Comparing Disjunctive Modal Transition Systems with Their One-Selecting Variant

Diplomarbeit

Heiko Schmidt

hsc@informatik.uni-kiel.de

Betreuer:
Dr. Harald Fecher,
Prof. Dr. Willem-Paul de Roever

Lehrstuhl Softwaretechnologie
Institut für Informatik und Praktische Mathematik
Christian-Albrechts-Universität zu Kiel

Kiel, 12.01.2006

Erklärung

Hiermit erkläre ich, Heiko Schmidt, diese Arbeit selbständig angefertigt und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Abstract

Refinement and abstraction are common concepts for the specification, analysis and verification of programs and dual to each other. Modal transition systems are an appropriate model for the abstraction of transition systems, since they explicitly specify necessary and possible behaviour by using two different kinds of transitions. A generalisation of modal transition systems, called disjunctive modal transition systems (DMTS), allows hypertransitions, which are transitions having a set of targets instead of a single target. These hypertransitions are interpreted such that an implementation must match *at least* one element of the target set. In this thesis, another generalisation, so-called 1-selecting modal transition systems (1MTS), is introduced, in which hypertransitions are interpreted such that *exactly* one element of the target set has to be matched.

In order to determine the relative expressive power of DMTSs and 1MTSs, a specialised concept of order embeddings is developed that demands the preservation of implementations. Although DMTSs and 1MTSs can describe the same sets of implementations, such an implementation-preserving order embedding can only be found in one direction, from DMTSs to 1MTSs. Thus regarding the refinement ordering structure, 1MTSs are found to be more expressive.

Keywords: underspecification, abstraction, refinement, transition systems, disjunctive modal transition systems, expressiveness

Contents

Chapter 1

Introduction

This chapter gives an introduction into the considered field of research. We describe the contribution of this thesis and point out, how the following chapters are organised.

1.1 Semantics of Concrete Processes

This work deals with the specification and underspecification of possibly concurrent systems. More exactly, we describe the observable behaviour of a system rather than the system itself. The behaviour of a system is called *process*. We consider processes that perform some kind of *actions*, i.e., steps of observable behaviour. It remains unspecified what kind of system we describe by defining a process and what kind of actions this system performs. For example, the system might be a program with actions being variable assignments or it might be an agent in a reactive system with actions being communications between agents.

It is a common approach to describe processes using so called *labelled transition systems* [24]. We will henceforth simply call them transition systems and abbreviate them by TS. A transition system is a directed graph with a designated root node and labelled edges. The nodes are identified with the states of the process. We typically use the variables s, t, u, ... for states. Edges are labelled by a, b, c, ... and stand for transitions of the system: An edge from state s to state s' having label a, shortly written $s \xrightarrow{a} s'$, means that if a process is in state s and an action a occurs, it will evolve into state s'. Non-determinism is allowed, i.e., there may be states s, s'_1, s'_2 and a label a such that $s \xrightarrow{a} s'_1$ and $s \xrightarrow{a} s'_2$. In this case, an action a can lead to state s'_1 or s'_2. We do not specify how this decision is made.

Figure 1.1 shows a simple example of a transition system. The root state of the transition system, in this case s, is marked by a small arrow pointing to it. From the root state, three transitions are possible; an action a leads to state t, an action b leads to u and an action c leads to state v. We call t, u, v *successor states* of u. To denote successor states, we often use primed variables, i.e., u', v', w', ...

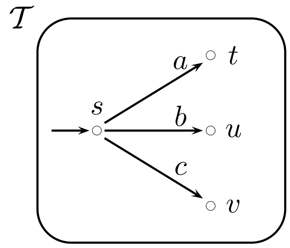

Figure 1.1: Transition System $\mathcal{T}$

In illustrations, we will sometimes omit state names, because the names have no effect on the observable behaviour of a transition system.

Usually there exist many transition systems that describe the same process. Thus in order to specify processes using transition systems, there is the need for an equivalence relation that defines which transition systems are to be identified. Then processes can be seen as equivalence classes of transition systems with regard to some chosen equivalence relation. As presented in [24], various equivalences have been established. The most common relations include *trace equivalence* and *bisimulation equivalence*.

Trace equivalence [9, 24] is defined as follows: The *trace set* of a transition system is the set of possible sequences of labels that are passed when starting in the root state and repeatedly following possible transitions. Then two transition systems are called *trace equivalent*, if they have equal trace sets.

Trace equivalence is not sufficient if the branching structure is of importance. Thus in most cases one prefers "finer" relations, e.g. the widely-used bisimulation equivalence [18, 24]: A bisimulation between two transition systems $\mathcal{T}_1$ and $\mathcal{T}_2$ is a relation between the two state sets such that the root states are related and the following property holds: For every transition $s_1 \xrightarrow{a} s_1'$ in $\mathcal{T}_1$ and every s_2 related to s_1, there exists some s_2' related to s_1' such that $s_2 \xrightarrow{a} s_2'$, and vice verca, for every transition $s_2 \xrightarrow{a} s_2'$ in $\mathcal{T}_2$ and every s_1 related to s_2, there exists some s_1' related to s_2' such that $s_1 \xrightarrow{a} s_1'$. Two transition systems are called *bisimilar* (or *bisimulation equivalent*) if there exists a bisimulation.

1.2 Underspecified Semantics

We have seen how a process can be specified using a transition system and an equivalence relation. However, in many applications it is useful to *underspecify*, i.e., not to specify the system in every detail, but instead leave parts of the system open to allow different implementations. For example, when developing a system in top-down manner, one starts with a general, underspecified framework of the system, leaving many parts open that are left to be implemented later. The term *refinement* describes the process of turning an underspecified system into a more concrete one. Stepwise refinement is a common technique in program development [25].

Another reason for having holes in the specification might be that the left out parts are simply not of interest for the task one wants to achieve. For example, in *model checking*, it is checked whether a formula in some modal logic holds for a given system. It might be possible to detect the validity of a formula without considering every detail of the given system. The given, fully specified system can be turned into an abstract, underspecified system with less states (possibly even turning an infinite state set into a finite one), which can be model checked more efficiently. This technique is called *abstraction*.

As a further application, underspecification can be useful if processes are used as a language to define semantics. For example, modeling languages, like UML [20, 21], are themselves underspecified, as they do not specify every detail of a program. Thus underspecification may be suitable for defining the semantics of such modeling languages.

1.2.1 Established Underspecification Techniques

If we want to underspecify, we need to find formalisms that do not describe a single process, but rather a set of processes, namely those that are supposed to be implementations of the underspecification. Furthermore, we need an efficient method to do abstraction/refinement checks: Is one system an abstraction/refinement of another? Which modifications of a given system yield systems that are refinements of the original?

One possible approach is to use a formula in some modal logic as specification and regard the class of transition systems satisfying the formula as implementations. Although this is useful in some applications, others demand more operational based approaches. These use transition systems and non-symmetric relations similar to trace and bisimulation equivalence, which have already been introduced in Section 1.1 as formalisms for (full) specification. In the following, we summarise some techniques used for underspecification in branching sensitive views:

Underspecification via Simulation

A common approach makes use of *simulations* [17]. A transition system $\mathcal{T}_1$ *(safety-)simulates* another transition system $\mathcal{T}_2$, if there exists a relation between the state sets of $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the root states are related and the following property holds: For every transition $s_1 \xrightarrow{a} s_1'$ in $\mathcal{T}_1$ and every s_2 related to s_1, there exists some s_2' related to s_1' such that $s_2 \xrightarrow{a} s_2'$. In this case, $\mathcal{T}_2$ *reverse-simulates* (or *liveness-simulates*) $\mathcal{T}_1$. Now underspecification can be introduced using one of these two notions: One can define some (concrete) transition system $\mathcal{T}_1$ to be an implementation of an (abstract) transition system $\mathcal{T}_2$, if $\mathcal{T}_1$ simulates (respectively reverse-simulates) $\mathcal{T}_2$. However, we have the problem of not being able to define combinations of *safety properties* ("something bad will

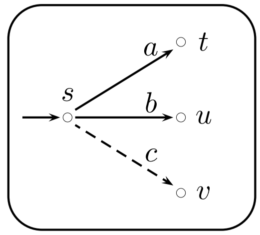

Figure 1.2: Transition System with Must And May Transitions

not happen") and liveness properties *liveness properties* ("eventually something good happens"), i.e., all behaviour can be neglected when the safety-simulation approach is used, respectively arbitrary behaviour can be added if the liveness-simulation approach is used. For example, it is not possible to express the class of all transition systems, where at the beginning actions a and b have to be possible, action c is allowed but no further actions are allowed at the beginning.

Underspecification via Must and May Transitions

In order to address this problem, one can use a modification of transition systems that features two kinds of transitions, one transition relation to denote the steps that are mandatory for the implementation, called *must transitions*, and the other to indicate those steps which may occur, but are not necessary for the implementation, called *may transitions*. This approach was followed by Larsen and Thomsen, who introduced *modal transition systems* [15], and Dams, who called his model *mixed transition systems* [4, 5]. We can solve the problem mentioned at the end of the previous section by turning the transition labelled with c into a may transition (Figure 1.2). Graphically, we represent must transitions as solid arrows, whereas may transitions are drawn as dashed arrows.

Underspecification via Disjunctive Modal Transition Systems

However, the approach with must and may transitions is still not sufficient to model all behaviours that appear in practice. Reconsider transition system $\mathcal{T}$ from Figure 1.1. Three transitions were possible in the initial state. One might like to express that two different implementations allow two different successor states after performing an action a. However, an action a shall be possible in every implementation. Thus we cannot use two may transitions to the two states, because this would also allow an implementation with no action a at all. Two must transitions would not serve either, because they would mean that both must be possible in *every* implementation. A solution to this problem is a further extension of transition systems. These are called *disjunctive modal transition systems* (DMTS) [16]. They allow *hypertransitions*, i.e., transitions starting in a single state, but ending in sets of states. Graphically, we represent hypertransitions by

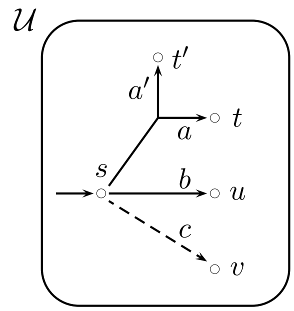

Figure 1.3: Disjunctive Modal Transition System $\mathcal{U}$

dividing the head of an arrow such that it points to all target states. For different targets of a single hypertransition we also allow different labels, thus we draw them behind the division. Figure 1.3 shows an example of a DMTS, named $\mathcal{U}$. It expresses that every implementation has to provide an action b, may provide an action c, and must provide one of the actions a or a'.[1]

The class of implementations of a DMTS is defined via a simulation relation that relates abstract DMTSs and concrete transition systems. The commonly used relation is called *disjunctive simulation*. For every hypertransition it demands that at least one of the targets can be reached in every implementation. Thus it also allows that an implementation can contain transitions to more than one target of the hypertransition.

1.2.2 A New Underspecification Technique

In this thesis, we introduce and examine a new technique for underspecification that takes an approach similar to DMTSs. We also use transition systems with must- and may-hypertransitions, but interpret hypertransitions differently, i.e., we use a modified simulation relation. This simulation relation is called *1-selecting simulation* and the corresponding transition systems are called *1-selecting modal transition systems* (1MTS). In contrast to the disjunctive approach, 1-selecting simulation demands, roughly spoken, that for every hypertransition *exactly one* of its targets corresponds to a transition in the implementation. Thus, if we understand example $\mathcal{U}$ in Figure 1.3 as an 1MTS, it does not allow an implementation in which both actions a and a' can be performed. If $\mathcal{U}$ is understood as DMTS, this is allowed.

The existing refinement notion on DMTSs is adapted for 1MTSs, yielding an abstraction/refinement framework based on 1MTSs. The aim of this work is

[1] In fact, this could have also been expressed with the must and may transition approach presented in the previous section, if we allow more than one root state. One could use a transition system with two components that both look like the one in Figure 1.2, where in one of them action a is replaced by a'. However, such a reduction is no longer possible without getting an infinite system, if, for example, we add to $\mathcal{U}$ in Figure 1.3 a transition from state t' to state s.

to compare the expressive power of the established DMTS and the new 1MTS approach. Both formalisms can express the same sets of implementations. However, because the refinement ordering structure should be taken into account, we specialise the concept of order embeddings to a notion for the comparison of abstraction/refinement frameworks that demands the preservation of implementations. We prove that, using this concept, the class of DMTSs can be embedded in the class of 1MTSs, but the class of 1MTSs cannot be embedded in the class of DMTSs. Thus, in this sense, we find 1MTSs to be more expressive than DMTSs.

1.3 Outline

In Chapter 2 of this thesis, we give a formal introduction into common, concrete transition systems and the two underspecified variants DMTS and 1MTS. We define a simulation relation between TSs and DMTSs and a simulation relation between TSs and 1MTSs. These relations define, whether a TS is an implementation of a DMTS, respectively 1MTS. In the last section of Chapter 2, fully determined DMTSs and 1MTSs, which correspond to TSs, are introduced. It is shown that the simulation notions on fully determined modal transition systems harmonise with the bisimulation notion on TSs and in this sense, the simulation relations are extensions of bisimulation.

Chapter 3 is dedicated to refinement in both DMTSs and 1MTSs. The introduced refinement notions are relations between DMTSs and DMTSs, respectively 1MTSs and 1MTSs, and define, whether one modal transition system $\mathcal{U}$ is a refinement of another modal transition system $\hat{\mathcal{U}}$ (or equivalently, whether $\hat{\mathcal{U}}$ is an abstraction of $\mathcal{U}$). A special case is that the more concrete system has "maximal concreteness" and can be identified with exactly one TS. We call such systems *fully determined*. It is shown that in cases, in which the more concrete system is fully determined, refinement (on DMTSs, respectively 1MTSs) harmonises with the simulation notion (on DMTSs, respectively 1MTSs) and in this sense, refinement is an extension of simulation. Equivalences on DMTSs and 1MTSs, that are induced by the corresponding refinement notions, are introduced and afterwards used in order to characterise all refinements of both a simple DMTS and a simple 1MTS.

Chapter 4 deals with the expressiveness of DMTSs and 1MTSs. First, the concept of order embeddings is specialised to so-called φ-preserving order embeddings that provide a general notion for the comparison of abstraction/refinement frameworks. This notion is then used in order to compare DMTSs and 1MTSs. It is shown that such a specialised order embedding exists from DMTSs to 1MTSs, but not from 1MTSs to DMTSs. This is the case, although DMTSs and 1MTSs can express the same sets of implementations.

Finally, Chapter 5 subsumes the main results of this work, gives an overview over related work and discusses future work.

Chapter 2

Transition System Variants and Modal Simulation

In this chapter, we give formal definitions of (labelled) transition systems (TSs), disjunctive modal transition systems (DMTSs), and 1-selecting modal transition systems (1MTSs), together with their corresponding simulation relations. The connection between TSs and fully determined DMTSs, respectively 1MTSs, is characterised.

2.1 Labelled Transition Systems

Labelled transition systems, which will in this work shortly be called transition systems, are directed graphs, where edges are labelled and we have a designated root state. Formally:

Definition 2.1 (TS)**.** *A* transition system *(TS) is a tuple* $(S, L, \longrightarrow, s^0)$, *where* S *is a set of states,* L *is a set of labels,* $\longrightarrow \subseteq S \times L \times S$ *is the* transition relation *and* $s^0 \in S$ *is the* root state. *We denote the set*[1] *of all TSs by* $\mathbb{TS}$.

Transitions are elements of the transition relation. For a state $s \in S$, we call $\{(a, s') \in L \times S \mid (s, a, s') \in \longrightarrow\}$ the *transition set* of u. Elements of the transition set (that are pairs of label and successor state) are called *targets*.

We give a formal definition of the most common equivalence relation used with TSs, *bisimulation*. Before that, we declare some conventions: For a binary relation $R \subseteq A \times B$ and $a \in A$, $b \in B$, we often write aRb instead of $(a, b) \in R$. Furthermore, we define:

Definition 2.2. *Let* A *and* B *be sets and* $\longrightarrow \subseteq A \times B$. *Let* $a \in A$. *Then define* $(a \longrightarrow) \stackrel{\text{def}}{=} \{b \in B \mid a \longrightarrow b\}$.

[1]Strictly speaking, this is a class, but a skeletal set may be chosen. This remark also applies to DMTSs and 1MTSs, that are defined later.

Now we can define the widely-used bisimulation notion as follows:

Definition 2.3 (Bisimulation). *Let* $\mathcal{T}_1 = (S_1, L, \longrightarrow_1, s_1^0), \mathcal{T}_2 = (S_2, L, \longrightarrow_2, s_2^0) \in \mathbb{TS}$*. A* bisimulation *between* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *is a relation* $R \subseteq S_1 \times S_2$ *such that the following properties hold:*

(i) $s_1^0 R s_2^0$

(ii) For all $(s_1, s_2) \in R$ *and* $a \in L$ *we have*

(a) $\forall (a_1, s_1') \in (s_1 \longrightarrow_1) : \exists (a_2, s_2') \in (s_2 \longrightarrow_2) : a_1 = a_2 \wedge s_1' R s_2'$

(b) $\forall (a_2, s_2') \in (s_2 \longrightarrow_2) : \exists (a_1, s_1') \in (s_1 \longrightarrow_1) : a_1 = a_2 \wedge s_1' R s_2'$

$\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are called* bisimilar *if there exists a bisimulation between them. In that case, we write* $\mathcal{T}_1 \sim \mathcal{T}_2$*.*

In the definition of bisimulation, we need to check the property $a_1 = a_2 \wedge s_1' R s_2'$ twice, where a_1, a_2 are labels and s_1', s_2' are states of the first and the second TS, respectively. Checks of this kind will often occur throughout this work. Thus in the following, we simplify the notation by extending any relation between state sets (say $R \subseteq S_1 \times S_2$) to $(L \times S_1) \times (L \times S_2)$ as follows: For targets $\vartheta_1 = (a_1, s_1')$ and $\vartheta_2 = (a_2, s_2')$, let

$$\vartheta_1 R \vartheta_2 \overset{\text{def}}{\Leftrightarrow} a_1 = a_2 \wedge s_1' R s_2'.$$

Then property (ii)(a) of bisimulation can be reformulated as

$$\forall \vartheta_1 \in (s_1 \longrightarrow_1) : \exists \vartheta_2 \in (s_2 \longrightarrow_2) : \vartheta_1 R \vartheta_2$$

and property (ii)(b) becomes

$$\forall \vartheta_2 \in (s_2 \longrightarrow_2) : \exists \vartheta_1 \in (s_1 \longrightarrow_1) : \vartheta_1 R \vartheta_2.$$

Bisimilarity, i.e., the relation $\sim \subseteq \mathbb{TS} \times \mathbb{TS}$, is obviously an equivalence relation. Consequently we can consider its equivalence classes:

Definition 2.4. *Define*

$$[\mathcal{T}]_\sim \overset{\text{def}}{=} \{\bar{\mathcal{T}} \in \mathbb{TS} \mid \bar{\mathcal{T}} \sim \mathcal{T}\}$$

to be the equivalence class that $\mathcal{T}$ *is a representative of. Furthermore, define*

$$\underline{\mathbb{TS}} \overset{\text{def}}{=} \{[\mathcal{T}]_\sim \mid \mathcal{T} \in \mathbb{TS}\}$$

to be the set of equivalence classes of TS*s.*

2.2 Disjunctive Modal Transition Systems

Disjunctive modal transition systems extend TSs. Instead of a single root state, they allow a set of root states. They feature two types of transitions: must and may transitions. In contrast to the transitions in TSs, a transition in a DMTS has a set of targets instead of a single target. We require this set to be non-empty, and, in the case of may transitions, the set has to have exactly one element. Every target in the target set of a must transition is required to appear also as the target of a may transition. This requirement seems reasonable, transitions required ("must") need to be allowed ("may"). The formal definition is as follows:

Definition 2.5 (DMTS)**.** *A* disjunctive modal transition system *(DMTS) is a tuple* $(U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U_\mathrm{D}^0)$*, where* U_D *is a set of states,* L *is a set of labels,* $\longmapsto_\mathrm{D} \subseteq U_\mathrm{D} \times (\mathcal{P}(L \times U_\mathrm{D}) \setminus \emptyset)$ *is the* must transition relation*,* $\dashrightarrow_\mathrm{D} \subseteq U_\mathrm{D} \times (\mathcal{P}(L \times U_\mathrm{D}) \setminus \emptyset)$ *is the* may transition relation*, and* $\emptyset \neq U_\mathrm{D}^0 \subseteq U_\mathrm{D}$ *is the set of* root states*, satisfying for all* $u \in U_\mathrm{D}$ *the conditions* $\forall \Theta_\mathrm{D} \in (u \dashrightarrow_\mathrm{D}) : |\Theta_\mathrm{D}| = 1$ *and*

$$\forall \Theta_\mathrm{D} \in (u \longmapsto_\mathrm{D}), \vartheta \in \Theta_\mathrm{D} : \{\vartheta\} \in (u \dashrightarrow_\mathrm{D}). \tag{2.1}$$

We denote the set of all DMTSs by $\mathbb{DMTS}$*.*

A transition, i.e., an element of the must or may transition relation, is called *hypertransition*, if and only if its target set contains more than one element. In graphical representations, hypertransitions are drawn as arrows having several heads, with every head having its own label. Must transitions are represented by solid arrows, whereas may transitions are drawn as dashed arrows. We do not draw a may transition from a state u, if there is a must (hyper-)transition starting in u that has a target set including the target of the may transition. Due to condition (2.1), such a may transition always exists implicitly.

Next, we define disjunctive simulation, the common simulation relation used with DMTSs. Similar to the bisimulation notion on TSs, a disjunctive simulation is a relation relating states in the TS with states in the DMTS, such that several properties are satisfied: The DMTS needs to have a root state corresponding to the root state of the TS. Furthermore, every state s' in the TS that can be reached by an action a needs to have an equivalent counterpart u' in the DMTS that is not forbidden to be reached with an action a. On the other hand, for every must transition to a set of targets Θ_D in the DMTS, there must be at least one target in Θ_D that has a corresponding step in the TS.

