

Def: Hilbert space $\rightarrow$ is a vector space H over complex numbers from the set C. Vectors are denoted as $|\Psi> \in C^n = H$. There is an inner | scalar product vector product $< .|. >$: $H * H \rightarrow C$ such that for every $\Phi_1,\ \Phi_2,\ \Psi \in H$ and any $c_1,\ c_2 \in C$ we have

$\leadsto < \Psi \mid \Phi > = < \Phi \mid \Psi >^*$
$\leadsto < \Psi \mid \Psi > \geq\ 0$ and $< \Psi \mid \Psi > =\ 0$ iff $\mid \Psi > =\ 0$
$\leadsto < \Psi \mid c_1 * \Phi_1\ +\ c_2 * \Phi_2 > =$

$$c_1 * < \Psi \mid \Phi_1 >\ + c_2 * < \Psi \mid \Phi_2 >$$

$\leadsto$ completeness of the norm $\parallel \Psi \parallel = < \Psi \mid \Psi >^{1/2}$

The inner product of two vectors | state descriptions
$\mid \Psi > =\ (u_1,\ \ldots\ ,\ u_n) \in C^n \mid$
$\mid \Phi > =\ (v_1,\ \ldots\ ,\ v_n) \in C^n$
is defined as
$< \Psi \mid \Phi > =\ \sum_{i=1}^{n}\ u_i^* * v_i$
where u_i^* is the conjugate complex of u_i.

Def: Two vectors $\Phi,\ \Psi \in H$ are orthogonal if $< \Phi \mid \Psi >= 0$. A set $S \subset H$ is orthogonal if for all $\Phi_i,\ \Psi_j \in S$ holds $< \Phi_i \mid \Psi_j >= 0$.
S is orthonormal if it is orthogonal and all $\Phi_i \in S$ have norm 1.

Def: A subspace G of H is a subset of H that is closed under addition and inner product.
For each closed subspace W of H there exists a unique subspace $W^{\perp}$ of H such that for all $\Phi \in W$ and $\Psi \in W^{\perp}$ holds $< \Phi \mid \Psi >= 0$, and each $\Phi \in H$ has a unique representation $\Phi = \Phi_1 \oplus \Phi_2$ with $\Phi_1 \in W$ and $\Phi_2 \in W^{\perp}$.

Def: An orthonormal set $B \subseteq H$ is an orthonormal basis of H if none of its supersets is orthonormal.

Def: If $B = \{\Phi_i \mid i \in \{o, \ldots, n-1\}\}$ is an orthonormal basis of an n-dimensional Hilbert space H and for $\Psi \in H$ holds

$$\Psi = \sum_{i=0}^{n-1} \alpha_i * \Phi_i$$

then $(\alpha_0, \ldots, \alpha_{(n-1)})$ is the representation of Ψ in B.

$< \Psi \mid$ is a bra-vector (represented as a row vector), $\mid \Psi >$ is a ket-vector (represented as a column vector)
$< \Psi \mid \Phi >$ is a scalar product, $\mid \Phi >< \Psi \mid$ is a tensor product

for instance, if $\mid \Phi > = \begin{pmatrix} \alpha_0 \\ \ldots \\ \alpha_{(n-1)} \end{pmatrix}$
and $< \Psi \mid = (\beta_0, \beta_1, \ldots, \beta_{(n-1)}$
then

$$\mid \Psi >< \Phi \mid = \begin{pmatrix} \alpha_0\beta_0 & \ldots & \alpha_0\beta_{(n-1)} \\ \ldots & \ldots & \ldots \\ \alpha_{(n-1)}\beta_0 & \ldots & \alpha_{(n-1)}\beta_{(n-1)} \end{pmatrix}$$

Linear Operators

Def: A linear operator of H is a mapping $A: \ H \ \rightarrow \ H$
Linearity is defined as
$A\,(\alpha\Phi + \beta\Psi) \ = \ \alpha A\Phi + \beta A\Psi$
The adjoint of A, denoted as A^*, is defined as
$< \Psi \mid A \mid \Phi > = < A^*\Psi \mid \Phi >$
If $A = A^*$, then the operator is self-adjoint, and then also has property
$< \Psi \mid A^* \mid \Phi > = < \Psi \mid A \mid \Phi >^*$
An operator A is represented as a matrix M_A of degree n for an n-dimensional Hilbert space H_n. If A is self-adjoint, then the matrix is called Hermitian, with the following properties:
$\rightsquigarrow$ all eigenvalues are real numbers
$\rightsquigarrow$ the determinant equals the product of its eigenvalues
$\rightsquigarrow$ the trace (sum over its diagonal elements) equals the sum of its eigenvalues

If A is self-adjoint in H_n, then there exist n eigenvalues λ such that there is a vector solution Φ_λ with

$$A * \Phi_\lambda \ = \ \lambda * \Phi_\lambda$$

Φ_λ is an eigenvector of A. The set of eigenvalues of A is called its spectrum. It is said to be non-degenerate if all pairs of eigenvalues from the set are different.