A TS *disjunctively simulates* a given DMTS, if there exists a disjunctive simulation between them. We define some TS $\mathcal{T}$ to be an implementation of an DMTS $\mathcal{U}_\mathrm{D}$, if and only if $\mathcal{T}$ disjunctively simulates $\mathcal{U}_\mathrm{D}$. The formal definition is as follows:

Definition 2.6 (Disjunctive simulation)**.** *Let* $\mathcal{T} = (S, L, \longrightarrow, s^0)$ *be a TS and* $\mathcal{U} = (U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U_\mathrm{D}^0)$ *be a DMTS over the same set of labels. A* disjunctive simulation *between* $\mathcal{T}$ *and* $\mathcal{U}_\mathrm{D}$ *is a relation* $R \subseteq S \times U_\mathrm{D}$ *such that the following properties hold:*

(i) $\exists u \in U^0_{\mathrm{D}} : s^0 R u$

(ii) For all $(s, u) \in R$ *we have*

(a) $\forall \vartheta \in (s \longrightarrow) : \exists \{\hat{\vartheta}\} \in (u \dashrightarrow_{\mathrm{D}}) : \vartheta R \hat{\vartheta}$

(b) $\forall \hat{\Theta}_{\mathrm{D}} \in (u \longmapsto_{\mathrm{D}}) : \exists \vartheta \in (s \longrightarrow), \hat{\vartheta} \in \hat{\Theta}_{\mathrm{D}} : \vartheta R \hat{\vartheta}$

$\mathcal{T}$ disjunctively simulates $\mathcal{U}_{\mathrm{D}}$ *if and only if there exists a disjunctive simulation between them. In that case, we write* $\mathcal{T} \prec_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$ *and call* $\mathcal{T}$ *an* implementation *of* $\mathcal{U}_{\mathrm{D}}$*.*

We conclude this section by taking a closer look at the requirement for DMTSs that may transitions are not allowed to be hypertransitions, i.e., that for all $\forall \Theta_{\mathrm{D}} \in (u \dashrightarrow_{\mathrm{D}})$, we require $|\Theta_{\mathrm{D}}| = 1$. In detail, we consider the following two questions: (a) What is the reason for making this restriction? And (b), why are singleton sets used instead of simple elements, i.e., why $\dashrightarrow_{\mathrm{D}} \subseteq U_{\mathrm{D}} \times (\mathcal{P}(L \times U_{\mathrm{D}}) \setminus \emptyset)$ instead of $\dashrightarrow_{\mathrm{D}} \subseteq U_{\mathrm{D}} \times L \times U_{\mathrm{D}}$?

To answer the first question, allowing may hypertransitions would simply not lead to additional expressiveness. After defining a generalised simulation notion for DMTSs with may hypertransitions, one can show that the set of implementations of a DMTS with may hypertransitions equals the set of implementations of the DMTS without may hypertransitions, in which each hypertransition was turned into a set of singleton may transitions [22]. This is quite intuitive: The disjunction of a set of possibilities (may hypertransition) coincides with the conjunction of a set of possibilities (set of singleton may transitions). In both cases, arbitrarily many possibilities can be taken.

The answer to the second question, why singleton sets are used in the may transition relation instead of simple elements, results from the fact that for 1MTSs, introduced in the following section, may hypertransitions make sense, since by means of them, expressiveness is gained. To keep the "signatures" of DMTSs and 1MTSs equal, we decided to use sets also in DMTSs.

2.3 1-Selecting Modal Transition Systems

Having defined the well-known DMTSs together with their simulation notion, we now continue with the definition of the new 1MTSs. The approach is similar, since the two notions essentially only differ in the interpretation of hypertransitions. As a result of this difference, may hypertransitions, that were not featured in DMTSs, make sense here and for this reason are allowed. As a consequence, the requirement that must transitions have to appear as may transitions (condition (2.1) in the case of DMTSs) then simplifies to $\longmapsto_1 \subseteq \dashrightarrow_1$ for 1MTSs. We adopt the following formal definition:

Definition 2.7 (1MTS)**.** *A* 1-selecting modal transition system *(1MTS) is a tuple* $(U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0)$*, where* U_1 *is a set of states,* L *is a set of labels,* $\longmapsto_1 \subseteq U_1 \times (\mathcal{P}(L \times U_1) \setminus \emptyset)$ *is the* must transition relation*,* $\dashrightarrow_1 \subseteq U_1 \times (\mathcal{P}(L \times U_1) \setminus \emptyset)$ *is the* may transition relation*, and* $\emptyset \neq U_1^0 \subseteq U_1$ *is the set of* root states*, satisfying the condition* $\longmapsto_1 \subseteq \dashrightarrow_1$*. We denote the set of all 1MTSs by* $\mathbb{1MTS}$*.*

For the definition of 1-selecting simulation, we use *choice functions*. Those are functions from a set of sets into a set of elements such that every set is mapped to an element of itself. Formally:

Definition 2.8 (Choice function)**.** *Let* A *be a set,* $\mathcal{B} \subseteq \mathcal{P}(A)$ *and* $\gamma : \mathcal{B} \to A$*.* γ *is called* choice function*, if and only if* $\forall B \in \mathcal{B} : \gamma(B) \in B$*. We denote the set of all choice functions on* $\mathcal{B}$ *by* $\mathsf{choice}(\mathcal{B})$*.*

The concept of 1-selecting simulation is similar to the one of disjunctive simulation. Again, we have a simulation relation that is required to satisfy several properties: As for the disjunctive approach, the 1MTS needs to have a root state corresponding to the root state of the TS. The interpretation of hypertransitions differs from disjunctive simulation: One needs to choose choice functions γ that pick for each hypertransition one of its targets. Then every state s' in the TS that can be reached by an action a needs to have an equivalent counterpart u' that has been chosen by γ and is not forbidden to be reached with an action a. On the other hand, for every chosen must transition target reachable with action a, there must be a corresponding state in the TS reachable with a.

A TS *1-selecting simulates* a given DMTS, if there exists a 1-selecting simulation between them. We define some TS $\mathcal{T}$ to be an implementation of an 1MTS $\mathcal{U}_1$, if and only if $\mathcal{T}$ 1-selecting simulates $\mathcal{U}_1$. The formal definition is as follows:

Definition 2.9 (1-selecting simulation)**.** *Let* $\mathcal{T} = (S, L, \longrightarrow, s^0)$ *be a TS and* $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0)$ *be a 1MTS over the same set of labels. An* 1-selecting simulation *between* $\mathcal{T}$ *and* $\mathcal{U}_1$ *is a relation* $R \subseteq S \times U_1$ *such that the following properties hold:*

(i) $\exists u \in U_1^0 : s^0 R u$

(ii) For all $(s, u) \in R$ *there exists* $\gamma \in \mathsf{choice}(u \hat{\dashrightarrow}_1)$ *such that*

(a) $\forall \vartheta \in (s \longrightarrow) : \exists \hat{\vartheta} \in \gamma(u \dashrightarrow_1) : \vartheta R \hat{\vartheta}$

(b) $\forall \hat{\vartheta} \in \gamma(u \longmapsto_1) : \exists \vartheta \in (s \longrightarrow) : \vartheta R \hat{\vartheta}$

$\mathcal{T}$ 1-selecting simulates $\mathcal{U}_1$ *if and only if there exists a 1-selecting simulation between them. In that case, we write* $\mathcal{T} \prec_1 \mathcal{U}_1$ *and call* $\mathcal{T}$ *an* implementation *of* $\mathcal{U}_1$*.*

Note that in parts (ii)(a) and (b) of the definition, the choice function γ is applied to a set of target sets and yields a set of (chosen) targets. Thus $\gamma(u \dashrightarrow_1) = \{\gamma(\Theta) \mid \Theta \in (u \dashrightarrow_1)\}$ and $\gamma(u \longmapsto_1) = \{\gamma(\Theta) \mid \Theta \in (u \longmapsto_1)\}$.

2.4 Fully Determined Modal Transition Systems

Fully determined DMTSs and 1MTSs are those systems that are completely described in the sense that they directly correspond to a TS. The correspondence to a TS is obviously established, if (i) there is only one root state, (ii) there are no hypertransitions and (iii) every may transition also appears as must transition. This holds for both DMTSs and 1MTSs, thus it is possible to define fully determined DMTSs and fully determined 1MTSs at once:

Definition 2.10 (Fully determined modal transition system)**.** *Let* $\mathcal{U} = (U, L, \longmapsto, \vdash\!\dashrightarrow, U^0) \in \mathbb{DMTS} \cup \mathbb{1MTS}$*. We call* $\mathcal{U}$ fully determined*, if and only if*

(i) $|U^0| = 1$*,*

(ii) $\forall u \in U, \Theta \in (u \longmapsto) : |\Theta| = 1$*, and*

(iii) $\vdash\!\dashrightarrow \subseteq \longmapsto$*.*

We denote the set of all fully determined DMTSs by $\mathbb{DMTS}^{\mathrm{det}}$ *and the set of all fully determined 1MTSs by* $\mathbb{1MTS}^{\mathrm{det}}$*.*

Any fully determined modal transition system satisfies $\vdash\!\dashrightarrow\, = \,\longmapsto$: 1MTSs always have the property $\longmapsto \subseteq \vdash\!\dashrightarrow$, which together with condition (iii) in the definition above implies $\vdash\!\dashrightarrow\, = \,\longmapsto$. DMTSs satisfy condition (2.1), i.e.,

$$\forall \Theta \in (u \longmapsto), \vartheta \in \Theta : \{\vartheta\} \in (u \vdash\!\dashrightarrow),$$

which by condition (ii) in the definition above yields $\longmapsto \subseteq \vdash\!\dashrightarrow$, resulting together with condition (iii) in $\vdash\!\dashrightarrow\, = \,\longmapsto$.

$\mathbb{DMTS}^{\mathrm{det}}$ and $\mathbb{1MTS}^{\mathrm{det}}$ are the same mathematical objects, i.e., $\mathbb{DMTS}^{\mathrm{det}} = \mathbb{1MTS}^{\mathrm{det}}$. $\mathbb{DMTS}^{\mathrm{det}}$ ($= \mathbb{1MTS}^{\mathrm{det}}$) corresponds to $\mathbb{TS}$ via the function

$$\pi : \mathbb{TS} \to \mathbb{DMTS}^{\mathrm{det}}; (S, L, \longrightarrow, s^0) \mapsto (S, L, \longmapsto, \longmapsto, \{s^0\}),$$

where $\longmapsto \stackrel{\mathrm{def}}{=} \{(s, \{(a, s')\}) \mid s \xrightarrow{a} s'\}$. The function π is obviously bijective. For better readability, we define the following functions:

Definition 2.11. *Define:*

$$\begin{aligned}
\mathsf{DMTS} :&\quad \mathbb{TS} \to \mathbb{DMTS}^{\mathrm{det}};\quad \mathcal{T} \mapsto \pi(\mathcal{T}),\\
\mathsf{1MTS} :&\quad \mathbb{TS} \to \mathbb{1MTS}^{\mathrm{det}};\quad \mathcal{T} \mapsto \pi(\mathcal{T}),\\
\mathsf{TS}_{\mathrm{D}} :&\quad \mathbb{DMTS}^{\mathrm{det}} \to \mathbb{TS};\quad \mathcal{U}_{\mathrm{D}} \mapsto \pi^{-1}(\mathcal{U}_{\mathrm{D}}),\\
\mathsf{TS}_{1} :&\quad \mathbb{1MTS}^{\mathrm{det}} \to \mathbb{TS};\quad \mathcal{U}_{1} \mapsto \pi^{-1}(\mathcal{U}_{1}).
\end{aligned}$$

Having established the tight connection between TSs and fully determined DMTSs (respectively 1MTSs), it is possible to compare the bisimulation notion with disjunctive (respectively 1-selecting) simulation on fully determined systems. In fact, the two notions coincide, as stated by the following proposition:

Proposition 2.12. *Let* $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{TS}$*. Then:*

$$\begin{aligned} \mathcal{T}_1 \sim \mathcal{T}_2 \quad &\Leftrightarrow \quad \mathcal{T}_1 \prec_{\mathrm{D}} \mathsf{DMTS}(\mathcal{T}_2) \\ &\Leftrightarrow \quad \mathcal{T}_1 \prec_1 \mathsf{1MTS}(\mathcal{T}_2). \end{aligned}$$

Proof. Consider the definition of disjunctive (respectively 1-selecting) simulation in the special case that the DMTS (respectively 1MTS) is a fully determined system and compare it with the definition of bisimulation. Then the proposition is obvious. □

Furthermore, disjunctive simulation and 1-selecting simulation are closed under bisimilarity:

Proposition 2.13. *Let* $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{TS}$*. Then*

(i) for all $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}$*, we have*

$$\mathcal{T}_1 \sim \mathcal{T}_2 \wedge \mathcal{T}_2 \prec_{\mathrm{D}} \mathcal{U}_{\mathrm{D}} \;\Rightarrow\; \mathcal{T}_1 \prec_{\mathrm{D}} \mathcal{U}_{\mathrm{D}},$$

(ii) for all $\mathcal{U}_1 \in \mathbb{1MTS}$*, we have*

$$\mathcal{T}_1 \sim \mathcal{T}_2 \wedge \mathcal{T}_2 \prec_1 \mathcal{U}_1 \;\Rightarrow\; \mathcal{T}_1 \prec_1 \mathcal{U}_1.$$

Proof. Let $\bar{R}$ be a bisimulation between $\mathcal{T}_1$ and $\mathcal{T}_2$ and R be a disjunctive simulation (respectively 1-selecting simulation) between $\mathcal{T}_2$ and $\mathcal{U}_{\mathrm{D}}$ (respectively $\mathcal{T}_2$ and $\mathcal{U}_1$). Then it is straightforwardly checked that $\bar{R} \circ R = \{(s_1, u) \mid \exists s_2 : (s_1, s_2) \in \bar{R} \wedge (s_2, u) \in R\}$ is a disjunctive simulation (respectively 1-selecting simulation) between $\mathcal{T}_1$ and $\mathcal{U}_{\mathrm{D}}$ (respectively $\mathcal{T}_1$ and $\mathcal{U}_1$). □

Chapter 3

Abstraction and Refinement

This chapter begins with a motivation for a formal refinement notion, as it is introduced afterwards for DMTSs and 1MTSs. After that, we give a characterisation of the connection between disjunctive, respectively 1-selecting, simulation on the one hand, and disjunctive, respectively 1-selecting, refinement on the other hand. We introduce equivalence relations that are induced by disjunctive, respectively 1-selecting, refinement and finally use these to completely characterise all refinements of a simple DMTS and a simple 1MTS.

3.1 Refinement Approaches

In Section 1.2, we have pointed out the need for refinement and abstraction formalisms in program development and verification. Refinement and abstraction can be regarded as being complementary notions: Whenever a system $\mathcal{U}$ refines a system $\hat{\mathcal{U}}$, the latter system $\hat{\mathcal{U}}$ is an abstraction of $\mathcal{U}$. For this reason it is enough, if we restrict ourselves to talking only about refinement (and thus making implicit statements about abstraction) in the following.

An important special case is that $\mathcal{U}$ is a concrete, *fully determined* system. Fully determined systems can be identified with TSs and, making use of this identification, we require a proper refinement notion to harmonise with the corresponding (disjunctive or 1-selecting) simulation notion, i.e., the corresponding fully determined system of any implementation of a system should be a refinement of that system, and vice versa, the corresponding TS of any fully determined refinement should be an implementation of the system.

This requirement is satisfied by a straightforward approach to define a refinement notion: One can define $\mathcal{U}$ to be a refinement of $\hat{\mathcal{U}}$, if and only if the set of implementations of $\mathcal{U}$ is a subset of the set of implementations of $\hat{\mathcal{U}}$. This type of refinement, which we will call *implementation-based refinement*, is considered in [22]. However, it has the disadvantage that refinement checks are inefficient. Consequently, there is a need for a direct refinement notion that approximates the straightforward refinement approach, but does not require considering all implementations. For DMTSs, such a refinement definition, based on simulation techniques, is presented in [16]. We call it *disjunctive refinement* in order to distinguish it from the refinement notion we introduce for 1MTSs, which will be called *1-selecting refinement.*

Note that the direct refinement notions do not coincide with the implementation-based approaches. Corollary 3.9 in Section 3.4 states that refinement in the direct sense implies implementation-based refinement. However, the opposite implication does not hold. After Corollary 3.9, a counter-example is presented.

3.2 Disjunctive Refinement

We define the refinement notion for DMTSs. A disjunctive refinement is a relation relating states of one DMTS, say $\mathcal{U}_\mathrm{D}$, with states of another DMTS, say $\hat{\mathcal{U}}_\mathrm{D}$, that satisfies several properties: For every root state of $\mathcal{U}_\mathrm{D}$ there exists a root state of $\hat{\mathcal{U}}_\mathrm{D}$ such that the two are related. Furthermore, for every target of a may transition in $\mathcal{U}_\mathrm{D}$, we must find a corresponding target of a may transition in $\hat{\mathcal{U}}_\mathrm{D}$. On the other hand, for every must (hyper-)transition in $\hat{\mathcal{U}}_\mathrm{D}$, we must find a must (hyper-)transition in $\mathcal{U}_\mathrm{D}$ such that for each target in the latter transition a corresponding target in the former transition can be found.

A DMTS $\mathcal{U}_\mathrm{D}$ disjunctively refines another DMTS $\hat{\mathcal{U}}_\mathrm{D}$, if and only if there is a disjunctive refinement between them. We will sometimes shortly say that $\mathcal{U}_\mathrm{D}$ refines $\hat{\mathcal{U}}_\mathrm{D}$, if it is clear that $\mathcal{U}_\mathrm{D}$ and $\hat{\mathcal{U}}_\mathrm{D}$ are DMTSs. Furthermore, $\mathcal{U}_\mathrm{D}$ is called a refinement of $\hat{\mathcal{U}}_\mathrm{D}$ and $\hat{\mathcal{U}}_\mathrm{D}$ is called an abstraction of $\mathcal{U}_\mathrm{D}$. We write $\mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$. The formal definition of disjunctive refinement is as follows:

Definition 3.1 (Disjunctive refinement)**.** *Let* $\mathcal{U}_\mathrm{D} = (U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U^0_\mathrm{D})$, $\hat{\mathcal{U}}_\mathrm{D} = (\hat{U}_\mathrm{D}, L, \hat{\longmapsto}_\mathrm{D}, \hat{\dashrightarrow}_\mathrm{D}, \hat{U}^0_\mathrm{D}) \in \mathbb{DMTS}$*. A* disjunctive refinement *between* $\mathcal{U}_\mathrm{D}$ *and* $\hat{\mathcal{U}}_\mathrm{D}$ *is a relation* $Q \subseteq U_\mathrm{D} \times \hat{U}_\mathrm{D}$ *such that the following properties hold:*

(i) $\forall u \in U^0_\mathrm{D} : \exists \hat{u} \in \hat{U}^0_\mathrm{D} : uQ\hat{u}$

(ii) For all $(u, \hat{u}) \in Q$ *we have*

  (a) $\forall \{\vartheta\} \in (u \dashrightarrow_\mathrm{D}) : \exists \{\hat{\vartheta}\} \in (\hat{u} \mathbin{\hat{\dashrightarrow}}_\mathrm{D}) : \vartheta Q \hat{\vartheta}$

  (b) $\forall \hat{\Theta} \in (\hat{u} \mathbin{\hat{\longmapsto}}_\mathrm{D}) : \exists \Theta \in (u \longmapsto_\mathrm{D}) : \forall \vartheta \in \Theta : \exists \hat{\vartheta} \in \hat{\Theta} : \vartheta Q \hat{\vartheta}$

$\mathcal{U}_\mathrm{D}$ *is said to be a* disjunctive refinement *of* $\hat{\mathcal{U}}_\mathrm{D}$ *(*$\mathcal{U}_\mathrm{D}$ *disjunctively refines* $\hat{\mathcal{U}}_\mathrm{D}$*, written* $\mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$*) if and only if there exists a disjunctive refinement between* $\mathcal{U}_\mathrm{D}$ *and* $\hat{\mathcal{U}}_\mathrm{D}$*.*

The formal introduction of disjunctive refinement motivates, why we allow root state sets in DMTSs instead of single root states. Having more than one root state is similar to having more than one target in a hypertransition. In both cases, for each concrete root state/target an abstract root state/target needs to be found in order to have a refinement relation. Thus in some sense, a root state set corresponds to an imaginary (unlabelled) hypertransition preceding the root states.

The disjunctive refinement relation is reflexive and transitive:

Proposition 3.2. $\lhd_{\mathrm{D}}$ *is reflexive.*

Proof. Take equality as disjunctive refinement. Then all properties to be checked are obvious. ∎

Proposition 3.3. $\lhd_{\mathrm{D}}$ *is transitive.*

Proof. For $i \in \{1,2,3\}$, let $\mathcal{U}_{\mathrm{D}}^i = (U_{\mathrm{D}}^i, L, \longmapsto_{\mathrm{D}}^i, \dashrightarrow_{\mathrm{D}}^i, U_{\mathrm{D}}^{0\,i}) \in \mathbb{DMTS}$ such that $\mathcal{U}_{\mathrm{D}}^1 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$ and $\mathcal{U}_{\mathrm{D}}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^3$. Choose disjunctive refinements Q_{12} between $\mathcal{U}_{\mathrm{D}}^1$ and $\mathcal{U}_{\mathrm{D}}^2$ and Q_{23} between $\mathcal{U}_{\mathrm{D}}^2$ and $\mathcal{U}_{\mathrm{D}}^3$. Define $Q_{13} \stackrel{\mathrm{def}}{=} Q_{12} \circ Q_{23} = \{(u_1, u_3) \mid \exists u_2 : (u_1, u_2) \in Q_{12} \wedge (u_2, u_3) \in Q_{23}\}$. We prove that Q_{13} is a disjunctive refinement between $\mathcal{U}_{\mathrm{D}}^1$ and $\mathcal{U}_{\mathrm{D}}^3$.

(i) Let $u_1 \in U_{\mathrm{D}}^{0\,1}$. We can choose $u_2 \in U_{\mathrm{D}}^{0\,2}$ such that $u_1 Q_{12} u_2$ and we can choose $u_3 \in U_{\mathrm{D}}^{0\,3}$ such that $u_2 Q_{23} u_3$. Then $u_1 Q_{13} u_3$, as required.

(ii) Let $(u_1, u_3) \in Q_{13}$. By definition of Q_{13}, we can choose $u_2 \in U_{\mathrm{D}}^2$ such that $u_1 Q_{12} u_2$ and $u_2 Q_{23} u_3$.

(a) Let $\{\vartheta_1\} \in (u_1 \dashrightarrow_{\mathrm{D}}^1)$. Choose $\{\vartheta_2\} \in (u_2 \dashrightarrow_{\mathrm{D}}^2)$ such that $\vartheta_1 Q_{12} \vartheta_2$. Furthermore, choose $\{\vartheta_3\} \in (u_3 \dashrightarrow_{\mathrm{D}}^3)$ such that $\vartheta_2 Q_{23} \vartheta_3$. Then $\vartheta_1 Q_{13} \vartheta_3$, as required.

(b) Let $\Theta_3 \in (u_3 \longmapsto_{\mathrm{D}}^3)$. Choose $\Theta_2 \in (u_2 \longmapsto_{\mathrm{D}}^2)$ such that

$$\forall \vartheta_2 \in \Theta_2 : \exists \vartheta_3 \in \Theta_3 : \vartheta_2 Q_{23} \vartheta_3. \tag{3.1}$$

Furthermore, choose $\Theta_1 \in (u_1 \longmapsto_{\mathrm{D}}^1)$ such that

$$\forall \vartheta_1 \in \Theta_1 : \exists \vartheta_2 \in \Theta_2 : \vartheta_1 Q_{12} \vartheta_2. \tag{3.2}$$

Now it remains to show

$$\forall \vartheta_1 \in \Theta_1 : \exists \vartheta_3 \in \Theta_3 : \vartheta_1 Q_{13} \vartheta_3.$$

Let $\vartheta_1 \in \Theta_1$. (3.2) allows us to choose $\vartheta_2 \in \Theta_2$ such that $\vartheta_1 Q_{12} \vartheta_2$ and (3.1) allows us to choose $\vartheta_3 \in \Theta_3$ such that $\vartheta_2 Q_{23} \vartheta_3$. Consequently $\vartheta_1 Q_{13} \vartheta_3$, as required. ∎

We have seen that disjunctive refinement is reflexive and transitive. Note that it is not antisymmetric and therefore not a partial order on $\mathbb{DMTS}$: Not for every $\mathcal{U}_{\mathrm{D}}^1, \mathcal{U}_{\mathrm{D}}^2 \in \mathbb{DMTS}$ with $\mathcal{U}_{\mathrm{D}}^1 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$ and $\mathcal{U}_{\mathrm{D}}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^1$, we have $\mathcal{U}_{\mathrm{D}}^1 = \mathcal{U}_{\mathrm{D}}^2$.

Further properties of disjunctive refinement will be shown in Section 3.4. There, we prove that disjunctive refinement (on fully determined DMTSs) is compatible with bisimulation and disjunctive simulation.

3.3 1-Selecting Refinement

Now we define the refinement notion for the newly introduced 1MTSs. An 1-selecting refinement is a relation relating states of one 1MTS, say $\mathcal{U}_1$, with states of another 1MTS, say $\hat{\mathcal{U}}_1$, that satisfies several properties: As for disjunctive refinement, we must find for every root state of $\mathcal{U}_1$ some corresponding root state in $\hat{\mathcal{U}}_1$. Furthermore, for each choice of a target in $\mathcal{U}_1$ there must be a choice of a target in $\hat{\mathcal{U}}_1$ such that for every may (hyper-)transition in $\mathcal{U}_1$, there is a may (hyper-)transition in $\hat{\mathcal{U}}_1$ such that the chosen targets of both transitions are related. On the other hand, for every must (hyper-)transition in $\hat{\mathcal{U}}_1$, there should be a must (hyper-)transition in $\mathcal{U}_1$ such that the chosen targets of both transitions are related.

An 1MTS $\mathcal{U}_1$ 1-selecting refines another 1MTS $\hat{\mathcal{U}}_1$, if and only if there is an 1-selecting refinement between them. We will sometimes shortly say that $\mathcal{U}_1$ refines $\hat{\mathcal{U}}_1$, if it is clear that $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$ are 1MTSs. Furthermore, $\mathcal{U}_1$ is called a refinement of $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_1$ is called an abstraction of $\mathcal{U}_1$. We write $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$. The formal definition of 1-selecting refinement is as follows:

Definition 3.4 (1-selecting refinement)**.** *Let* $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0), \hat{\mathcal{U}}_1 = (\hat{U}_1, L, \hat{\longmapsto}_1, \hat{\dashrightarrow}_1, \hat{U}_1^0) \in \mathbb{1MTS}$*. An* 1-selecting refinement *between* $\mathcal{U}_1$ *and* $\hat{\mathcal{U}}_1$ *is a relation* $Q \subseteq U_1 \times \hat{U}_1$ *such that the following properties hold:*

(i) $\forall u \in U_1^0 : \exists \hat{u} \in \hat{U}_1^0 : uQ\hat{u}$.

(ii) For all $(u, \hat{u}) \in Q$ *and all* $\gamma \in \mathsf{choice}(u \dashrightarrow_1)$ *there exists* $\hat{\gamma} \in \mathsf{choice}(\hat{u} \,\hat{\dashrightarrow}_1)$ *such that*

  (a) $\forall \Theta \in (u \dashrightarrow_1) : \exists \hat{\Theta} \in (\hat{u} \,\hat{\dashrightarrow}_1) : \gamma(\Theta) \; Q \; \hat{\gamma}(\hat{\Theta})$

  (b) $\forall \hat{\Theta} \in (\hat{u} \,\hat{\longmapsto}_1) : \exists \Theta \in (u \longmapsto_1) : \gamma(\Theta) \; Q \; \hat{\gamma}(\hat{\Theta})$

$\mathcal{U}_1$ *is said to be an* 1-selecting refinement *of* $\hat{\mathcal{U}}_1$ *(*$\mathcal{U}_1$ *1-selecting refines* $\hat{\mathcal{U}}_1$*, written* $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$*) if and only if there exists a 1-selecting refinement between* $\mathcal{U}_1$ *and* $\hat{\mathcal{U}}_1$*.*

The 1-selecting refinement relation is reflexive and transitive:

Proposition 3.5. $\lhd_1$ *is reflexive.*

Proof. Take equality as 1-selecting refinement. Then all properties to be checked are obvious. □

Proposition 3.6. $\lhd_1$ *is transitive.*

Proof. For $i \in \{1,2,3\}$, let $\mathcal{U}_1^i = (U_1^i, L, \longmapsto_1^i, \dashrightarrow_1^i, U_1^{0^i}) \in \mathbb{1MTS}$ such that $\mathcal{U}_1^1 \lhd_1 \mathcal{U}_1^2$ and $\mathcal{U}_1^2 \lhd_1 \mathcal{U}_1^3$. Choose 1-selecting refinements Q_{12} between $\mathcal{U}_1^1$ and $\mathcal{U}_1^2$ and Q_{23} between $\mathcal{U}_1^2$ and $\mathcal{U}_1^3$. Define $Q_{13} \stackrel{\text{def}}{=} Q_{12} \circ Q_{23} = \{(u_1, u_3) \mid \exists u_2 : (u_1, u_2) \in Q_{12} \wedge (u_2, u_3) \in Q_{23}\}$. We prove that Q_{13} is an 1-selecting refinement between $\mathcal{U}_1^1$ and $\mathcal{U}_1^3$.

(i) Let $u_1 \in U_1^{0^1}$. We can choose $u_2 \in U_1^{0^2}$ such that $u_1 Q_{12} u_2$ and we can choose $u_3 \in U_1^{0^3}$ such that $u_2 Q_{23} u_3$. Then $u_1 Q_{13} u_3$, as required.

(ii) Let $(u_1, u_3) \in Q_{13}$. Choose $u_2 \in U_1^2$ such that $u_1 Q_{12} u_2$ and $u_2 Q_{23} u_3$. Furthermore, let $\gamma_1 \in \mathsf{choice}(u_1 \dashrightarrow^1_1)$. Choose $\gamma_2 \in \mathsf{choice}(u_2 \dashrightarrow^2_1)$ such that γ_1 and γ_2 satisfy properties (ii)(a) and (ii)(b) of the 1-selecting refinement Q_{12}. Furthermore, choose $\gamma_3 \in \mathsf{choice}(u_3 \dashrightarrow^3_1)$ such that γ_2 and γ_3 satisfy properties (ii)(a) and (ii)(b) of the 1-selecting refinement Q_{23}. We prove these two properties for Q_{13} with respect to γ_1 and γ_3:

(a) Let $\Theta_1 \in (u_1 \dashrightarrow^1_1)$. We can choose $\Theta_2 \in (u_2 \dashrightarrow^2_1)$ such that $\gamma_1(\Theta_1)\ Q_{12}\ \gamma_2(\Theta_2)$. Furthermore, we can choose $\Theta_3 \in (u_3 \dashrightarrow^3_1)$ such that $\gamma_2(\Theta_2)\ Q_{23}\ \gamma_3(\Theta_3)$. Then we get $\gamma_1(\Theta_1)\ Q_{13}\ \gamma_3(\Theta_3)$, as required.

(b) Let $\Theta_3 \in (u_3 \longmapsto^3_1)$. We can choose $\Theta_2 \in (u_2 \longmapsto^2_1)$ such that $\gamma_2(\Theta_2)\ Q_{23}\ \gamma_3(\Theta_3)$. Furthermore, we can choose $\Theta_1 \in (u_1 \longmapsto^1_1)$ such that $\gamma_1(\Theta_1)\ Q_{12}\ \gamma_2(\Theta_2)$. Then we get $\gamma_1(\Theta_1)\ Q_{13}\ \gamma_3(\Theta_3)$, as required. ☐

We have seen that 1-selecting refinement is reflexive and transitive. Note that it is not antisymmetric and therefore not a partial order on $\mathbb{1MTS}$: Not for every $\mathcal{U}_1^1, \mathcal{U}_1^2 \in \mathbb{1MTS}$ with $\mathcal{U}_1^1 \lhd_1 \mathcal{U}_1^2$ and $\mathcal{U}_1^2 \lhd_1 \mathcal{U}_1^1$, we have $\mathcal{U}_1^1 = \mathcal{U}_1^2$.

3.4 Refinement to Fully Determined Systems

It is desirable that disjunctive and 1-selecting refinement harmonise with existing notions like bisimulation and disjunctive (respectively 1-selecting) simulation. We first state that, in the case of two fully determined systems, disjunctive and 1-selecting refinement coincide with bisimulation:

Proposition 3.7. *Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{TS}$. Then:*

$$\begin{aligned} \mathcal{T}_1 \sim \mathcal{T}_2 \;&\Leftrightarrow\; \mathsf{DMTS}(\mathcal{T}_1) \lhd_{\mathrm{D}} \mathsf{DMTS}(\mathcal{T}_2) \\ &\Leftrightarrow\; \mathsf{1MTS}(\mathcal{T}_1) \lhd_1 \mathsf{1MTS}(\mathcal{T}_2) \end{aligned}$$

Proof. Consider the definition of disjunctive (1-selecting) refinement in the special case that both DMTSs (1MTSs) are fully determined, respectively. Then the proposition is obvious. ☐

If one of the systems is fully determined, disjunctive refinement coincides with disjunctive simulation, and 1-selecting refinement coincides with 1-selecting simulation:

Proposition 3.8. *Let $\mathcal{T} \in \mathbb{TS}$. Then*

(i) for all $\hat{\mathcal{U}}_{\mathrm{D}} \in \mathbb{DMTS}$, we have

$$\mathsf{DMTS}(\mathcal{T}) \lhd_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}} \;\Leftrightarrow\; \mathcal{T} \prec_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}},$$

(ii) for all $\hat{\mathcal{U}}_1 \in \mathbb{1MTS}$*, we have*

$$\mathsf{1MTS}(\mathcal{T}) \lhd_1 \hat{\mathcal{U}}_1 \;\Leftrightarrow\; \mathcal{T} \prec_1 \hat{\mathcal{U}}_1.$$

Proof. Consider the definition of disjunctive (1-selecting) refinement in the special case that $\mathcal{U}_\mathrm{D}$ ($\mathcal{U}_1$) is a fully determined DMTS (1MTS) and compare it with the definition of disjunctive (1-selecting) simulation, respectively. Then the proposition is obvious. □

As a corollary, we get that refinement respects the simulation notion in the following sense:

Corollary 3.9. *Let* $\mathcal{T} \in \mathbb{TS}$*. Then*

(i) for all $\mathcal{U}_\mathrm{D}, \hat{\mathcal{U}}_\mathrm{D} \in \mathbb{DMTS}$*, we have*

$$\mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D} \wedge \mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D} \;\Rightarrow\; \mathcal{T} \prec_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D},$$

(ii) for all $\mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MTS}$*, we have*

$$\mathcal{T} \prec_1 \mathcal{U}_1 \wedge \mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1 \;\Rightarrow\; \mathcal{T} \prec_1 \hat{\mathcal{U}}_1.$$

Proof. (i) Since $\mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D}$ is equivalent to $\mathsf{DMTS}(\mathcal{T}) \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}$ and $\mathcal{T} \prec_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$ is equivalent to $\mathsf{DMTS}(\mathcal{T}) \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$, the corollary follows directly from the transitivity of $\lhd_\mathrm{D}$.

(ii) Since $\mathcal{T} \prec_1 \mathcal{U}_1$ is equivalent to $\mathsf{1MTS}(\mathcal{T}) \lhd_1 \mathcal{U}_1$ and $\mathcal{T} \prec_1 \hat{\mathcal{U}}_1$ is equivalent to $\mathsf{1MTS}(\mathcal{T}) \lhd_1 \hat{\mathcal{U}}_1$, the corollary follows directly from the transitivity of $\lhd_1$. □

Corollary 3.9(i) can be reformulated as follows: For all $\mathcal{U}_\mathrm{D}, \hat{\mathcal{U}}_\mathrm{D} \in \mathbb{DMTS}$ we have

$$\mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D} \;\Rightarrow\; \{\mathcal{T} \in \mathbb{TS} \mid \mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D}\} \subseteq \{\mathcal{T} \in \mathbb{TS} \mid \mathcal{T} \prec_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}\}.$$

The analogous statement for 1MTSs also holds. Note however, that in both cases the other implication ("$\Leftarrow$") does not hold. A counter-example from [26], pp. 87–88, is illustrated in Figure 3.1. $\mathcal{U}$ and $\hat{\mathcal{U}}$ have the same sets of implementations, regardless of whether we interpret them as DMTSs or 1MTSs: For both $\mathcal{U}$ and $\hat{\mathcal{U}}$, possible implementations have no transition, a transition labelled with a or a transition labelled with a and afterwards a transition labelled with b. However, $\mathcal{U}$ does not refine $\hat{\mathcal{U}}$, because for the must transition in $\hat{\mathcal{U}}$, no corresponding must transition in $\mathcal{U}$ can be found.

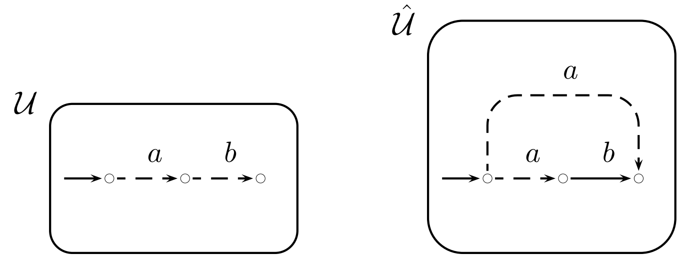

Figure 3.1: A counter-example: $\mathcal{U}$ does not refine $\hat{\mathcal{U}}$

3.5 Refinement Equivalence

As TSs come with bisimulation, it also makes sense for modal transition systems to have corresponding equivalences that allow us to identify systems that are "essentially the same". It is of course a question of definition, what "essentially the same" should actually mean. We take the refinement notions as a foundation and define two systems to be equivalent, if and only if they refine each other in both directions. Formally:

Definition 3.10 (Refinement equivalence). *(i) Let $\mathcal{U}_{\mathrm{D}}^1, \mathcal{U}_{\mathrm{D}}^2 \in \mathbb{DMTS}$. We call $\mathcal{U}_{\mathrm{D}}^1$ and $\mathcal{U}_{\mathrm{D}}^2$* disjunctive refinement equivalent *(or shortly* DR-equivalent*), if and only if $\mathcal{U}_{\mathrm{D}}^1 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$ and $\mathcal{U}_{\mathrm{D}}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^1$. In that case, we write $\mathcal{U}_{\mathrm{D}}^1 \approx_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$.*

(ii) Let $\mathcal{U}_1^1, \mathcal{U}_1^2 \in \mathbb{1MTS}$. We call $\mathcal{U}_1^1$ and $\mathcal{U}_1^2$ 1-selecting refinement equivalent *(or shortly* 1R-equivalent*), if and only if $\mathcal{U}_1^1 \lhd_1 \mathcal{U}_1^2$ and $\mathcal{U}_1^2 \lhd_1 \mathcal{U}_1^1$. In that case, we write $\mathcal{U}_1^1 \approx_1 \mathcal{U}_1^2$.*

It is easy to see that both DR- and 1R-equivalence are in fact equivalence relations: They are reflexive and transitive, because disjunctive and 1-selecting refinement are reflexive and transitive. Furthermore, the relations are obviously symmetric by definition. We introduce notations for the corresponding equivalence classes:

Definition 3.11. *For $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}$ and $\mathcal{U}_1 \in \mathbb{1MTS}$, we define*

$$[\mathcal{U}_{\mathrm{D}}]_{\approx_{\mathrm{D}}} \stackrel{\text{def}}{=} \{\bar{\mathcal{U}}_{\mathrm{D}} \in \mathbb{DMTS} \mid \bar{\mathcal{U}}_{\mathrm{D}} \approx_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}\},$$
$$[\mathcal{U}_1]_{\approx_1} \stackrel{\text{def}}{=} \{\bar{\mathcal{U}}_1 \in \mathbb{1MTS} \mid \bar{\mathcal{U}}_1 \approx_1 \mathcal{U}_1\}$$

to be the DR-, respectively 1R-equivalence classes that $\mathcal{U}_{\mathrm{D}}$, respectively $\mathcal{U}_1$ are representatives of. Furthermore, define

$$\underline{\mathbb{DMTS}} \stackrel{\text{def}}{=} \{[\mathcal{U}_{\mathrm{D}}]_{\approx_{\mathrm{D}}} \mid \mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}\},$$
$$\underline{\mathbb{1MTS}} \stackrel{\text{def}}{=} \{[\mathcal{U}_1]_{\approx_1} \mid \mathcal{U}_1 \in \mathbb{1MTS}\}$$

to be the sets of equivalence classes of DMTSs, respectively $\mathsf{1MTS}$s. Finally, define

$$\underline{\mathbb{DMTS}}^{\text{det}} \stackrel{\text{def}}{=} \{\mathcal{K}_{\mathrm{D}} \in \underline{\mathbb{DMTS}} \mid \exists \mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}} : \mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}^{\text{det}}\},$$
$$\underline{\mathbb{1MTS}}^{\text{det}} \stackrel{\text{def}}{=} \{\mathcal{K}_1 \in \underline{\mathbb{1MTS}} \mid \exists \mathcal{U}_1 \in \mathcal{K}_1 : \mathcal{U}_1 \in \mathbb{1MTS}^{\text{det}}\}.$$

Note that for $\mathcal{K}_{\mathrm{D}} \in \underline{\mathbb{DMTS}}^{\mathrm{det}}$, not every $\mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}}$ needs to be fully determined (and the analogue statement is true for $\mathsf{1MTS}$s). For example, the DMTS with only one must transition labelled with a is DR-equivalent to the DMTS with only one must hypertransition that has two targets both labelled with a. The first is fully determined, whereas the second is not.

In a straightforward manner, the refinement relations can be defined on equivalence classes of DMTSs, respectively $\mathsf{1MTS}$s:

Definition 3.12. *(i) Define* $\trianglelefteq_{\mathrm{D}}$: $\underline{\mathbb{DMTS}} \times \underline{\mathbb{DMTS}}$ *as follows: Let* $\mathcal{K}_{\mathrm{D}}, \hat{\mathcal{K}}_{\mathrm{D}} \in \underline{\mathbb{DMTS}}$ *and choose* $\mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}}$, $\hat{\mathcal{U}}_{\mathrm{D}} \in \hat{\mathcal{K}}_{\mathrm{D}}$. *Then we set*

$$\mathcal{K}_{\mathrm{D}} \trianglelefteq_{\mathrm{D}} \hat{\mathcal{K}}_{\mathrm{D}} \overset{\text{def}}{\Leftrightarrow} \mathcal{U}_{\mathrm{D}} \triangleleft_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}}.$$

(ii) Define $\trianglelefteq_1$: $\underline{\mathbb{1MTS}} \times \underline{\mathbb{1MTS}}$ *as follows: Let* $\mathcal{K}_1, \hat{\mathcal{K}}_1 \in \underline{\mathbb{1MTS}}$ *and choose* $\mathcal{U}_1 \in \mathcal{K}_1$, $\hat{\mathcal{U}}_1 \in \hat{\mathcal{K}}_1$. *Then we set*

$$\mathcal{K}_1 \trianglelefteq_1 \hat{\mathcal{K}}_1 \overset{\text{def}}{\Leftrightarrow} \mathcal{U}_1 \triangleleft_1 \hat{\mathcal{U}}_1.$$

The definitions are obviously well-defined, because $\triangleleft_{\mathrm{D}}$ and $\triangleleft_1$ are transitive. For the relations $\trianglelefteq_{\mathrm{D}}$ and $\trianglelefteq_1$, we also use the terms *disjunctive*, respectively *1-selecting refinement*. Note that $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$ and $(\underline{\mathbb{1MTS}}, \trianglelefteq_1)$ are partially ordered sets. This is not the case for $(\mathbb{DMTS}, \triangleleft_{\mathrm{D}})$ and $(\mathbb{1MTS}, \triangleleft_1)$. As it was already remarked, these are reflexive and transitive, but they are not antisymmetric, i.e.,

$$\forall \mathcal{U}_{\mathrm{D}}^1, \mathcal{U}_{\mathrm{D}}^2 \in \mathbb{DMTS} : \; \mathcal{U}_{\mathrm{D}}^1 \triangleleft_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2 \wedge \mathcal{U}_{\mathrm{D}}^2 \triangleleft_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^1 \;\not\Rightarrow\; \mathcal{U}_{\mathrm{D}}^1 = \mathcal{U}_{\mathrm{D}}^2$$

(respectively the analogue for $\mathsf{1MTS}$s). Obviously the property holds, when considering equivalence classes:

$$\forall \mathcal{K}_{\mathrm{D}}^1, \mathcal{K}_{\mathrm{D}}^2 \in \underline{\mathbb{DMTS}} : \; \mathcal{K}_{\mathrm{D}}^1 \trianglelefteq_{\mathrm{D}} \mathcal{K}_{\mathrm{D}}^2 \wedge \mathcal{K}_{\mathrm{D}}^2 \trianglelefteq_{\mathrm{D}} \mathcal{K}_{\mathrm{D}}^1 \;\Rightarrow\; \mathcal{K}_{\mathrm{D}}^1 = \mathcal{K}_{\mathrm{D}}^2$$

(respectively the analogue for $\mathsf{1MTS}$s). Thus considering equivalence classes enables us to reason with partially ordered sets.