For any Hilbert space H and any self-adjoint operator A there is an orthonormal basis consisting of eigenvectors of A.

Projections

$\rightarrow$ are special operators that map vectors into subspaces

Def: Let H be a Hilbert space with subspaces W and $W^{\perp}$ such that $H = W \oplus W^{\perp}$, then each $\Phi \in H$ has a unique representation $\Phi = \Phi_W + \Phi_{W^{\perp}}$ such that $\Phi_W \in W$ and $\Phi_{W^{\perp}} \in W^{\perp}$.
The projections are then

$$P_W(\Phi) \ = \ \Phi_W \mid P_{W^{\perp}}(\Phi) \ = \ \Phi_{W^{\perp}}$$

If W is a subspace spanned by a single vector Ψ, then the projection of Φ is denoted as $P_{\Psi}(\Phi)$

Def: If $\{ \mid \Phi_0 >, \ \ldots \ , \mid \Phi_{(n-1)} > \}$ is an orthogonal basis, and some $\mid \Psi >$ is given relative to this basis as

$$\mid \Psi > = \sum_{i=0}^{n-1} < \Phi_i \mid \Psi > \ \mid \Phi_i >$$

then the projection P_{Φ_i} is defined as

$$P_{\Phi_i} \mid \Psi > = < \Phi_i \mid \Psi > \ \mid \Phi_i >$$

For any $\Phi \ \in \ H$ with $\| \ \Phi \ \| = \ 1$, the operator
$P_{\Phi} = \mid \Phi >< \Phi \mid$ with

$$P_{\Phi} \mid \Psi > = \ \mid \Phi >< \Phi \mid (\mid \Psi >) \ = < \Phi \mid \Psi > \ \mid \Phi >$$

is the projection (of Ψ) onto the one-dimensional space $\mid \Phi >$)

If $\lambda_0, \ \ldots \ , \ \lambda_{(n-1)}$ are eigenvalues of an operator A in a Hilbert space H_n, then A has the spectral representation

$$A \ = \ \sum_{i=0}^{n-1} \lambda_i \mid \Phi_i >< \Phi_i \mid \ = \ \sum_{i=0}^{n-1} \lambda_i P_{\Phi_i}$$

where Φ_i are the eigenvectors associated with λ_i.

Observables of quantum systems

$\rightarrow$ measurements of quantum states are the most difficult problem in quantum physics.
$\leadsto$ in classical physics a measurement always is an answer to the question: what is the value of a particular state | property of an object of the real world $\leadsto$ it relates to some pre-existing measurable reality
$\leadsto$ in quantum physics the question must be rephrased: is there an object whose property | state takes on a specific value ?
$\leadsto$ only by being measured does an object assume such value | property, otherwise this value | property is completely undetermined.

$\rightarrow$ measurements in quantum mechanics are specified by a special operator – called observable – applied to a quantum state.
$\rightarrow$ an observable is a self-adjoint operator A with

$$A = \sum_{i=0}^{n-1} a_i * P_i$$

where a_i are the eigenvalues of A and the P_i are projections onto the eigenvectors | eigenstates associated with the eigenvalues a_i.
With Φ_i being an eigenvector, we have

$$P_i = \mid \Phi_i >< \Phi_i \mid$$

$\rightarrow$ a measurement of an observable A has a numerical outcome which is an eigenvalue a_i of A
$\rightarrow$ after a measurement the quantum state collapses into $a_i * \mid \Phi_i >$ (a_i is the eigenvalue measured, Φ_i is the eigenvector associated with it).
$\rightarrow$ if the quantum state at the time of the measurement was Ψ, then the outcome a_i is obtained with probability

$$p(a_i) \ = \| \ P_i \mid \Psi > \|^2 = < \Psi \mid P_i \mid \Psi >$$

and the normalized quantum state becomes

$$P_i \mid \Psi > / < \Psi \mid P_i \mid \Psi >^2$$

$\rightarrow$ another measurement of the state results in the same state with probability $p(a_i) \ = \ 1$.
$\rightarrow$ since a non-degenerate operator A of degree n produces n eigenvalues, one gets one of n possible outcomes of a measurement $\rightsquigarrow$ the state collapses into one of n possible states specified by the eigenvectors
$\rightarrow$ the choice of the observable | the operator A determines what the eigenvectors and the projections to them are $\rightsquigarrow$ the operator A in fact specifies the apparatus which does the measurement.