We generalise the functions DMTS, $\mathsf{1MTS}$, TS_{D} and TS_1, that were introduced in Definition 2.11, to equivalence classes:

Definition 3.13. *(i) Define* $\underline{\mathsf{DMTS}} : \underline{\mathbb{TS}} \to \underline{\mathbb{DMTS}}^{\mathrm{det}}$ *as follows: For* $\mathcal{K} \in \underline{\mathbb{TS}}$, *choose* $\mathcal{T} \in \mathcal{K}$ *and define* $\underline{\mathsf{DMTS}}(\mathcal{K}) \overset{\text{def}}{=} [\mathsf{DMTS}(\mathcal{T})]_{\approx_{\mathrm{D}}}$.

(ii) Define $\underline{\mathsf{1MTS}} : \underline{\mathbb{TS}} \to \underline{\mathbb{1MTS}}^{\mathrm{det}}$ *as follows: For* $\mathcal{K} \in \underline{\mathbb{TS}}$, *choose* $\mathcal{T} \in \mathcal{K}$ *and define* $\underline{\mathsf{1MTS}}(\mathcal{K}) \overset{\text{def}}{=} [\mathsf{1MTS}(\mathcal{T})]_{\approx_1}$.

(iii) Define $\underline{\mathsf{TS}}_{\mathrm{D}} : \underline{\mathbb{DMTS}}^{\mathrm{det}} \to \underline{\mathbb{TS}}$ *as follows: For* $\mathcal{K}_{\mathrm{D}} \in \underline{\mathbb{DMTS}}^{\mathrm{det}}$, *choose a fully determined* $\mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}} \cap \mathbb{DMTS}^{\mathrm{det}}$ *and define* $\underline{\mathsf{TS}}_{\mathrm{D}}(\mathcal{K}_{\mathrm{D}}) \overset{\text{def}}{=} [\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}})]_{\sim}$.

(iv) Define $\underline{\mathsf{TS}}_1 : \underline{\mathbb{1MTS}}^{\mathrm{det}} \to \underline{\mathbb{TS}}$ *as follows: For* $\mathcal{K}_1 \in \underline{\mathbb{1MTS}}^{\mathrm{det}}$, *choose a fully determined* $\mathcal{U}_1 \in \mathcal{K}_1 \cap \mathbb{1MTS}^{\mathrm{det}}$ *and define* $\underline{\mathsf{TS}}_1(\mathcal{K}_1) \overset{\text{def}}{=} [\mathsf{TS}_1(\mathcal{U}_1)]_{\sim}$.

Note that the functions are well-defined, i.e.,

- $\underline{\mathsf{DMTS}}(\mathcal{K})$ does not depend on the choice of $\mathcal{T} \in \mathcal{K}$,
- $\underline{\mathsf{1MTS}}(\mathcal{K})$ does not depend on the choice of $\mathcal{T} \in \mathcal{K}$,
- $\underline{\mathsf{TS}}_{\mathrm{D}}(\mathcal{K}_{\mathrm{D}})$ does not depend on the choice of $\mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}} \cap \mathbb{DMTS}^{\mathrm{det}}$, and
- $\underline{\mathsf{TS}}_{1}(\mathcal{K}_{1})$ does not depend on the choice of $\mathcal{U}_{1} \in \mathcal{K}_{1} \cap \mathbb{1MTS}^{\mathrm{det}}$,

due to the following proposition that states that on fully determined systems, the notions bisimulation and refinement equivalence coincide:

Proposition 3.14. *Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{TS}$. Then:*

$$\begin{aligned} \mathcal{T}_1 \sim \mathcal{T}_2 \;&\Leftrightarrow\; \mathsf{DMTS}(\mathcal{T}_1) \approx_{\mathrm{D}} \mathsf{DMTS}(\mathcal{T}_2) \\ &\Leftrightarrow\; \mathsf{1MTS}(\mathcal{T}_1) \approx_{1} \mathsf{1MTS}(\mathcal{T}_2). \end{aligned}$$

Proof. Immediately follows from Proposition 3.7. □

In the next two sections, we examine the partially ordered sets $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$ and $(\underline{\mathbb{1MTS}}, \trianglelefteq_{1})$ by giving all refinements of (the DR-equivalence class of) a simple DMTS and (the 1R-equivalence class of) a simple 1MTS. Thus we completely characterise finite portions of the infinite Hasse diagrams of $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$ and $(\underline{\mathbb{1MTS}}, \trianglelefteq_{1})$.

3.6 All Refinements of a Simple DMTS

We consider the simple DMTS

$$\hat{\mathcal{U}}_{\mathrm{D}} \stackrel{\mathrm{def}}{=} (\{0,1\}, \{a,b\}, \{(0,\{(a,1),(b,1)\}\}, \{(0,\{(a,1)\}),(0,\{(b,1)\})\}, \{0\}),$$

which is illustrated at the top of Figure 3.2. It consists of a single (must) hypertransition. By condition (2.1) of DMTSs, it has two "implicit" may transitions, which we do not draw in illustrations. We want to give a complete overview over all refinements of $[\hat{\mathcal{U}}_{\mathrm{D}}]_{\approx_{\mathrm{D}}}$. We claim that Figure 3.2 gives such a complete overview. Any DMTS illustrated in the figure stands for the DR-equivalence class of which it is a representative. Lines between the systems represent the disjunctive refinement relation on DR-equivalence classes, $\trianglelefteq_{\mathrm{D}}$. Classes below disjunctively refine classes above. Consequently, Figure 3.2 forms a Hasse diagram of all disjunctive refinements of $[\hat{\mathcal{U}}_{\mathrm{D}}]_{\approx_{\mathrm{D}}}$, ordered by $\trianglelefteq_{\mathrm{D}}$.

The fact that all lines indeed represent refinement relations can easily be checked. In order to show that Figure 3.2 lists all refinements of $[\hat{\mathcal{U}}_{\mathrm{D}}]_{\approx_{\mathrm{D}}}$, we take an arbitrary refinement of $\hat{\mathcal{U}}_{\mathrm{D}}$ and prove that it is DR-equivalent to one of the DMTSs in the figure, i.e., that it belongs to one of the equivalence classes represented by the figure. We achieve this by considering the *components* of the refinement, which are defined using the notion of *reachability*:

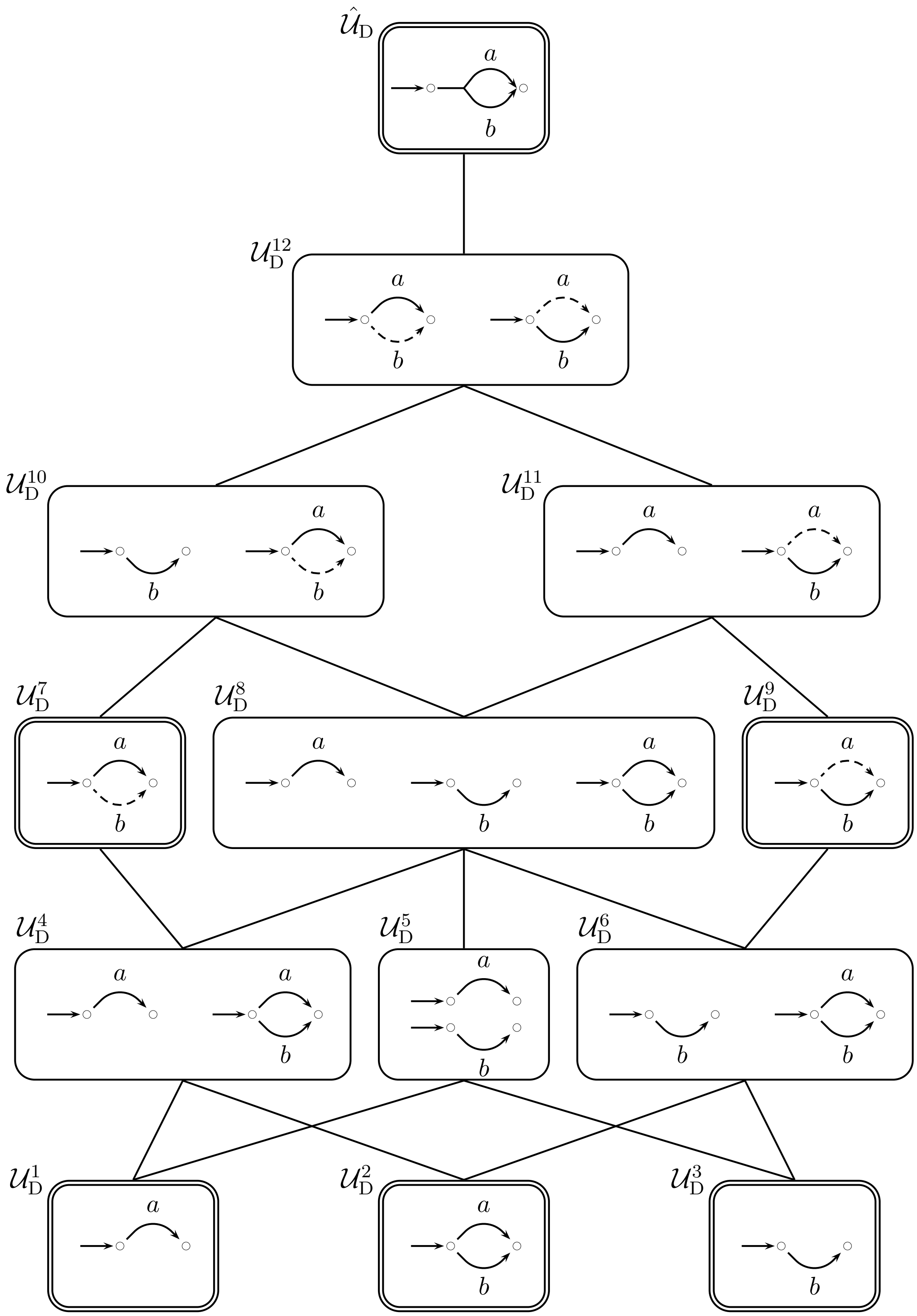

Figure 3.2: $[\hat{\mathcal{U}}_D]_{\approx_D}$ and all its refinements in a Hasse diagram

Definition 3.15 ($\mathcal{U}_\mathrm{D}$-reachable)**.** *Let* $\mathcal{U}_\mathrm{D} = (U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U^0_\mathrm{D}) \in \mathbb{DMTS}$ *and* $\bar{u} \in U_\mathrm{D}$*. Recursively define*

$$\begin{aligned} V_1^{\bar{u}} &\stackrel{\text{def}}{=} \{\bar{u}\} \\ V_n^{\bar{u}} &\stackrel{\text{def}}{=} V_{n-1}^{\bar{u}} \cup \{u' \in U_\mathrm{D} \mid \exists u \in V_{n-1}^{\bar{u}}, a \in L : u \dashrightarrow_\mathrm{D} \{(a,u')\}\} \end{aligned}$$

and let $V^{\bar{u}} \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} V_n^{\bar{u}}$*. Then* $u \in U_\mathrm{D}$ *is called* $\mathcal{U}_\mathrm{D}$-reachable from $\bar{u}$*, if and only if* $u \in V^{\bar{u}}$*. Furthermore,* $u \in U_\mathrm{D}$ *is called* $\mathcal{U}_\mathrm{D}$-reachable*, if and only if there exists* $u^0 \in U^0_\mathrm{D}$ *such that* u *is* $\mathcal{U}_\mathrm{D}$*-reachable from* u^0*.*

The definition of reachability is based on the may transition relation. This includes reachability via must transitions due to condition (2.1) of DMTSs.

A *component* is a DMTS with only one root state in which all states are reachable. For a DMTS $\mathcal{U}_\mathrm{D}$, a *component of* $\mathcal{U}_\mathrm{D}$ is a component that is a "part" of $\mathcal{U}_\mathrm{D}$. Formally:

Definition 3.16 (Component)**.** *A DMTS* $\mathcal{C} = (U_\mathrm{C}, L, \longmapsto_\mathrm{C}, \dashrightarrow_\mathrm{C}, U^0_\mathrm{C}) \in \mathbb{DMTS}$ *is called* component*, if and only if there exists* $u^0 \in U^0_\mathrm{C}$ *such that* $U^0_\mathrm{C} = \{u^0\}$ *and each* $u \in U_\mathrm{C}$ *is* $\mathcal{C}$*-reachable.*

Definition 3.17 (Component of $\mathcal{U}_\mathrm{D}$)**.** *Let* $\mathcal{U}_\mathrm{D} = (U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U^0_\mathrm{D}) \in \mathbb{DMTS}$*. A DMTS* $\mathcal{C} = (U_\mathrm{C}, L, \longmapsto_\mathrm{C}, \dashrightarrow_\mathrm{C}, U^0_\mathrm{C}) \in \mathbb{DMTS}$ *is called* component of $\mathcal{U}_\mathrm{D}$*, if and only if there exists* $u^0 \in U^0_\mathrm{D}$ *such that*

$$\begin{aligned} U^0_\mathrm{C} &= \{u^0\}, \\ U_\mathrm{C} &= \{u \in U_\mathrm{D} \mid u \text{ is } \mathcal{U}_\mathrm{D}\text{-reachable from } u^0\}, \\ u \longmapsto_\mathrm{C} \Theta &\Leftrightarrow u \in U_\mathrm{C} \wedge u \longmapsto_\mathrm{D} \Theta, \\ u \dashrightarrow_\mathrm{C} \{(a,u')\} &\Leftrightarrow u \in U_\mathrm{C} \wedge u \dashrightarrow_\mathrm{D} \{(a,u')\}. \end{aligned}$$

We denote the set of all components of $\mathcal{U}_\mathrm{D}$ *by* $\mathbb{C}\mathsf{omp}(\mathcal{U}_\mathrm{D})$*.*

For any DMTS $\mathcal{U}_\mathrm{D}$, each component of $\mathcal{U}_\mathrm{D}$ is obviously a component. The following lemmata describe some properties of components and their connection to disjunctive refinement:

Lemma 3.18. *Let* $\mathcal{U}_\mathrm{D}, \hat{\mathcal{U}}_\mathrm{D} \in \mathbb{DMTS}$*. Then* $\mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$ *if and only if* $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_\mathrm{D}) : \mathcal{C} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$*.*

Proof. Obvious by definition of disjunctive refinement. □

Lemma 3.19. *Let* $\mathcal{U}_\mathrm{D} \in \mathbb{DMTS}$*. Then* $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_\mathrm{D}) : \mathcal{C} \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}$*.*

Proof. Apply Lemma 3.18 to $\mathcal{U}_\mathrm{D}$ and $\mathcal{U}_\mathrm{D}$. □

Lemma 3.20. *Let* $\mathcal{U}_\mathrm{D}, \hat{\mathcal{U}}_\mathrm{D} \in \mathbb{DMTS}$ *such that* $\exists \mathcal{C} \in \mathbb{C}\mathsf{omp}(\hat{\mathcal{U}}_\mathrm{D}) : \mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \mathcal{C}$*. Then* $\mathcal{U}_\mathrm{D} \lhd_\mathrm{D} \hat{\mathcal{U}}_\mathrm{D}$*.*

Proof. Obvious by definition of disjunctive refinement. □

Now being able to split up DMTSs into their components, we also need a way to perform the opposite operation, i.e., to merge components obtaining more complex DMTSs. To achieve this, we introduce the merge operator $\otimes$:

Definition 3.21. *Let I be an index set and $\mathcal{U}_{\mathrm{D}}^i = (U_{\mathrm{D}}^i, L^i, \longmapsto_{\mathrm{D}}^i, \dashrightarrow_{\mathrm{D}}^i, {U_{\mathrm{D}}^0}^i) \in \mathbb{DMTS}$ for all $i \in I$. Then $\bigotimes_{i \in I} \mathcal{U}_{\mathrm{D}}^i$ is defined to be the DMTS $(U_\otimes, L, \longmapsto_\otimes, \dashrightarrow_\otimes, U_\otimes^0)$, where*

$$
\begin{aligned}
U_\otimes &\stackrel{\text{def}}{=} \bigcup_{i \in I} \left(U_{\mathrm{D}}^i \times \{i\}\right), \\
L &\stackrel{\text{def}}{=} \bigcup_{i \in I} L^i, \\
(u,i) \longmapsto_\otimes \Theta &\stackrel{\text{def}}{\Leftrightarrow} (\forall (a,(u',i')) \in \Theta : i = i') \wedge \\
&\qquad u \longmapsto_{\mathrm{D}}^i \{(a,u') \mid (a,(u',i)) \in \Theta\}, \\
(u,i) \dashrightarrow_\otimes \{(a,(u',i'))\} &\stackrel{\text{def}}{\Leftrightarrow} i = i' \wedge u \dashrightarrow_{\mathrm{D}}^i \{(a,u')\}, \\
U_\otimes^0 &\stackrel{\text{def}}{=} \bigcup_{i \in I} \left({U_{\mathrm{D}}^0}^i \times \{i\}\right).
\end{aligned}
$$

For any set of DMTSs $\mathbb{U}_{\mathrm{D}}$, we shortly write $\bigotimes \mathbb{U}_{\mathrm{D}}$, which is defined to be $\bigotimes_{\mathcal{U}_{\mathrm{D}} \in \mathbb{U}_{\mathrm{D}}} \mathcal{U}_{\mathrm{D}}$. If $\mathbb{U}_{\mathrm{D}}$ has exactly two elements, say $\mathcal{U}_{\mathrm{D}}^1$ and $\mathcal{U}_{\mathrm{D}}^2$, we usually write $\mathcal{U}_{\mathrm{D}}^1 \otimes \mathcal{U}_{\mathrm{D}}^2$.

The following two lemmata allows us to reason with components and merging. If we merge all components of a DMTS, we get a DMTS that is DR-equivalent to the original one, i.e., we remain in the same equivalence class. Furthermore, exchanging one component of a DMTS with a DR-equivalent component does not leave the DR-equivalence class either.

Lemma 3.22. *Let $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}$. Then $\mathcal{U}_{\mathrm{D}} \approx_{\mathrm{D}} \bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$.*

Proof. Let $\mathcal{U}_{\mathrm{D}} = (U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \dashrightarrow_{\mathrm{D}}, U_{\mathrm{D}}^0) \in \mathbb{DMTS}$ and $\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) = \bigotimes_{\mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})} \mathcal{C} = (U_\otimes, L, \longmapsto_\otimes, \dashrightarrow_\otimes, U_\otimes^0)$. For any component $\mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$, set $\mathcal{C} = (U_{\mathrm{C}}^{\mathcal{C}}, L, \longmapsto_{\mathrm{C}}^{\mathcal{C}}, \dashrightarrow_{\mathrm{C}}^{\mathcal{C}}, {U_{\mathrm{C}}^0}^{\mathcal{C}})$.

We start with the proof of $\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} \bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$ by showing that $Q \subseteq U_{\mathrm{D}} \times U_\otimes$, defined by

$$u Q (\hat{u}, \mathcal{C}) \stackrel{\text{def}}{\Leftrightarrow} u = \hat{u},$$

is a disjunctive refinement between $\mathcal{U}_{\mathrm{D}}$ and $\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$.

(i) Let $u \in U_{\mathrm{D}}^0$. Then there exists $\mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$ such that $u \in {U_{\mathrm{C}}^0}^{\mathcal{C}}$ and $(u, \mathcal{C}) \in U_\otimes^0$. We have $uQ(u, \mathcal{C})$, as required.

(ii) Let $(u, (\hat{u}, \mathcal{C})) \in Q$. Then $u = \hat{u}$.

(a) Let $\{(a,u')\} \in (u \dashrightarrow_{\mathrm{D}})$. We have $u \in U^{\mathcal{C}}_{\mathrm{C}}$, $u' \in U^{\mathcal{C}}_{\mathrm{C}}$, and $u \dashrightarrow^{\mathcal{C}}_{\mathrm{C}} \{(a,u')\}$. Consequently $(u,\mathcal{C}) \dashrightarrow_{\otimes} \{(a,(u',\mathcal{C})\}$. We have $(a,u')Q(a,(u',\mathcal{C}))$, as required.

(b) Let $\hat{\Theta} \in ((u,\mathcal{C}) \longmapsto_{\otimes})$. Then $u \in \mathcal{U}_{\mathrm{D}}$ and $u \longmapsto_{\mathrm{D}} \{(a,u') \mid (a,(u',\mathcal{C})) \in \hat{\Theta}\}$. For each (a,u') with $(a,(u',\mathcal{C})) \in \hat{\Theta}$, we have $(a,u')Q(a,(u',\mathcal{C}))$, as required.

It remains to prove $\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$. Define $Q \subseteq U_{\mathrm{D}} \times U_{\otimes}$ as follows:

$$(u,\mathcal{C})Q\hat{u} \overset{\text{def}}{\Leftrightarrow} u = \hat{u}.$$

We prove that Q is a disjunctive refinement between $\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$ and $\mathcal{U}_{\mathrm{D}}$.

(i) Let $(u,\mathcal{C}) \in U^0_{\otimes}$. Then $u \in U^0_{\mathrm{D}}$ and $(u,\mathcal{C})Qu$.

(ii) Let $((u,\mathcal{C}),\hat{u}) \in Q$. Then $u = \hat{u}$.

(a) Let $\{(a,(u',\mathcal{C}'))\} \in ((u,\mathcal{C}) \dashrightarrow_{\otimes})$. Then $\mathcal{C} = \mathcal{C}'$ and $u \dashrightarrow^{\mathcal{C}}_{\mathrm{C}} \{(a,u')\}$. Consequently $u \dashrightarrow_{\mathrm{D}} \{(a,u')\}$. We have $(a,(u',\mathcal{C}))Q(a,u')$, as required.

(b) Let $\hat{\Theta} \in (u \longmapsto_{\mathrm{D}})$. We have $u \in U^{\mathcal{C}}_{\mathrm{C}}$, $u' \in U^{\mathcal{C}}_{\mathrm{C}}$ for all $(a,u') \in \hat{\Theta}$, and $u \longmapsto^{\mathcal{C}}_{\mathrm{C}} \hat{\Theta}$. Consequently $(u,\mathcal{C}) \longmapsto_{\otimes} \{(a,(u',\mathcal{C})) \mid (a,u') \in \hat{\Theta}\}$. For each $(a,(u',\mathcal{C}))$ with $(a,u') \in \hat{\Theta}$, we have $(a,(u',\mathcal{C}))Q(a,u')$, as required. □

Lemma 3.23. *Let $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}$, $\mathcal{C}^1 \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$ and $\mathcal{C}^2$ be a component such that $\mathcal{C}^1 \approx_{\mathrm{D}} \mathcal{C}^2$. Then $\mathcal{U}_{\mathrm{D}} \approx_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$.*

Proof. We start with the proof of $(\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$. We have $\mathcal{C}^2 \lhd_{\mathrm{D}} \mathcal{C}^1$ and by Lemma 3.19 $\mathcal{C}^1 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$. Transitivity of $\lhd_{\mathrm{D}}$ implies $\mathcal{C}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$. Since all components of $\mathcal{U}_{\mathrm{D}}$ disjunctively refine $\mathcal{U}_{\mathrm{D}}$ (Lemma 3.19) and $\mathcal{C}^2$ does as well, we have $\forall \mathcal{C} \in (\mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2 : \mathcal{C} \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$ and Lemma 3.18 implies $(\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$.

Now it remains to prove $\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$. For each $\mathcal{C} \in \bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1$, there obviously exists $\mathcal{C}' \in (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$ such that $\mathcal{C} \approx_{\mathrm{D}} \mathcal{C}'$ (simply choose $\mathcal{C}' = \mathcal{C}$). Then Lemma 3.20 implies $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1 : \mathcal{C} \lhd_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$. Again with Lemma 3.20, $\mathcal{C}^1 \lhd_{\mathrm{D}} \mathcal{C}^2$ implies $\mathcal{C}^1 \lhd_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$. Consequently $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) : \mathcal{C} \lhd_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$ and Lemma 3.18 implies $\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} (\bigotimes \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}}) \setminus \mathcal{C}^1) \cup \mathcal{C}^2$. □

We are now ready to prove the main theorem of this section:

Theorem 3.24. *Each refinement of*

$$\hat{\mathcal{U}}_{\mathrm{D}} \overset{\text{def}}{=} (\{0,1\},\{a,b\},\{(0,\{(a,1),(b,1)\}\},\{(0,\{(a,1),(b,1)\}\},\{0\})$$

is DR-equivalent to one of the DMTS*s shown in Figure 3.2.*

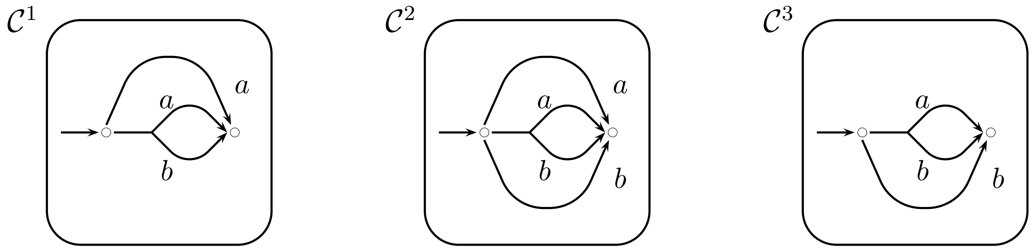

Figure 3.3: Components $\mathcal{C}^1, \mathcal{C}^2$ and $\mathcal{C}^3$

In other words, if all DMTSs in Figure 3.2 are interpreted as their DR-equivalence classes, each refinement of $[\hat{\mathcal{U}}_{\mathrm{D}}]_{\approx_{\mathrm{D}}}$ appears in the Figure.

Proof of Theorem 3.24. Let $\mathcal{U}_{\mathrm{D}} = (U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \dashrightarrow_{\mathrm{D}}, U^0_{\mathrm{D}}) \in \mathbb{DMTS}$ such that $\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}}$. Choose a disjunctive refinement $Q \subseteq U_{\mathrm{D}} \times \{0,1\}$ between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$. We examine an arbitrary component of $\mathcal{U}_{\mathrm{D}}$.

Thus let $\mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_{\mathrm{D}})$ and let u be the root state of $\mathcal{C}$. Then we have $uQ0$. $\mathcal{C}$ satisfies the following properties:

(i) There are no may transitions (and consequently no must transitions) starting in u with a label different from a and b, because otherwise, Q would not be a refinement between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$.

(ii) There are no may transitions (and consequently no must transitions) starting in a state that is a target of a transition starting in u, because this would again contradict the fact that Q is a disjunctive refinement between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$.

For this reason, $\mathcal{C}$ is DR-equivalent to a component that has only two states, where one of them is the root state and the other does not have any outgoing transitions. Due to Lemma 3.23, it is enough to consider components with exactly two states.

(iii) There is a must transition starting in u, because otherwise, Q would not be a disjunctive refinement between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$.

Due to these properties, the only possible components of $\mathcal{U}_{\mathrm{D}}$ are $\mathcal{U}^1_{\mathrm{D}}$, $\mathcal{U}^2_{\mathrm{D}}$, $\mathcal{U}^3_{\mathrm{D}}$, $\mathcal{U}^7_{\mathrm{D}}$, $\mathcal{U}^9_{\mathrm{D}}$, $\hat{\mathcal{U}}_{\mathrm{D}}$ (those are marked in Figure 3.2 by double-lined frames) and the three components $\mathcal{C}^1$, $\mathcal{C}^2$, $\mathcal{C}^3$ shown in Figure 3.3. However, $\mathcal{C}^1$ is DR-equivalent to $\mathcal{U}^7_{\mathrm{D}}$, $\mathcal{C}^2$ is DR-equivalent to $\mathcal{U}^2_{\mathrm{D}}$ and $\mathcal{C}^3$ is DR-equivalent to $\mathcal{U}^9_{\mathrm{D}}$. Consequently we need not consider the components from Figure 3.3, because due to Lemma 3.23, refinements including those components are DR-equivalent to other refinements considered.

Thus all refinements of $\hat{\mathcal{U}}_{\mathrm{D}}$ are DR-equivalent to a DMTS built up from the components $\mathcal{U}^1_{\mathrm{D}}, \mathcal{U}^2_{\mathrm{D}}, \mathcal{U}^3_{\mathrm{D}}, \mathcal{U}^7_{\mathrm{D}}, \mathcal{U}^9_{\mathrm{D}}, \hat{\mathcal{U}}_{\mathrm{D}}$. Now we already have that all refinements with a single component are DR-equivalent to a DMTS shown in Figure 3.2.

DMTSs with more than one component that have $\hat{\mathcal{U}}_{\text{D}}$ as one of their components need not be considered, because they either do not refine $\hat{\mathcal{U}}_{\text{D}}$ or are DR-equivalent to $\hat{\mathcal{U}}_{\text{D}}$. We consider the remaining DMTSs built up out of two components:

$$\begin{array}{rcl}
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^2 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^4 \\
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^3 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^5 \\
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^7 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^7 \\
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{11} \\
\hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^3 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^6 \\
\hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^7 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{11} \\
\hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{10} \\
\hat{\mathcal{U}}_{\text{D}}^3 \otimes \hat{\mathcal{U}}_{\text{D}}^7 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{10} \\
\hat{\mathcal{U}}_{\text{D}}^3 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^9 \\
\hat{\mathcal{U}}_{\text{D}}^7 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{12}
\end{array}$$

All refinements built up out of two components are DR-equivalent to a DMTS shown in Figure 3.2.

We continue with remaining DMTSs built up out of three components. We need not consider combinations including both $\hat{\mathcal{U}}_{\text{D}}^1$ and $\hat{\mathcal{U}}_{\text{D}}^7$, and both $\hat{\mathcal{U}}_{\text{D}}^3$ and $\hat{\mathcal{U}}_{\text{D}}^9$, because both of these pairs is DR-equivalent to a single component from our remaining set and consequently all such combinations have already been considered above.

$$\begin{array}{rcl}
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^3 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^8 \\
\hat{\mathcal{U}}_{\text{D}}^1 \otimes \hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{11} \\
\hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^3 \otimes \hat{\mathcal{U}}_{\text{D}}^7 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{10} \\
\hat{\mathcal{U}}_{\text{D}}^2 \otimes \hat{\mathcal{U}}_{\text{D}}^7 \otimes \hat{\mathcal{U}}_{\text{D}}^9 & \approx_{\text{D}} & \hat{\mathcal{U}}_{\text{D}}^{12}
\end{array}$$

All refinements built up out of three components are DR-equivalent to a DMTS shown in Figure 3.2.

Next, we would have to consider DMTSs build up out of four, respectively five components, where combinations including both $\hat{\mathcal{U}}_{\text{D}}^1$ and $\hat{\mathcal{U}}_{\text{D}}^7$, and both $\hat{\mathcal{U}}_{\text{D}}^3$ and $\hat{\mathcal{U}}_{\text{D}}^9$ need not be considered, because they are DR-equivalent to a DMTS built up out of three, respectively four components. However, each such combination includes $\hat{\mathcal{U}}_{\text{D}}^1$ and $\hat{\mathcal{U}}_{\text{D}}^7$, or $\hat{\mathcal{U}}_{\text{D}}^3$ and $\hat{\mathcal{U}}_{\text{D}}^9$, thus there are no further refinements of $\hat{\mathcal{U}}_{\text{D}}$. □

3.7 All Refinements of a Simple 1MTS

In this section, we change over to the 1-selecting formalism and want to give a complete overview over all (1-selecting) refinements of a simple 1MTS. The 1MTS of interest is

$$\hat{\mathcal{U}}_1 \stackrel{\text{def}}{=} (\{0,1\}, \{a,b\}, \{(0, \{(a,1),(b,1)\}\}, \{(0, \{(a,1),(b,1)\}\}, \{0\}).$$

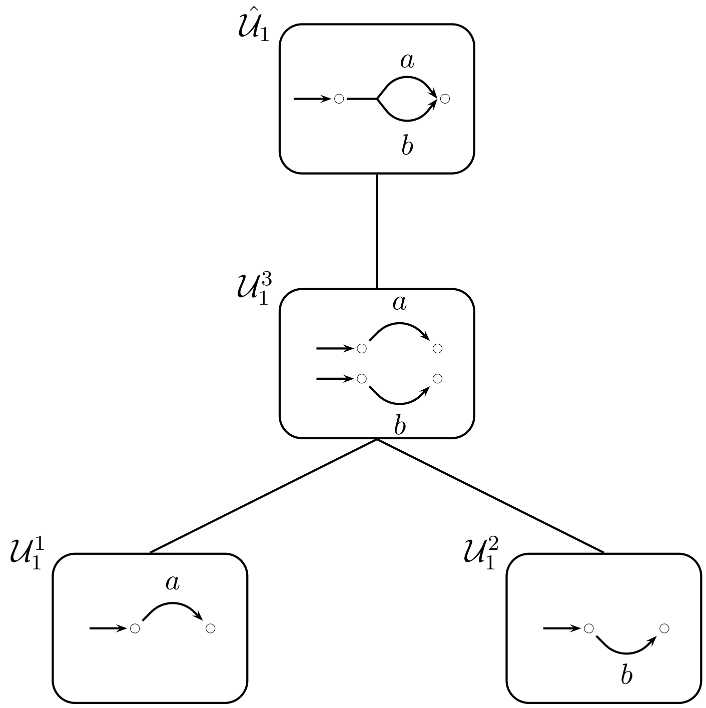

Figure 3.4: $[\hat{\mathcal{U}}_1]_{\approx_1}$ and all its refinements in a Hasse diagram

This 1MTS is illustrated at the top of Figure 3.4. As usual, we do not draw may (hyper-)transitions, if they also exist as must (hyper-)transitions. For this reason, the may hypertransition of $\hat{\mathcal{U}}_1$ does not appear in the drawing. We claim that Figure 3.4 gives a complete overview over all refinements of $[\hat{\mathcal{U}}_1]_{\approx_1}$. Any 1MTS illustrated in the figure stands for the 1R-equivalence class it is a representative of. Lines between the systems represent the 1-selecting refinement relation on 1R-equivalence classes, $\trianglelefteq_1$. Classes below 1-selecting refine classes above. Consequently, Figure 3.4 forms a Hasse diagram of all 1-selecting refinements of $[\hat{\mathcal{U}}_1]_{\approx_1}$, ordered by $\trianglelefteq_1$.

It is easy to see that all lines in Figure 3.4 indeed represent 1-selecting refinement relations. In order to show that the figure lists all 1-selecting refinements of $[\hat{\mathcal{U}}_1]_{\approx_1}$, we take an approach analogous to the one in the previous section: We take an arbitrary refinement of $\hat{\mathcal{U}}_1$ and prove that it is 1R-equivalent to one of the DMTSs in the figure, i.e., it belongs to one of the equivalence classes represented by the figure. Again, we make use of *components*, which are defined using the notion of *reachability*. These definitions are analogue to the ones in the disjunctive formalism. Differences are required only due to the fact that 1MTSs feature may hypertransitions, whereas DMTSs do not.

Definition 3.25 ($\mathcal{U}_1$-reachable)**.** *Let* $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MTS}$ *and*

$\bar{u} \in U_1$. *Recursively define*

$$V_1^{\bar{u}} \stackrel{\text{def}}{=} \{\bar{u}\}$$
$$V_n^{\bar{u}} \stackrel{\text{def}}{=} V_{n-1}^{\bar{u}} \cup \{u' \in U_1 \mid \exists u \in V_{n-1}^{\bar{u}}, \Theta \in \mathcal{P}(L \times U_1) \setminus \emptyset, a \in L : u \dashrightarrow_{\text{D}} \Theta \wedge (a, u') \in \Theta\}$$

and let $V^{\bar{u}} \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} V_n^{\bar{u}}$. *Then* $u \in U_{\text{D}}$ *is called* $\mathcal{U}_1$-reachable from $\bar{u}$, *if and only if* $u \in V^{\bar{u}}$. *Furthermore,* $u \in U_{\text{D}}$ *is called* $\mathcal{U}_1$-reachable, *if and only if there exists* $u^0 \in U_{\text{D}}^0$ *such that* u *is* $\mathcal{U}_1$*-reachable from* u^0.

The definition of reachability is based on the may transition relation. This includes reachability via must transitions due to the condition of 1MTSs that the must transition relation is required to be a subset of the may transition relation.

A *component* is an 1MTS with only one root state in which all states are reachable. For an 1MTS $\mathcal{U}_1$, a *component of* $\mathcal{U}_1$ is a component that is a "part" of $\mathcal{U}_1$. Formally:

Definition 3.26 (Component)**.** *An 1MTS* $\mathcal{C} = (U_{\text{C}}, L, \longmapsto_{\text{C}}, \dashrightarrow_{\text{C}}, U_{\text{C}}^0) \in \mathbb{1MT\$}$ *is called* component, *if and only if there exists* $u^0 \in U_{\text{C}}^0$ *such that* $U_{\text{C}}^0 = \{u^0\}$ *and each* $u \in U_{\text{C}}$ *is* $\mathcal{C}$*-reachable.*

Definition 3.27 (Component of $\mathcal{U}_1$)**.** *Let* $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MT\$}$. *A 1MTS* $\mathcal{C} = (U_{\text{C}}, L, \longmapsto_{\text{C}}, \dashrightarrow_{\text{C}}, U_{\text{C}}^0) \in \mathbb{1MT\$}$ *is called* component of $\mathcal{U}_1$, *if and only if there exists* $u^0 \in U_1^0$ *such that*

$$\begin{aligned} U_{\text{C}}^0 &= \{u^0\}, \\ U_{\text{C}} &= \{u \in U_1 \mid u \text{ is } \mathcal{U}_1\text{-reachable from } u^0\}, \\ u \longmapsto_{\text{C}} \Theta &\Leftrightarrow u \in U_{\text{C}} \wedge u \longmapsto_1 \Theta, \\ u \dashrightarrow_{\text{C}} \Theta &\Leftrightarrow u \in U_{\text{C}} \wedge u \dashrightarrow_1 \Theta. \end{aligned}$$

We denote the set of all components of $\mathcal{U}_1$ *by* $\mathbb{C}\mathsf{omp}(\mathcal{U}_1)$.

For any 1MTS $\mathcal{U}_1$, each component of $\mathcal{U}_1$ is obviously a component. The following lemmata, that have analogous counterparts in the disjunctive formalism, hold also in the 1-selecting formalism:

Lemma 3.28. *Let* $\mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MT\$}$. *Then* $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$ *if and only if* $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_1) : \mathcal{C} \lhd_1 \hat{\mathcal{U}}_1$.

Proof. Obvious by definition of 1-selecting refinement. □

Lemma 3.29. *Let* $\mathcal{U}_1 \in \mathbb{1MT\$}$. *Then* $\forall \mathcal{C} \in \mathbb{C}\mathsf{omp}(\mathcal{U}_1) : \mathcal{C} \lhd_1 \mathcal{U}_1$.

Proof. Apply Lemma 3.28 to $\mathcal{U}_1$ and $\mathcal{U}_1$. □

Lemma 3.30. *Let* $\mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MT\$}$ *such that* $\exists \mathcal{C} \in \mathbb{C}\mathsf{omp}(\hat{\mathcal{U}}_1) : \mathcal{U}_1 \lhd_1 \mathcal{C}$. *Then* $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$.

Proof. Obvious by definition of 1-selecting refinement. □

There are only slight differences between the definitions of the merge operator for DMTSs and the merge operator for 1MTSs. These differences are again required only due to the fact that 1MTSs feature may hypertransitions. For 1MTS, the merge operator $\otimes$ is defined as follows:

Definition 3.31. *Let I be an index set and $\mathcal{U}_1^i = (U_1^i, L^i, \longmapsto_1^i, \dashrightarrow_1^i, U_1^{0^i}) \in \mathbb{1MTS}$ for all $i \in I$. Then $\bigotimes_{i\in I} \mathcal{U}_1^i$ is defined to be the 1MTS $(U_\otimes, L, \longmapsto_\otimes, \dashrightarrow_\otimes, U_\otimes^0)$, where*

$$
\begin{aligned}
U_\otimes &\stackrel{\text{def}}{=} \bigcup_{i\in I} \left(U_1^i \times \{i\}\right), \\
L &\stackrel{\text{def}}{=} \bigcup_{i\in I} L^i, \\
(u,i) \longmapsto_\otimes \Theta &\stackrel{\text{def}}{\Leftrightarrow} (\forall (a,(u',i')) \in \Theta : i = i') \wedge \\
&\qquad u \longmapsto_{\mathrm{D}}^i \{(a,u') \mid (a,(u',i)) \in \Theta\}, \\
(u,i) \dashrightarrow_\otimes \Theta &\stackrel{\text{def}}{\Leftrightarrow} (\forall (a,(u',i')) \in \Theta : i = i') \wedge \\
&\qquad u \dashrightarrow_{\mathrm{D}}^i \{(a,u') \mid (a,(u',i)) \in \Theta\}, \\
U_\otimes^0 &\stackrel{\text{def}}{=} \bigcup_{i\in I} \left(U_1^{0^i} \times \{i\}\right).
\end{aligned}
$$

For any set of 1MTSs $\mathbb{U}_1$, we shortly write $\bigotimes \mathbb{U}_1$, which is defined to be $\bigotimes_{\mathcal{U}_1\in\mathbb{U}_1} \mathcal{U}_1$. If $\mathbb{U}_1$ has exactly two elements, say $\mathcal{U}_1^1$ and $\mathcal{U}_1^2$, we usually write $\mathcal{U}_1^1 \otimes \mathcal{U}_1^2$.

As in the disjunctive formalism, we present lemmata that allow us to reason with components and merging. If we merge all components of an 1MTS, we get an 1MTS that is 1R-equivalent to the original one, i.e., we do not leave the equivalence class. Furthermore, exchanging one component of a 1MTS with an 1R-equivalent component does not leave the 1R-equivalence class either.

Lemma 3.32. *Let $\mathcal{U}_1 \in \mathbb{1MTS}$. Then $\mathcal{U}_1 \approx_1 \bigotimes \mathsf{Comp}(\mathcal{U}_1)$.*

Proof. Let $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MTS}$ and $\bigotimes \mathsf{Comp}(\mathcal{U}_1) = \bigotimes_{\mathcal{C}\in\mathsf{Comp}(\mathcal{U}_\mathrm{D})} \mathcal{C} = (U_\otimes, L, \longmapsto_\otimes, \dashrightarrow_\otimes, U_\otimes^0)$. For any component $\mathcal{C} \in \mathsf{Comp}(\mathcal{U}_1)$, set $\mathcal{C} = (U_\mathrm{C}^\mathcal{C}, L, \longmapsto_\mathrm{C}^\mathcal{C}, \dashrightarrow_\mathrm{C}^\mathcal{C}, U_\mathrm{C}^{0^\mathcal{C}})$.

We start with the proof of $\mathcal{U}_1 \lhd_1 \bigotimes \mathsf{Comp}(\mathcal{U}_1)$ by showing that $Q \subseteq U_1 \times U_\otimes$, defined by

$$u Q(\hat{u}, \mathcal{C}) \stackrel{\text{def}}{\Leftrightarrow} u = \hat{u},$$

is an 1-selecting refinement between $\mathcal{U}_1$ and $\bigotimes \mathsf{Comp}(\mathcal{U}_1)$.

(i) Let $u \in U_1^0$. Then there exists $\mathcal{C} \in \mathsf{Comp}(\mathcal{U}_1)$ such that $u \in U_\mathrm{C}^{0^\mathcal{C}}$ and $(u, \mathcal{C}) \in U_\otimes^0$. We have $uQ(u,\mathcal{C})$, as required.

(ii) Let $(u, (\hat{u}, \mathcal{C})) \in Q$. Then $u = \hat{u}$. Furthermore, let $\gamma \in \mathsf{choice}(u \dashrightarrow_1)$. Define $\hat{\gamma} \in \mathsf{choice}((u, \mathcal{C}) \dashrightarrow_\otimes)$ as follows: For $\hat{\Theta} \in ((u, \mathcal{C}) \dashrightarrow_\otimes)$, define $\hat{\gamma}(\hat{\Theta}) \stackrel{\text{def}}{=} (a, (u', \mathcal{C}))$, where $(a, u') \stackrel{\text{def}}{=} \gamma(\{(a, u') \mid (a, (u', \mathcal{C})) \in \hat{\Theta}\})$.

(a) Let $\Theta \in (u \dashrightarrow_1)$. We have $u \in U_{\mathrm{C}}^{\mathcal{C}}$, $u' \in U_{\mathrm{C}}^{\mathcal{C}}$ for all $(a, u') \in \Theta$ and $u \dashrightarrow_{\mathrm{C}}^{\mathcal{C}} \Theta$. Consequently $(u, \mathcal{C}) \dashrightarrow_\otimes \hat{\Theta}$, where $\hat{\Theta} \stackrel{\text{def}}{=} \{(a, (u', \mathcal{C})) \mid (a, u') \in \Theta\}$. We have $\gamma(\Theta)\ Q\ \hat{\gamma}(\hat{\Theta})$, as required.

(b) Let $\hat{\Theta} \in ((u, \mathcal{C}) \longmapsto_\otimes)$. Then $u \in U_1$ and $u \longmapsto_1 \Theta$, where $\Theta \stackrel{\text{def}}{=} \{(a, u') \mid (a, (u', \mathcal{C})) \in \hat{\Theta}\}$. We have $\gamma(\Theta)\ Q\ \hat{\gamma}(\hat{\Theta})$, as required.

It remains to prove $\bigotimes \mathsf{Comp}(\mathcal{U}_1) \lhd_1 \mathcal{U}_1$. Define $Q \subseteq U_1 \times U_\otimes$ as follows:

$$(u, \mathcal{C}) Q \hat{u} \stackrel{\text{def}}{\Leftrightarrow} u = \hat{u}.$$

We prove that Q is an 1-selecting refinement between $\bigotimes \mathsf{Comp}(\mathcal{U}_1)$ and $\mathcal{U}_1$.

(i) Let $(u, \mathcal{C}) \in U_\otimes^0$. Then $u \in U_1^0$ and $(u, \mathcal{C}) Q u$.

(ii) Let $((u, \mathcal{C}), \hat{u}) \in Q$. Then $u = \hat{u}$. Furthermore, let $\gamma \in \mathsf{choice}((u, \mathcal{C}) \dashrightarrow_\otimes)$. Define $\hat{\gamma} \in \mathsf{choice}(u \dashrightarrow_1)$ as follows: For $\hat{\Theta} \in (u \dashrightarrow_1)$, define $\hat{\gamma}(\hat{\Theta}) \stackrel{\text{def}}{=} (a, u')$, where $(a, (u', \mathcal{C})) \stackrel{\text{def}}{=} \gamma(\{(a, (u', \mathcal{C})) \mid (a, u') \in \hat{\Theta}\})$.

(a) Let $\Theta \in ((u, \mathcal{C}) \dashrightarrow_\otimes)$. Then $u \in U_1$ and $u \dashrightarrow_1 \hat{\Theta}$, where $\hat{\Theta} \stackrel{\text{def}}{=} \{(a, u') \mid (a, (u', \mathcal{C})) \in \Theta\}$. We have $\gamma(\Theta)\ Q\ \hat{\gamma}(\hat{\Theta})$, as required.

(b) Let $\hat{\Theta} \in (u \longmapsto_1)$. We have $u \in U_{\mathrm{C}}^{\mathcal{C}}$, $u' \in U_{\mathrm{C}}^{\mathcal{C}}$ for all $(a, u') \in \hat{\Theta}$ and $u \longmapsto_{\mathrm{C}}^{\mathcal{C}} \hat{\Theta}$. Consequently $(u, \mathcal{C}) \longmapsto_\otimes \Theta$, where $\Theta \stackrel{\text{def}}{=} \{(a, (u', \mathcal{C})) \mid (a, u') \in \hat{\Theta}\}$. We have $\gamma(\Theta)\ Q\ \hat{\gamma}(\hat{\Theta})$, as required. □

Lemma 3.33. *Let $\mathcal{U}_1 \in \mathbb{1MTS}$, $\mathcal{U}_1^1 \in \mathsf{Comp}(\mathcal{U}_1)$ and $\mathcal{U}_1^2$ be a component such that $\mathcal{U}_1^1 \approx_1 \mathcal{U}_1^2$. Then $\mathcal{U}_1 \approx_1 (\bigotimes \mathsf{Comp}(\mathcal{U}_1) \setminus \mathcal{U}_1^1) \cup \mathcal{U}_1^2$.*

Proof. Analogue to the proof of Lemma 3.23. □

We are now ready to prove the main theorem of this section:

Theorem 3.34. *Each refinement of*

$$\hat{\mathcal{U}}_1 \stackrel{\text{def}}{=} (\{0,1\}, \{a,b\}, \{(0, \{(a,1),(b,1)\}\}, \{(0, \{(a,1),(b,1)\}\}, \{0\}).$$

is 1R-equivalent to one of the 1MTS*s shown in Figure 3.4.*

In other words, if all 1MTSs in Figure 3.4 are interpreted as their 1R-equivalence classes, each refinement of $[\hat{\mathcal{U}}_1]_{\approx_1}$ appears in the Figure.

Proof of Theorem 3.34. Let $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MTS}$ such that $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$. Choose an 1-selecting refinement $Q \subseteq U_1 \times \{0,1\}$ between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$. We examine an arbitrary component of $\mathcal{U}_1$.

Thus let $\mathcal{C} \in \mathsf{Comp}(\mathcal{U}_1)$ and let u be the root state of $\mathcal{C}$. Then we have $uQ0$. $\mathcal{C}$ satisfies the following properties:

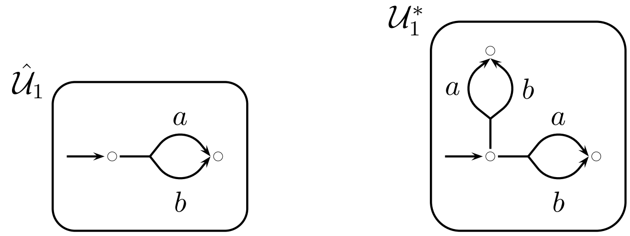

Figure 3.5: DMTSs $\hat{\mathcal{U}}_1$ and $\mathcal{U}_1^*$: $\mathcal{U}_1^*$ does not refine $\hat{\mathcal{U}}_1$

(i) There are no may (hyper-)transitions starting in u with a label different from a and b, because otherwise, Q would not be an 1-selecting refinement between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$.

(ii) There are no may (hyper-)transitions starting in a state targeted by a transition starting in u, because this would again contradict the fact that Q is an 1-selecting refinement between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$.

Note that this does *not* imply that we can assume what we could assume in the disjunctive formalism, namely that there are only two states in the component. For example, the 1MTSs $\hat{\mathcal{U}}_1$ and $\mathcal{U}_1^*$ illustrated in Figure 3.5 are *not* 1-selecting equivalent: $\mathcal{U}_1^*$ does *not* refine $\hat{\mathcal{U}}_1$.

(iii) There is a must transition starting in u, because otherwise, Q would not be an 1-selecting refinement between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$.

(iv) There are no two may (hyper-)transitions starting in u, where one of them includes a label a and the other includes a label b. If two such transitions existed, then one could not find an appropriate choice function in $\hat{\mathcal{U}}_1$ for the choice function that selects a in one of the transitions in $\mathcal{C}$ and b in the other.

Property (iii) states that there is a must (hyper-)transition starting in u. By property (i), it has labels a, b or both. Property (iv) implies that further transitions can only exist, if all labels of the component are the same. Then the only possible components are the following:

- $\hat{\mathcal{U}}_1$.
- Components, that have arbitrarily many (hyper-)transitions starting in the root state, where each transition has only labels a. Due to property (ii), all these are 1R-equivalent to $\mathcal{U}_1^1$.
- Components, that have arbitrarily many (hyper-)transitions starting in the root state, where each transition has only labels b. Due to property (ii), all these are 1R-equivalent to $\mathcal{U}_1^2$.

We now have that for all refinements with only one component there is an 1MTS in Figure 3.4 that is 1R-equivalent to it. 1MTSs with more than one component that have $\hat{\mathcal{U}}_1$ as one of their components need not be considered, because they either do not refine $\hat{\mathcal{U}}_1$ or are 1R-equivalent to $\hat{\mathcal{U}}_1$. The only remaining 1MTSs built up out of two components is $\mathcal{U}_1^1 \otimes \mathcal{U}_1^2$, which is 1R-equivalent to $\mathcal{U}_1^3$. Refinements built up out of more components are 1R-equivalent to refinements already considered. $\square$

Chapter 4

Expressiveness

In this chapter, we formalise a general notion of relative expressiveness for abstraction/refinement frameworks that takes the refinement ordering structure into account and demands the preservation of implementations. Afterwards, we use it in order to compare 1MTSs with DMTSs. We will see that, when regarding this notion, the 1MTS-formalism is strictly more expressive than the DMTS-formalism, although DMTSs and 1MTSs can express the same sets of implementations.

4.1 Order Homomorphisms and Expressiveness

We would like to compare the expressiveness of DMTSs and 1MTSs. To achieve this, we first need to develop an idea of what exactly we mean by the term *(relative) expressiveness*. Aiming at a formal definition of this term, we start with a consideration of partially ordered sets (which can be abstraction/refinement formalisms like $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$ or $(\underline{\mathbb{1MTS}}, \trianglelefteq_1)$) and the following existing notions of structure-preserving functions on them:

Definition 4.1 (Order homomorphism)**.** *Let* $(A, \preceq)$*,* $(B, \sqsubseteq)$ *be partially ordered sets. A function* $f : A \to B$ *is called* order homomorphism *(or simply* monotonic*), if and only if*

$$\forall a_1, a_2 \in A : a_1 \preceq a_2 \Rightarrow f(a_1) \sqsubseteq f(a_2).$$

Definition 4.2 (Order embedding)**.** *Let* $(A, \preceq)$*,* $(B, \sqsubseteq)$ *be partially ordered sets. A function* $f : A \to B$ *is called* order embedding*, if and only if*

$$\forall a_1, a_2 \in A : a_1 \preceq a_2 \Leftrightarrow f(a_1) \sqsubseteq f(a_2).$$

Definition 4.3 (Order isomorphism)**.** *An* order isomorphism *is a surjective order embedding.*

Another characterisation of order isomorphisms is the following: An order homomorphism f is an order isomorphism, if and only if f is bijective and its inverse is an order homomorphism as well.

Roughly speaking, if an order embedding exists, then one partially ordered set can be included into the other. If the two partially ordered sets are order isomorphic, they can be considered to be essentially the same in the sense that one of the orders can be obtained from the other just by renaming of elements.

Note that we cannot directly apply the notions of order embeddings and isomorphisms to $(\mathbb{DMTS}, \lhd_{\mathrm{D}})$ and $(\mathbb{1MTS}, \lhd_1)$, because these are not partially ordered sets. This is because they are not antisymmetric: Not for every $\mathcal{U}_{\mathrm{D}}^1, \mathcal{U}_{\mathrm{D}}^2 \in \mathbb{DMTS}$ with $\mathcal{U}_{\mathrm{D}}^1 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$ and $\mathcal{U}_{\mathrm{D}}^2 \lhd_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^1$, we have $\mathcal{U}_{\mathrm{D}}^1 = \mathcal{U}_{\mathrm{D}}^2$. However, we do have $\mathcal{U}_{\mathrm{D}}^1 \approx_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}^2$. Thus we consider the DR-, respectively 1R-equivalence classes of DMTSs, respectively 1MTSs: $(\underline{\mathbb{DMTS}}, \unlhd_{\mathrm{D}})$ and $(\underline{\mathbb{1MTS}}, \unlhd_1)$ are partially ordered sets and therefore the notions of order homomorphisms, embeddings and isomorphisms can be applied.

However, when the expressiveness of ordered sets is compared in the context of abstraction, the concepts of order embeddings are in general not distinguishing enough. This is illustrated in the following example: Consider

$$\underline{\mathbb{DMTS}}^* \stackrel{\text{def}}{=} \{[\mathcal{U}_{\mathrm{D}}^1]_{\approx_{\mathrm{D}}}, [\mathcal{U}_{\mathrm{D}}^2]_{\approx_{\mathrm{D}}}, [\mathcal{U}_{\mathrm{D}}^3]_{\approx_{\mathrm{D}}}, [\mathcal{U}_{\mathrm{D}}^4]_{\approx_{\mathrm{D}}}, [\mathcal{U}_{\mathrm{D}}^7]_{\approx_{\mathrm{D}}}\} \subseteq \underline{\mathbb{DMTS}}$$

on the one hand, where the DMTSs are those from Figure 3.2 in Section 3.6, and

$$\underline{\mathbb{1MTS}}^* \stackrel{\text{def}}{=} \{[\mathcal{U}_1^1]_{\approx_1}, [\mathcal{U}_1^2]_{\approx_1}, [\mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}}^2))]_{\approx_1}, [\mathcal{U}_1^3]_{\approx_1}, [\hat{\mathcal{U}}_1]_{\approx_1}\} \subseteq \underline{\mathbb{1MTS}}$$

on the other hand, where the 1MTSs are those from Figure 3.4 in Section 3.7. Figures 3.2 and 3.4 illustrate that the two ordered sets are order isomorphic. A possible order isomorphism from $\underline{\mathbb{DMTS}}^*$ to $\underline{\mathbb{1MTS}}^*$ maps the elements as follows:

$$\begin{array}{lcl} [\mathcal{U}_{\mathrm{D}}^1]_{\approx_{\mathrm{D}}} & \mapsto & [\mathcal{U}_1^1]_{\approx_1} \\ [\mathcal{U}_{\mathrm{D}}^2]_{\approx_{\mathrm{D}}} & \mapsto & [\mathcal{U}_1^2]_{\approx_1} \\ [\mathcal{U}_{\mathrm{D}}^3]_{\approx_{\mathrm{D}}} & \mapsto & [\mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}}^2))]_{\approx_1} \\ [\mathcal{U}_{\mathrm{D}}^4]_{\approx_{\mathrm{D}}} & \mapsto & [\mathcal{U}_1^3]_{\approx_1} \\ [\mathcal{U}_{\mathrm{D}}^7]_{\approx_{\mathrm{D}}} & \mapsto & [\hat{\mathcal{U}}_1]_{\approx_1} \end{array}$$

Although there is an order isomorphism, it does not seem appropriate to call the two sets equally expressive. For instance, $\mathcal{U}_{\mathrm{D}}^2$ can be abstracted in $(\underline{\mathbb{DMTS}}^*, \unlhd_{\mathrm{D}}|_{\underline{\mathbb{DMTS}}^*})$, but $\mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}}^2))$ cannot be abstracted in $(\underline{\mathbb{1MTS}}^*, \unlhd_1|_{\underline{\mathbb{1MTS}}^*})$, although these two fully determined systems are basically the same (up to the change of formalism that is performed by $\mathsf{1MTS} \circ \mathsf{TS}_{\mathrm{D}}$). Thus there is a need for an order embedding concept that requires concrete systems to be mapped to "themselves", i.e., one that preserves a given function on minimal (i.e., concrete) elements (which will often be the identity function). This concept is formalised in the following definitions, where $\mathsf{Min}(A)$ for an ordered set A denotes the set of minimal elements of A:

Definition 4.4 (φ-preserving order homomorphism)**.** *Let* $(A, \preceq)$*,* $(B, \sqsubseteq)$ *be partially ordered sets, and* $\varphi : \mathsf{Min}(A) \to \mathsf{Min}(B)$*. A function* $f : A \to B$ *is called* φ-preserving order homomorphism*, if and only if* *(i) it is an order homomorphism, and*

(ii) $\forall a \in \mathsf{Min}(A) : f(a) = \varphi(a)$.

Definition 4.5 (φ-preserving order embedding)**.** *Let* $(A, \preceq)$, $(B, \sqsubseteq)$ *be partially ordered sets, and* $\varphi : \mathsf{Min}(A) \to \mathsf{Min}(B)$. *A function* $f : A \to B$ *is called* φ-preserving order embedding, *if and only if*

(i) it is an order embedding, and

(ii) $\forall a \in \mathsf{Min}(A) : f(a) = \varphi(a)$.

Definition 4.6 (φ-preserving order isomorphism)**.** *A* φ-preserving order isomorphism *is a* φ*-preserving order embedding that is surjective.*

In our setting, the sets $\underline{\mathbb{DMTS}}^{\mathrm{det}}$ and $\underline{\mathbb{1MTS}}^{\mathrm{det}}$ are identified. Making use of the functions $\underline{\mathsf{DMTS}}$, $\underline{\mathsf{1MTS}}$, $\underline{\mathsf{TS}}_{\mathrm{D}}$, and $\underline{\mathsf{TS}}_1$ from Definition 3.13, we consider $(\underline{\mathsf{1MTS}} \circ \underline{\mathsf{TS}}_{\mathrm{D}})$-preserving homomorphisms, embeddings, and isomorphisms from $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$ to $(\underline{\mathbb{1MTS}}, \trianglelefteq_1)$, that we will shortly call *D1-homomorphism*s, *D1-embedding*s, and *D1-isomorphism*s, and in the other direction, $(\underline{\mathsf{DMTS}} \circ \underline{\mathsf{TS}}_1)$-preserving homomorphisms, embeddings, and isomorphisms from $(\underline{\mathbb{1MTS}}, \trianglelefteq_1)$ to $(\underline{\mathbb{DMTS}}, \trianglelefteq_{\mathrm{D}})$, that we will shortly call *1D-homomorphism*s, *1D-embedding*s, and *1D-isomorphism*s.

In Section 4.2, we will show that a D1-embedding exists, getting the result that 1MTSs are at least as expressive as DMTSs. In Section 4.3, a straightforward 1D-homomorphism is presented that is however not an 1D-embedding. Finally in Section 4.4, we show that there exists no 1D-embedding at all and consequently no 1D-isomorphism either, getting the result that 1MTSs are more expressive than DMTSs.

4.2 A D1-Embedding

We define a function f from $\mathbb{DMTS}$ to $\mathbb{1MTS}$ and show that it induces a D1-embedding f^* from $\underline{\mathbb{DMTS}}$ to $\underline{\mathbb{1MTS}}$. First, we concentrate on f and show the two properties

$$\forall \mathcal{U}_{\mathrm{D}}, \hat{\mathcal{U}}_{\mathrm{D}} \in \mathbb{DMTS} : \mathcal{U}_{\mathrm{D}} \trianglelefteq_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}} \Leftrightarrow f(\mathcal{U}_{\mathrm{D}}) \trianglelefteq_1 f(\hat{\mathcal{U}}_{\mathrm{D}})$$

(Lemma 4.8) and

$$\forall \mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}^{\mathrm{det}} : \mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}})) \approx_1 f(\mathcal{U}_{\mathrm{D}})$$

(Lemma 4.9). Then we derive the function $f^* : \underline{\mathbb{DMTS}} \to \underline{\mathbb{1MTS}}$ from f by switching over to DR-equivalence classes and prove that f^* is a D1-embedding, which essentially requires showing that the two properties shown for f persist the transition to equivalence classes.

Before giving the formal definition of function f, we describe informally the idea behind it: In a DMTS, arbitrarily many targets of a hypertransition can be "taken", whereas in 1MTSs, only one target per hypertransition can be "taken". The idea is to turn every DMTS-hypertransition with n targets into n 1MTS-hypertransitions. To achieve this, we need to introduce "state copies" (every DMTS-state becomes two 1MTS-states), because it is not possible to have exactly the same hypertransition more than once in the 1MTS (sets allow no duplicates). Using these copies, we can have transitions, that are "behaviourally" the same, but in fact lead to different states (different copies of the same DMTS-state). The formal definition of f is as follows:

Definition 4.7. *Define* $f : \mathbb{DMTS} \to \mathbb{1MTS}$; $(U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \dashrightarrow_{\mathrm{D}}, U^0_{\mathrm{D}}) \mapsto (U_1, L, \longmapsto_1, \dashrightarrow_1, U^0_1)$, *with*

$$\begin{aligned}
U_1 &\overset{\text{def}}{=} U_{\mathrm{D}} \times \{0,1\}, \textit{ where we usually write } u_i \textit{ instead of } (u,i),\\
u_i \longmapsto_1 \Theta_1 &\overset{\text{def}}{\Leftrightarrow} \exists \Theta_{\mathrm{D}} \in (u \longmapsto_{\mathrm{D}}) : \exists (\bar{a}, \bar{u'}) \in \Theta_{\mathrm{D}} :\\
&\quad\ \Theta_1 = \{(a, u'_0) \mid (a, u') \in \Theta_{\mathrm{D}}\} \cup \{(\bar{a}, \bar{u'}_1)\},\\
u_i \dashrightarrow_1 \Theta_1 &\overset{\text{def}}{\Leftrightarrow} \exists \{(a,u')\} \in (u \dashrightarrow_{\mathrm{D}}) : \Theta_1 = \{(a, u'_0), (a, u'_1)\}\\
&\quad\ \vee (u_i \longmapsto_1 \Theta_1),\\
U^0_1 &\overset{\text{def}}{=} U^0_{\mathrm{D}} \times \{0,1\}.
\end{aligned}$$

Note that, in analogy to the definition of the must transition relation, we could have also written the following for the may transition relation:

$$\begin{aligned}
u_i \dashrightarrow_1 \Theta_1 &\overset{\text{def}}{\Leftrightarrow} \exists \Theta_{\mathrm{D}} \in (u \dashrightarrow_{\mathrm{D}}) : \exists (\bar{a}, \bar{u'}) \in \Theta_{\mathrm{D}} :\\
&\quad\ \Theta_1 = \{(a, u'_0) \mid (a, u') \in \Theta_{\mathrm{D}}\} \cup \{(\bar{a}, \bar{u'}_1)\}\\
&\quad\ \vee (u_i \longmapsto_1 \Theta_1)
\end{aligned}$$

f doubles the number of states. We already explained that the only reason for this is to be able to have certain hypertransitions (up to state indices) more than once. However, in most cases, less states are sufficient to achieve this, as illustrated by the following example of a DMTS-hypertransition with four targets $(a,s), (a,t), (a,u)$ and (a,v). The translation should turn this hypertransition into four 1MTS-hypertransitions, where it is essential that each 1MTS-hypertransition targets each DMTS-target up to equivalence. f takes the straightforward approach to include every DMTS-target in every 1MTS-hypertransition with an index 0 and add one DMTS-target with an index 1, different for each 1MTS-hypertransition. This approach is illustrated in the second column of Table 4.1. The result of an advanced translation is shown in the third column. As required, every DMTS-target appears in every 1MTS-hypertransition. This is achieved without using the state copies u_1 and v_1, thus this hypertransition does not require these to appear. Note however, that other hypertransitions might require one or both of them. Nevertheless, we see that in many cases it should be possible to leave some states uncopied. It only needs to be guaranteed that for every

DMTS-targets	straightforward translation into 1MTS-hypertransitions	advanced translation into 1MTS-hypertransitions
(a,s),	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,s_1)\}$,	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0)\}$,
(a,t),	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,t_1)\}$,	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,s_1)\}$,
(a,u),	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,u_1)\}$,	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,t_1)\}$,
(a,v)	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,v_1)\}$	$\{(a,s_0),(a,t_0),(a,u_0),(a,v_0),(a,s_1),(a,t_1)\}$

Table 4.1: Straightforward and advanced translation

hypertransition with n targets in the DMTS, at least $\lceil \log_2(n) \rceil$ of these targets appear with an index 1 in the state set of the 1MTS. Nevertheless, for the sake of ease and better readability, we consider the straightforward translation function f in the following. Note that the overall complexity is not increased by the straightforward transformation (it is linear in both cases).

Lemma 4.8. *For all* $\mathcal{U}_{\mathrm{D}}, \hat{\mathcal{U}}_{\mathrm{D}} \in \mathbb{DMTS}$, *we have*

$$\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}} \Leftrightarrow f(\mathcal{U}_{\mathrm{D}}) \lhd_1 f(\hat{\mathcal{U}}_{\mathrm{D}}). \tag{4.1}$$

Proof. Let $\mathcal{U}_{\mathrm{D}} = (U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \dashrightarrow_{\mathrm{D}}, U^0_{\mathrm{D}}), \hat{\mathcal{U}}_{\mathrm{D}} = (\hat{U}_{\mathrm{D}}, L, \hat{\longmapsto}_{\mathrm{D}}, \hat{\dashrightarrow}_{\mathrm{D}}, \hat{U}^0_{\mathrm{D}}) \in \mathbb{DMTS}$. Define $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U^0_1) \stackrel{\text{def}}{=} f(\mathcal{U}_{\mathrm{D}})$ and $\hat{\mathcal{U}}_1 = (\hat{U}_1, L, \hat{\longmapsto}_1, \hat{\dashrightarrow}_1, \hat{U}^0_1) \stackrel{\text{def}}{=} f(\hat{\mathcal{U}}_{\mathrm{D}})$. Before proving the two implications of (4.1), we introduce some useful notation.

- For a target $\vartheta_{\mathrm{D}} = (a, u')$ in the DMTS and $k \in \{0,1\}$, define

  $$\vartheta_{\mathrm{D}} \uparrow_k \stackrel{\text{def}}{=} (a, u'_k).$$

  Using this notation, we can turn targets in the DMTS into targets in the 1MTS by addressing either the "state copy" indexed with 0 or the one indexed with 1 and leaving the label unchanged.

- For a target $\vartheta_1 = (a, u'_k)$ in the 1MTS, define

  $$\vartheta_1 \downarrow \stackrel{\text{def}}{=} (\bar{a}, \bar{u}').$$

  This is in some sense the reverse operation to the one introduced before. It turns a target in the 1MTS into the target in the DMTS that it was generated from via f. The index is removed and the label remains unchanged.

- For a target set Θ_1 in the 1MTS, define

  $$\Theta_1 \downarrow \stackrel{\text{def}}{=} \{\vartheta \downarrow \mid \vartheta \in \Theta_1\}.$$

  This generalises the notation above to target sets. If $\Theta_1 \in (\bar{u}_i \longmapsto_1)$, we have $\Theta_1 \downarrow \in (\bar{u} \longmapsto_{\mathrm{D}})$. If $\Theta_1 \in (\bar{u}_i \dashrightarrow_1) \setminus (\bar{u}_i \longmapsto_1)$, $\Theta_1 \downarrow$ is a singleton set and $\Theta_1 \downarrow \in (\bar{u} \dashrightarrow_{\mathrm{D}})$.

- For a target set Θ_{D} in the DMTS, define

$$[\Theta_{\mathrm{D}}]_1 \stackrel{\text{def}}{=} \{\{\vartheta\!\uparrow_0 \mid \vartheta \in \Theta_{\mathrm{D}}\} \cup \{\bar{\vartheta}\!\uparrow_1\} \mid \bar{\vartheta} \in \Theta_{\mathrm{D}}\}.$$

$[\Theta_{\mathrm{D}}]_1$ gives the set of all target sets that are "induced by" Θ_{D}. Thus we have

$$u_i \longmapsto_1 \Theta_1 \;\Leftrightarrow\; \exists \Theta_{\mathrm{D}} \in (u \longmapsto_{\mathrm{D}}) : \Theta_1 \in [\Theta_{\mathrm{D}}]_1$$

and

$$\begin{aligned} u_i \dashrightarrow_1 \Theta_1 \;&\Leftrightarrow\; \exists \Theta_{\mathrm{D}} \in (u \dashrightarrow_{\mathrm{D}}) : \Theta_1 \in [\Theta_{\mathrm{D}}]_1 \vee (u_i \longmapsto_1 \Theta_1) \\ &\Leftrightarrow\; \exists \Theta_{\mathrm{D}} \in (u \dashrightarrow_{\mathrm{D}}) \cup (u \longmapsto_{\mathrm{D}}) : \Theta_1 \in [\Theta_{\mathrm{D}}]_1. \end{aligned}$$

We prove the implication "⇒" of (4.1). Suppose $\mathcal{U}_{\mathrm{D}} \lhd_{\mathrm{D}} \hat{\mathcal{U}}_{\mathrm{D}}$. Choose a disjunctive refinement Q_{D} between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$. Define $Q_1 \subseteq U_1 \times \hat{U}_1$ by

$$u_i Q_1 \hat{u}_j \stackrel{\text{def}}{\Leftrightarrow} u Q_{\mathrm{D}} \hat{u}.$$

We prove that Q_1 is an 1-selecting refinement between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$:

(i) Let $u_i \in U_1^0$. Then $u \in U_{\mathrm{D}}^0$ and due to property (i) of the disjunctive refinement Q_{D}, we can choose $\hat{u} \in \hat{U}_{\mathrm{D}}^0$ such that $u Q_{\mathrm{D}} \hat{u}$. Then $\hat{u}_0 \in \hat{U}_1^0$ and $u_i Q_1 \hat{u}_0$.

(ii) Let $(u_i, \hat{u}_j) \in Q_1$. Then $u Q_{\mathrm{D}} \hat{u}$ and due to property (ii)(a) of the disjunctive refinement Q_{D}, we can choose a function $\iota : (u \dashrightarrow_{\mathrm{D}}) \to (\hat{u} \,\hat{\dashrightarrow}_{\mathrm{D}})$ such that for each $\{\vartheta\} \in (u \dashrightarrow_{\mathrm{D}})$ we have $\vartheta Q_{\mathrm{D}} \hat{\vartheta}$, where $\{\hat{\vartheta}\} \stackrel{\text{def}}{=} \iota(\{\vartheta\})$. Furthermore, due to property (ii)(b) of the disjunctive refinement Q_{D}, we can choose a function $\bar{\iota} : (\hat{u} \,\hat{\longmapsto}_{\mathrm{D}}) \to (u \longmapsto_{\mathrm{D}})$ such that for each $\hat{\Theta}_{\mathrm{D}} \in (\hat{u} \,\hat{\longmapsto}_{\mathrm{D}})$ we have

$$\forall \vartheta \in \bar{\iota}(\hat{\Theta}_{\mathrm{D}}) : \exists \hat{\vartheta} \in \hat{\Theta}_{\mathrm{D}} : \vartheta Q_{\mathrm{D}} \hat{\vartheta},$$

and for each $\hat{\Theta}_{\mathrm{D}} \in (\hat{u} \,\hat{\longmapsto}_{\mathrm{D}})$, we can choose another function $\bar{\kappa}_{\bar{\iota}} : \bar{\iota}(\hat{\Theta}_{\mathrm{D}}) \to \hat{\Theta}_{\mathrm{D}}$ such that for all $\vartheta \in \bar{\iota}(\hat{\Theta}_{\mathrm{D}})$ we have $\vartheta Q_{\mathrm{D}} \bar{\kappa}_{\bar{\iota}}(\vartheta)$.

Let $\gamma \in \mathsf{choice}(u_i \dashrightarrow_1)$. We need to find an appropriate $\hat{\gamma} \in \mathsf{choice}(\hat{u}_j \,\hat{\dashrightarrow}_1)$. Thus let $\hat{\Theta}_1 \in (\hat{u}_j \,\hat{\dashrightarrow}_1)$. In order to define $\hat{\gamma}(\hat{\Theta}_1)$, we distinguish the following cases:

Case 1. $\hat{\Theta}_1 \in (\hat{u}_j \,\hat{\dashrightarrow}_1) \setminus (\hat{u}_j \,\hat{\longmapsto}_1)$. In this case, choose an arbitrary $\hat{\vartheta} \in \hat{\Theta}_1$ and define $\hat{\gamma}(\hat{\Theta}_1) \stackrel{\text{def}}{=} \hat{\vartheta}$.

Case 2. $\hat{\Theta}_1 \in (\hat{u}_j \,\hat{\longmapsto}_1)$. In this case, choose some $\vartheta \in \gamma([\bar{\iota}(\hat{\Theta}_1\!\downarrow)]_1)$ and define $\hat{\gamma}(\hat{\Theta}_1) \stackrel{\text{def}}{=} \bar{\kappa}_{\bar{\iota}}(\vartheta\!\downarrow)\!\uparrow_0$.

We have defined $\hat{\gamma}$ and proceed with checking the properties (ii)(a) and (ii)(b) of a 1-selecting refinement.

(a) Let $\Theta_1 \in (u_i \dashrightarrow_1)$.

Case 1. $\Theta_1 \notin (u_i \longmapsto_1)$. Then there exists ϑ such that $\{\vartheta\} = \Theta_1{\downarrow} \in (u \dashrightarrow_{\mathrm{D}})$, and there exists $\hat{\vartheta}$ such that $\{\hat{\vartheta}\} = \iota(\{\vartheta\}) \in (\hat{u} \mathrel{\hat{\dashrightarrow}}_{\mathrm{D}})$. Define $\hat{\Theta}_1 \stackrel{\text{def}}{=} \{\hat{\vartheta}{\uparrow_0}, \hat{\vartheta}{\uparrow_1}\}$. By definition of ι, we have $\vartheta Q_{\mathrm{D}} \hat{\vartheta}$. By definition of Q_1, this implies $\gamma(\Theta_1)\ Q_1\ \hat{\gamma}(\hat{\Theta}_1)$, as required.

Case 2. $\Theta_1 \in (u_i \longmapsto_1)$. Define $\vartheta \stackrel{\text{def}}{=} \gamma(\Theta_1){\downarrow}$. Then $\vartheta \in \Theta_1{\downarrow} \in (u \longmapsto_{\mathrm{D}})$. Condition (2.1) of DMTSs implies $\{\vartheta\} \in (u \dashrightarrow_{\mathrm{D}})$. There exists $\hat{\vartheta}$ such that $\{\hat{\vartheta}\} = \iota(\{\vartheta\}) \in (\hat{u} \mathrel{\hat{\dashrightarrow}}_{\mathrm{D}})$. Define $\hat{\Theta}_1 \stackrel{\text{def}}{=} \{\hat{\vartheta}{\uparrow_0}, \hat{\vartheta}{\uparrow_1}\}$. By definition of ι, we have $\vartheta Q_{\mathrm{D}} \hat{\vartheta}$. By definition of Q_1, this implies $\gamma(\Theta_1)\ Q_1\ \hat{\gamma}(\hat{\Theta}_1)$, as required.

(b) Let $\hat{\Theta}_1 \in (\hat{u}_j \mathrel{\hat{\longmapsto}}_1)$. By definition of $\hat{\gamma}$, we have $\vartheta \in \gamma([\bar{\iota}(\hat{\Theta}_1{\downarrow})]_1)$ such that $\hat{\gamma}(\hat{\Theta}_1) = \bar{\kappa}_{\bar{\iota}}(\vartheta{\downarrow}){\uparrow_0}$. Choose $\Theta_1 \in [\bar{\iota}(\hat{\Theta}_1{\downarrow})]_1$ such that $\vartheta = \gamma(\Theta_1)$. By definition of $\bar{\kappa}_{\bar{\iota}}$, we have $\vartheta{\downarrow}\ Q_{\mathrm{D}}\ \bar{\kappa}_{\bar{\iota}}(\vartheta{\downarrow})$. Since $\vartheta{\downarrow} = \gamma(\Theta_1){\downarrow}$ and $\bar{\kappa}_{\bar{\iota}}(\vartheta{\downarrow}) = \hat{\gamma}(\hat{\Theta}_1){\downarrow}$, we get $\gamma(\Theta_1){\downarrow}\ Q_{\mathrm{D}}\ \hat{\gamma}(\hat{\Theta}_1){\downarrow}$ and by definition of Q_1, this implies $\gamma(\Theta_1)\ Q_1\ \hat{\gamma}(\hat{\Theta}_1)$, as required.

It remains to show the implication "$\Leftarrow$" of (4.1). Suppose $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$. Choose an 1-selecting refinement Q_1 between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$. Define $Q_{\mathrm{D}} \subseteq U_{\mathrm{D}} \times \hat{U}_{\mathrm{D}}$ by

$$u Q_{\mathrm{D}} \hat{u} \stackrel{\text{def}}{\Leftrightarrow} \exists i, j \in \{0,1\} : u_i Q_1 \hat{u}_j.$$

We prove that Q_{D} is a disjunctive refinement between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$.

(i) Let $u \in U_{\mathrm{D}}^0$. Then $u_0 \in U_1^0$ and due to property (i) of the 1-selecting refinement Q_1, we can choose $\hat{u}_i \in \hat{U}_1^0$ such that $u_0 Q_1 \hat{u}_i$. Then $\hat{u} \in \hat{U}_{\mathrm{D}}^0$ and $u Q_{\mathrm{D}} \hat{u}$.

(ii) Let $(u, \hat{u}) \in Q_{\mathrm{D}}$. Then we can choose $i, j \in \{0,1\}$ such that $u_i Q_1 \hat{u}_j$. Since Q_1 is a 1-selecting refinement, we can choose a function $\sigma : \mathsf{choice}(u_i \dashrightarrow_1) \to \mathsf{choice}(\hat{u}_j \mathrel{\hat{\dashrightarrow}}_1)$ such that for all $\gamma \in \mathsf{choice}(u_i \dashrightarrow_1)$ two functions $\iota : (u_i \dashrightarrow_1) \to (\hat{u}_j \mathrel{\hat{\dashrightarrow}}_1)$ and $\bar{\iota} : (\hat{u}_j \mathrel{\hat{\longmapsto}}_1) \to (u_i \longmapsto_1)$ can be chosen such that

$$\forall \Theta_1 \in (u_i \dashrightarrow_1) : \gamma(\Theta_1)\ Q_1\ \sigma(\gamma)(\iota(\Theta_1)) \tag{4.2}$$

and

$$\forall \hat{\Theta}_1 \in (\hat{u} \mathrel{\hat{\longmapsto}}_1) : \gamma(\bar{\iota}(\hat{\Theta}_1))\ Q_1\ \sigma(\gamma)(\hat{\Theta}_1). \tag{4.3}$$

(a) Let $\{\vartheta\} \in (u \dashrightarrow_{\mathrm{D}})$ and $\gamma \in \mathsf{choice}(u_i \dashrightarrow_1)$. Define $\Theta_1 \stackrel{\text{def}}{=} \{\vartheta{\uparrow_0}, \vartheta{\uparrow_1}\}$ and $\hat{\Theta}_1 \stackrel{\text{def}}{=} \iota(\Theta_1)$. Then $\Theta_1 \in (u_i \dashrightarrow_1)$ and $\hat{\Theta}_1 \in (\hat{u}_j \mathrel{\hat{\dashrightarrow}}_1)$. Due to (4.2), we have

$$\gamma(\Theta_1)\ Q_1\ \sigma(\gamma)(\hat{\Theta}_1). \tag{4.4}$$

Define $\hat{\vartheta} \stackrel{\text{def}}{=} \sigma(\gamma)(\hat{\Theta}_1){\downarrow}$. We have $\{\hat{\vartheta}\} \in (\hat{u} \mathrel{\hat{\dashrightarrow}}_{\mathrm{D}})$:

Case 1. If $\hat{\Theta}_1 \in (\hat{u}_j \mathrel{\hat{\dashrightarrow}}_1) \setminus (\hat{u}_j \mathrel{\hat{\longmapsto}}_1)$, we know $\{\hat{\vartheta}\} \in (\hat{u} \mathrel{\hat{\dashrightarrow}}_{\mathrm{D}})$ (and $\hat{\Theta}_1 = \{\hat{\vartheta}{\uparrow_0}, \hat{\vartheta}{\uparrow_1}\}$).

Case 2. If $\hat{\Theta}_1 \in (\hat{u}_j \mathrel{\hat{\longmapsto}}_1)$, then $\hat{\Theta}_1{\downarrow} \in (\hat{u} \mathrel{\hat{\longmapsto}}_{\mathrm{D}})$ and condition (2.1) of DMTSs implies $\{\hat{\vartheta}\} \in (\hat{u} \mathrel{\hat{\dashrightarrow}}_{\mathrm{D}})$.

By definition of Q_{D}, (4.4) implies $\gamma(\Theta_1)\!\downarrow\; Q_{\mathrm{D}}\; \sigma(\gamma)(\hat{\Theta}_1)\!\downarrow$, thus $\vartheta Q_{\mathrm{D}}\hat{\vartheta}$, as required.

(b) Let $\hat{\Theta}_{\mathrm{D}} \in (\hat{u} \mathrel{\hat{\longmapsto}}_{\mathrm{D}})$. Choose arbitrary $\hat{\Theta}_1 \in [\hat{\Theta}_{\mathrm{D}}]_1$. Then $\hat{\Theta}_1 \in (\hat{u}_j \mathrel{\hat{\longmapsto}}_1)$. Let $\Theta_{\mathrm{D}} \stackrel{\text{def}}{=} \bar{\iota}(\hat{\Theta}_1)\downarrow$. Then $\Theta_{\mathrm{D}} \in (u \longmapsto_{\mathrm{D}})$. Let $\vartheta \in \Theta_{\mathrm{D}}$ and consider $\gamma \in \mathsf{choice}(u_i \dashrightarrow_1)$ that satisfies $\gamma(\bar{\iota}(\hat{\Theta}_1))\downarrow = \vartheta$. Define $\hat{\vartheta} \stackrel{\text{def}}{=} \sigma(\gamma)(\hat{\Theta}_1)\downarrow$. Then $\hat{\vartheta} \in \hat{\Theta}_{\mathrm{D}}$. By definition of Q_{D}, (4.3) implies $\vartheta Q_{\mathrm{D}}\hat{\vartheta}$, as required. □

Lemma 4.9. *For all fully determined* $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}^{\mathrm{det}}$, *we have*

$$\mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}})) \approx_1 f(\mathcal{U}_{\mathrm{D}}).$$

Proof. Let $\mathcal{U}_{\mathrm{D}} \in \mathbb{DMTS}^{\mathrm{det}}$, $\mathcal{U}_1^1 = (U_1^1, L, \longmapsto_1^1, \dashrightarrow_1^1, U_1^{01}) \stackrel{\text{def}}{=} \mathsf{1MTS}(\mathsf{TS}_{\mathrm{D}}(\mathcal{U}_{\mathrm{D}}))$ and $\mathcal{U}_1^2 = (U_1^2, L, \longmapsto_1^2, \dashrightarrow_1^2, U_1^{02}) \stackrel{\text{def}}{=} f(\mathcal{U}_{\mathrm{D}})$.

First, we prove $\mathcal{U}_1^1 \lhd_1 \mathcal{U}_1^2$. We claim that $Q \subseteq U_1^1 \times U_1^2$, defined by

$$uQ\hat{u}_i \stackrel{\text{def}}{\Leftrightarrow} u = \hat{u},$$

is an 1-selecting refinement between $\mathcal{U}_1^1$ and $\mathcal{U}_1^2$.

(i) Property (i) of an 1-selecting refinement is obvious.

(ii) Let $(u, \hat{u}_i) \in Q$. Then $u = \hat{u}$. Let $\gamma \in \mathsf{choice}(u \dashrightarrow_1^1)$. Since $\mathcal{U}_1^1$ is fully determined, there is only one possible choice for γ. Define $\hat{\gamma} \in \mathsf{choice}(u_i \dashrightarrow_1^2)$ arbitrarily.

  (a) Let $\Theta_1 \in (u \dashrightarrow_1^1)$. Define $\hat{\Theta}_1 \stackrel{\text{def}}{=} \{(a, u_i') \mid \{(a, u')\} = \Theta_1 \wedge i \in \{0,1\}\}$. Then we have $\gamma(\Theta_1)\; Q\; \hat{\gamma}(\hat{\Theta}_1)$, as required.

  (b) Let $\hat{\Theta}_1 \in (u_i \longmapsto_1^2)$. Define $\Theta_1 \stackrel{\text{def}}{=} \{(a, u') \mid \exists i \in \{0,1\} : (a, u_i') \in \hat{\Theta}_1\}$. Then we have $\gamma(\Theta_1)\; Q\; \hat{\gamma}(\hat{\Theta}_1)$, as required.

It remains to prove $\mathcal{U}_1^2 \lhd_1 \mathcal{U}_1^1$. We claim that $Q \subseteq U_1^2 \times U_1^1$, defined by

$$u_iQ\hat{u} \stackrel{\text{def}}{\Leftrightarrow} u = \hat{u},$$

is an 1-selecting refinement between $\mathcal{U}_1^2$ and $\mathcal{U}_1^1$.

(i) Again, property (i) is obvious.

(ii) Let $(u_i, \hat{u}) \in Q$. Then $u = \hat{u}$. Let $\gamma \in \mathsf{choice}(u_i \dashrightarrow_1^2)$. Define $\hat{\gamma} \in \mathsf{choice}(u \dashrightarrow_1^1)$ arbitrarily (however, since $\mathcal{U}_1^1$ is fully determined, there is only one possible choice). As in the first part of this proof, the properties (ii)(a) and (ii)(b) of an 1-selecting refinement can now easily be checked. □

Definition 4.10. *Define* $f^* : \underline{\mathbb{DMTS}} \to \underline{\mathbb{1MTS}}$ *as follows: For* $\mathcal{K}_{\mathrm{D}} \in \underline{\mathbb{DMTS}}$, *choose* $\mathcal{U}_{\mathrm{D}} \in \mathcal{K}_{\mathrm{D}}$ *and define* $f^*(\mathcal{K}_{\mathrm{D}}) \stackrel{\text{def}}{=} [f(\mathcal{U}_{\mathrm{D}})]_{\approx_1}$.

Note that $f^*(\mathcal{K}_\mathrm{D})$ does not depend on the choice of $\mathcal{U}_\mathrm{D} \in \mathcal{K}_\mathrm{D}$: Two DMTSs $\mathcal{U}_\mathrm{D}^1, \mathcal{U}_\mathrm{D}^2 \in \mathcal{K}_\mathrm{D}$ satisfy $\mathcal{U}_\mathrm{D}^1 \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}^2$ and $\mathcal{U}_\mathrm{D}^2 \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}^1$. Then Lemma 4.8 implies $f(\mathcal{U}_\mathrm{D}^1) \lhd_1 f(\mathcal{U}_\mathrm{D}^2)$ and $f(\mathcal{U}_\mathrm{D}^2) \lhd_1 f(\mathcal{U}_\mathrm{D}^1)$, consequently $[f(\mathcal{U}_\mathrm{D}^1)]_{\approx_1} = [f(\mathcal{U}_\mathrm{D}^2)]_{\approx_1}$.

Theorem 4.11. *f^* is a D1-homomorphism.*

Proof. We need to prove the following two properties:

(i) $\forall \mathcal{K}_\mathrm{D}, \hat{\mathcal{K}}_\mathrm{D} \in \underline{\mathbb{DMTS}} : \mathcal{K}_\mathrm{D} \unlhd_\mathrm{D} \hat{\mathcal{K}}_\mathrm{D} \Leftrightarrow f^*(\mathcal{K}_\mathrm{D}) \unlhd_1 f^*(\hat{\mathcal{K}}_\mathrm{D})$

(ii) $\forall \mathcal{K}_\mathrm{D} \in \underline{\mathbb{DMTS}}^{\mathrm{det}} : f^*(\mathcal{K}_\mathrm{D}) = \underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\mathrm{D}(\mathcal{K}_\mathrm{D}))$

The first property follows directly from Lemma 4.8. For the proof of the second property, let $\mathcal{K}_\mathrm{D} \in \underline{\mathbb{DMTS}}^{\mathrm{det}}$. Choose $\mathcal{U}_\mathrm{D} \in \mathcal{K}_\mathrm{D} \cap \mathbb{DMTS}^{\mathrm{det}}$. With Lemma 4.9, we get

$$\begin{aligned} f^*(\mathcal{K}_\mathrm{D}) &= [f(\mathcal{U}_\mathrm{D})]_{\approx_1} \\ &= [\mathsf{1MTS}(\mathsf{TS}_\mathrm{D}(\mathcal{U}_\mathrm{D}))]_{\approx_1} \\ &= \underline{\mathsf{1MTS}}([\mathsf{TS}_\mathrm{D}(\mathcal{U}_\mathrm{D})]_{\sim}) \\ &= \underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\mathrm{D}([\mathcal{U}_\mathrm{D}]_{\approx_\mathrm{D}})) \\ &= \underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\mathrm{D}(\mathcal{K}_\mathrm{D})), \end{aligned}$$

as required. □

We have seen that f^* is a $\mathsf{D1}$-embedding, i.e. the function embeds DMTSs into $\mathsf{1MTS}$s such that the identification of fully determined systems and refinement orders are respected. Note that as a corollary, we get that f is an embedding with respect to the alternative implementation-based refinement, in which one system refines another, if and only if the set of implementations of the first is a subset of the set of implementations of the second. This is expressed in the following corollary that concludes this section:

Corollary 4.12. *For all $\mathcal{T} \in \mathbb{TS}$ and $\mathcal{U}_\mathrm{D} \in \mathbb{DMTS}$, we have*

$$\mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D} \;\Leftrightarrow\; \mathcal{T} \prec_1 f(\mathcal{U}_\mathrm{D}).$$

Proof. We start with the implication "⇒": Let $\mathcal{T} \in \mathbb{TS}$ and $\mathcal{U}_\mathrm{D} \in \mathbb{DMTS}$ such that $\mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D}$. Then, by Proposition 3.8(i), $\mathsf{DMTS}(\mathcal{T}) \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}$, which implies by Lemma 4.8 $f(\mathsf{DMTS}(\mathcal{T})) \lhd_1 f(\mathcal{U}_\mathrm{D})$. $f(\mathsf{DMTS}(\mathcal{T}))$ is not fully determined, since it has more than one root state. However, it is 1R-equivalent to a fully determined $\mathsf{1MTS}$, say $\mathcal{U}_1$, because f^* maps $[\mathsf{DMTS}(\mathcal{T})]_{\approx_\mathrm{D}}$ to an element of $\underline{\mathbb{1MTS}}^{\mathrm{det}}$. Proposition 3.8(ii) yields

$$\mathsf{TS}_1(\mathcal{U}_1) \prec_1 f(\mathcal{U}_\mathrm{D}). \tag{4.5}$$

By Lemma 4.9, we have $\mathsf{1MTS}(\mathcal{T}) \approx_1 f(\mathsf{DMTS}(\mathcal{T}))$ and since $f(\mathsf{DMTS}(\mathcal{T}))$ is 1R-equivalent to the fully determined $\mathsf{1MTS}$ $\mathcal{U}_1$, Proposition 3.14 implies $\mathcal{T} \sim$

$\mathsf{TS}_1(\mathcal{U}_1)$. This, together with (4.5), implies $\mathcal{T} \prec_1 f(\mathcal{U}_\mathrm{D})$ by Proposition 2.13, as required.

The implication "$\Leftarrow$" is proven similarly: Let $\mathcal{T} \in \mathbb{TS}$ and $\mathcal{U}_\mathrm{D} \in \mathbb{DMTS}$ such that $\mathcal{T} \prec_1 f(\mathcal{U}_\mathrm{D})$. Then by Proposition 3.8(ii), we have $\mathsf{1MTS}(\mathcal{T}) \lhd_1 f(\mathcal{U}_\mathrm{D})$. By Lemma 4.9, we know $\mathsf{1MTS}(\mathcal{T}) \approx_1 f(\mathsf{DMTS}(\mathcal{T}))$ and consequently $f(\mathsf{DMTS}(\mathcal{T}))$ $\lhd_1 f(\mathcal{U}_\mathrm{D})$. Now Lemma 4.8 yields $\mathsf{DMTS}(\mathcal{T}) \lhd_\mathrm{D} \mathcal{U}_\mathrm{D}$ and by Proposition 3.8(i), this is equivalent to $\mathcal{T} \prec_\mathrm{D} \mathcal{U}_\mathrm{D}$, as required. □

4.3 An 1D-Homomorphism

We define a function g from $\mathbb{1MTS}$ to $\mathbb{DMTS}$ and show that it induces an 1D-homomorphism g^* from $\underline{\mathbb{1MTS}}$ to $\underline{\mathbb{DMTS}}$. First, we concentrate on g and show the two properties

$$\forall \mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MTS} : \mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1 \Rightarrow g(\mathcal{U}_1) \lhd_\mathrm{D} g(\hat{\mathcal{U}}_1)$$

(Lemma 4.14) and

$$\forall \mathcal{U}_1 \in \mathbb{1MTS}^{\mathrm{det}} : \mathsf{DMTS}(\mathsf{TS}_1(\mathcal{U}_1)) \approx_1 g(\mathcal{U}_1)$$

(Lemma 4.15). Then we derive the function $g^* : \underline{\mathbb{1MTS}} \to \underline{\mathbb{DMTS}}$ from g by switching over to 1R-equivalence classes and prove that g^* is an 1D-homomorphism, which essentially requires showing that the two properties shown for g persist the transition to equivalence classes. We proceed with showing that g^* is not an 1D-embedding. Nevertheless, g embeds 1MTSs into DMTSs with respect to the alternative implementation-based refinement that was introduced in Section 3.1, as will be shown at the end of this section.

The idea behind function g is straightforward: Every state of the 1MTS, say u, is turned into several states in the DMTS, one for each possible choice function $\gamma \in \mathsf{choice}(u \dashrightarrow_1)$. Thus the states of the DMTS are pairs of a 1MTS-state and a choice function. Then (u, γ) plays the part of u in the case that choice function γ would be considered in the 1MTS. Formally, the function is defined as follows:

Definition 4.13. *Define* $g : \mathbb{1MTS} \to \mathbb{DMTS};\ (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \mapsto (U_\mathrm{D}, L, \longmapsto_\mathrm{D}, \dashrightarrow_\mathrm{D}, U_\mathrm{D}^0)$*, with*

$$\begin{aligned}
U_\mathrm{D} &\stackrel{\mathrm{def}}{=} \{(u, \gamma) \mid u \in U_1 \wedge \gamma \in \mathsf{choice}(u \dashrightarrow_1)\}, \\
(u, \gamma) \longmapsto_\mathrm{D} \Theta &\stackrel{\mathrm{def}}{\Leftrightarrow} \exists (a, u') \in \gamma(u \longmapsto_1) : \\
&\qquad \Theta = \{(a, (u', \gamma')) \mid \gamma' \in \mathsf{choice}(u' \dashrightarrow_1)\}, \\
(u, \gamma) \dashrightarrow_\mathrm{D} \{\vartheta\} &\stackrel{\mathrm{def}}{\Leftrightarrow} \exists (a, u') \in \gamma(u \dashrightarrow_1) : \\
&\qquad \vartheta \in \{(a, (u', \gamma')) \mid \gamma' \in \mathsf{choice}(u' \dashrightarrow_1)\}, \\
U_\mathrm{D}^0 &\stackrel{\mathrm{def}}{=} \{(u, \gamma) \mid u \in U_1^0 \wedge \gamma \in \mathsf{choice}(u \dashrightarrow_1)\}.
\end{aligned}$$

Lemma 4.14. *For all* $\mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MTS}$ *we have*

$$\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1 \Rightarrow g(\mathcal{U}_1) \lhd_{\mathrm{D}} g(\hat{\mathcal{U}}_1).$$

Proof. Let $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \vdash\!\dashrightarrow_1, U_1^0), \hat{\mathcal{U}}_1 = (\hat{U}_1, L, \stackrel{\wedge}{\longmapsto}_1, \stackrel{\wedge}{\vdash\!\dashrightarrow}_1, \hat{U}_1^0) \in \mathbb{1MTS}$. Define $\mathcal{U}_{\mathrm{D}} = (U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \vdash\!\dashrightarrow_{\mathrm{D}}, U_{\mathrm{D}}^0) \stackrel{\text{def}}{=} g(\mathcal{U}_1)$ and $\hat{\mathcal{U}}_{\mathrm{D}} = (\hat{U}_{\mathrm{D}}, L, \stackrel{\wedge}{\longmapsto}_{\mathrm{D}}, \stackrel{\wedge}{\vdash\!\dashrightarrow}_{\mathrm{D}}, \hat{U}_{\mathrm{D}}^0) \stackrel{\text{def}}{=} g(\hat{\mathcal{U}}_1)$. Suppose $\mathcal{U}_1 \lhd_1 \hat{\mathcal{U}}_1$. Choose an 1-selecting refinement Q_1 between $\mathcal{U}_1$ and $\hat{\mathcal{U}}_1$. For all $(u, \hat{u}) \in Q_1$, we can choose a function $\sigma_{(u,\hat{u})} : \mathsf{choice}(u \vdash\!\dashrightarrow_1) \to \mathsf{choice}(\hat{u} \stackrel{\wedge}{\vdash\!\dashrightarrow}_1)$ such that for all $\gamma \in \mathsf{choice}(u \vdash\!\dashrightarrow_1)$ two functions $\iota : (u \vdash\!\dashrightarrow_1) \to (\hat{u} \stackrel{\wedge}{\vdash\!\dashrightarrow}_1)$ and $\bar{\iota} : (\hat{u} \stackrel{\wedge}{\longmapsto}_1) \to (u \longmapsto_1)$ can be chosen such that

$$\forall \Theta_1 \in (u \vdash\!\dashrightarrow_1) : \gamma(\Theta_1)\ Q_1\ \sigma_{(u,\hat{u})}(\gamma)(\iota(\Theta_1)) \tag{4.6}$$

and

$$\forall \hat{\Theta}_1 \in (\hat{u} \stackrel{\wedge}{\longmapsto}_1) : \gamma(\bar{\iota}(\hat{\Theta}_1))\ Q_1\ \sigma_{(u,\hat{u})}(\gamma)(\hat{\Theta}_1). \tag{4.7}$$

We prove that $Q_{\mathrm{D}} \subseteq U_{\mathrm{D}} \times \hat{U}_{\mathrm{D}}$, defined by

$$(u, \gamma)\ Q_{\mathrm{D}}\ (\hat{u}, \hat{\gamma}) \ \stackrel{\text{def}}{\Leftrightarrow}\ uQ_1\hat{u} \ \wedge\ \hat{\gamma} = \sigma_{(u,\hat{u})}(\gamma),$$

is a disjunctive refinement between $\mathcal{U}_{\mathrm{D}}$ and $\hat{\mathcal{U}}_{\mathrm{D}}$.

(i) Let $(u, \gamma) \in U_{\mathrm{D}}^0$. Then $u \in U_1^0$. Since Q_1 is an 1-selecting refinement, we can choose $\hat{u} \in \hat{U}_1^0$ such that $uQ_1\hat{u}$. Then $(u, \gamma)\ Q_{\mathrm{D}}\ (\hat{u}, \sigma_{(u,\hat{u})}(\gamma))$.

(ii) Let $((u, \gamma), (\hat{u}, \hat{\gamma})) \in Q_{\mathrm{D}}$. Then $uQ_1\hat{u}$ and $\hat{\gamma} = \sigma_{(u,\hat{u})}(\gamma)$.

(a) Let $\{(a, (u', \gamma'))\} \in ((u, \gamma) \vdash\!\dashrightarrow_{\mathrm{D}})$. Then $\{(a, u')\} \in (u \vdash\!\dashrightarrow_1)$ and (4.6) yields

$$\gamma(\{(a, u')\})\ Q_1\ \hat{\gamma}(\iota(\{(a, u')\})).$$

Define $(\hat{a}, \hat{u'}) \stackrel{\text{def}}{=} \hat{\gamma}(\iota(\{(a, u')\}))$. Since obviously $\gamma(\{(a, u')\}) = (a, u')$, we get $(a, u')\ Q_1\ (\hat{a}, \hat{u'})$. Thus $a = \hat{a}$ and $u'Q_1\hat{u'}$. By definition of Q_{D}, we get $(u', \gamma')\ Q_{\mathrm{D}}\ (\hat{u'}, \sigma_{(u',\hat{u'})}(\gamma'))$ and since $a = \hat{a}$,

$$(a, (u', \gamma'))\ Q_{\mathrm{D}}\ (\hat{a}, (\hat{u'}, \sigma_{(u',\hat{u'})}(\gamma'))). \tag{4.8}$$

We have $(\hat{a}, \hat{u'}) \in \hat{\gamma}(\hat{u} \stackrel{\wedge}{\vdash\!\dashrightarrow}_1)$ and $\sigma_{(u',\hat{u'})}(\gamma') \in \mathsf{choice}(\hat{u'} \stackrel{\wedge}{\vdash\!\dashrightarrow}_1)$, which implies

$$\{(\hat{a}, (\hat{u'}, \sigma_{(u',\hat{u'})}(\gamma')))\} \in ((\hat{u}, \hat{\gamma}) \stackrel{\wedge}{\vdash\!\dashrightarrow}_{\mathrm{D}}).$$

Consequently, (4.8) completes the proof of property (ii)(a).

(b) Let $\hat{\Theta}_{\mathrm{D}} \in ((\hat{u}, \hat{\gamma}) \stackrel{\wedge}{\longmapsto}_{\mathrm{D}})$. Then there exists $(\hat{a}, \hat{u'}) \in \hat{\gamma}(\hat{u} \stackrel{\wedge}{\longmapsto}_1)$ such that

$$\hat{\Theta}_{\mathrm{D}} = \{(\hat{a}, (\hat{u'}, \hat{\gamma'})) \mid \hat{\gamma'} \in \mathsf{choice}(\hat{u'} \stackrel{\wedge}{\vdash\!\dashrightarrow}_1)\}.$$

We can choose $\hat{\Theta}_1 \in (\hat{u} \stackrel{\wedge}{\longmapsto}_1)$ such that $(\hat{a}, \hat{u'}) = \hat{\gamma}(\hat{\Theta}_1)$. Define $\Theta_1 \stackrel{\text{def}}{=} \bar{\iota}(\hat{\Theta}_1)$ and

$$\Theta_{\mathrm{D}} \stackrel{\text{def}}{=} \{(a, (u', \gamma')) \mid (a, u') = \gamma(\Theta_1) \wedge \gamma' \in \mathsf{choice}(u' \vdash\!\dashrightarrow_1)\}.$$

Then $\Theta_D \in ((u,\gamma) \longmapsto_D)$. Let $(a,(u',\gamma')) \in \Theta_D$. Then $(a,u') = \gamma(\Theta_1)$ and $\gamma' \in \mathsf{choice}(u' \dashrightarrow_1)$. (4.7) implies $\gamma(\Theta_1)\ Q_1\ \hat{\gamma}(\hat{\Theta}_1)$, thus $(a,u')\ Q_1\ (\hat{a},\hat{u'})$. Consequently $a = \hat{a}$ and $u' Q_1 \hat{u'}$. By definition of Q_D, we get $(u',\gamma')\ Q_D\ (\hat{u'},\sigma_{(u',\hat{u'})}(\gamma'))$ and since $a = \hat{a}$,

$$(a,(u',\gamma'))\ Q_D\ (\hat{a},(\hat{u'},\sigma_{(u',\hat{u'})}(\gamma'))).$$

Since $(\hat{a},(\hat{u'},\sigma_{(u',\hat{u'})}(\gamma'))) \in \hat{\Theta}_D$, this completes the proof of property (ii)(b). $\square$

Lemma 4.15. *For all fully determined* $\mathcal{U}_1 \in \mathbb{1MTS}^{\mathrm{det}}$*, we have*

$$\mathsf{DMTS}(\mathsf{TS}_1(\mathcal{U}_1)) \approx_D g(\mathcal{U}_1).$$

Proof. Let $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MTS}^{\mathrm{det}}$. Then for any $u \in U_1$, there is only one possible $\gamma \in \mathsf{choice}(u \dashrightarrow_1)$, because there are no hypertransitions in $\mathcal{U}_1$. Consequently, g only renames the states of $\mathcal{U}_1$, and the property to be shown is obvious. $\square$

Definition 4.16. *Define* $g^* : \underline{\mathbb{1MTS}} \to \underline{\mathbb{DMTS}}$ *as follows: For* $\mathcal{K}_1 \in \underline{\mathbb{1MTS}}$*, choose* $\mathcal{U}_1 \in \mathcal{K}_1$ *and define* $g^*(\mathcal{K}_1) \stackrel{\mathrm{def}}{=} [g(\mathcal{U}_1)]_{\approx_D}$*.*

Note that $g^*(\mathcal{K}_1)$ does not depend on the choice of $\mathcal{U}_1 \in \mathcal{K}_1$: Two 1MTSs $\mathcal{U}_1^1, \mathcal{U}_1^2 \in \mathcal{K}_1$ satisfy $\mathcal{U}_1^1 \lhd_1 \mathcal{U}_1^2$ and $\mathcal{U}_1^2 \lhd_1 \mathcal{U}_1^1$. Then Lemma 4.14 implies $g(\mathcal{U}_1^1) \lhd_D g(\mathcal{U}_1^2)$ and $g(\mathcal{U}_1^2) \lhd_D g(\mathcal{U}_1^1)$, consequently $[g(\mathcal{U}_1^1)]_{\approx_D} = [g(\mathcal{U}_1^2)]_{\approx_D}$.

Theorem 4.17. g^* *is an 1D-homomorphism.*

Proof. We need to prove the following two properties:

(i) $\forall \mathcal{K}_1, \hat{\mathcal{K}}_1 \in \underline{\mathbb{1MTS}} : \mathcal{K}_1 \trianglelefteq_1 \hat{\mathcal{K}}_1 \Rightarrow g^*(\mathcal{K}_1) \trianglelefteq_D g^*(\hat{\mathcal{K}}_1)$

(ii) $\forall \mathcal{K}_1 \in \underline{\mathbb{1MTS}}^{\mathrm{det}} : g^*(\mathcal{K}_1) = \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1))$

The first property follows immediately from Lemma 4.14. The second property follows from Lemma 4.15, similarly to the way that was presented in the proof of Theorem 4.11. $\square$

The following proposition states that g^* is not an 1D-embedding. In Section 4.4, we will even show that there is no 1D-embedding at all, which of course implies the following proposition. It is nevertheless interesting to make the following consideration, because, using this example, one can directly see where a specific 1D-homomorphism fails.

Proposition 4.18. g^* *is not an 1D-embedding.*

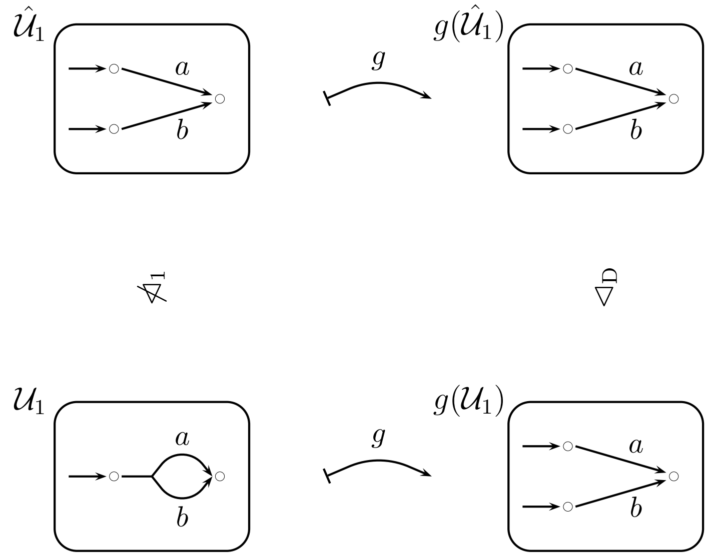

Figure 4.1: A counter-example to show that g^* is not an 1D-embedding

Proof. It is enough to find $\mathcal{U}_1, \hat{\mathcal{U}}_1 \in \mathbb{1MTS}$ such that $g(\mathcal{U}_1) \trianglelefteq_D g(\hat{\mathcal{U}}_1)$, but $\mathcal{U}_1 \not\trianglelefteq_1 \hat{\mathcal{U}}_1$. Such a counter-example is illustrated in Figure 4.1.

First consider $\mathcal{U}_1$. The root state of $\mathcal{U}_1$ is duplicated by g, attaching to one copy the choice function that chooses the a branch of the hypertransition and to the other the choice function that chooses the b branch of the hypertransition. The other state of $\mathcal{U}_1$ is simply renamed, as the empty choice function is attached to it. We get the illustrated DMTS $g(\mathcal{U}_1)$.

Now consider $\hat{\mathcal{U}}_1$. The fact that each state of $\hat{\mathcal{U}}_1$ allows only one possible choice function is the reason, why g only renames the states, attaching the unique choice functions to them.

We have $\mathcal{U}_1 \not\trianglelefteq_1 \hat{\mathcal{U}}_1$, because for the two possible choice functions in $\mathcal{U}_1$, we only have one possible choice function $\hat{\mathcal{U}}_1$. Thus either for the choice function that chooses the a branch in $\mathcal{U}_1$, or for the choice function that chooses the b branch in $\mathcal{U}_1$, a corresponding transition in $\hat{\mathcal{U}}_1$ can be found; but not for both. Furthermore, we have $g(\mathcal{U}_1) \trianglelefteq_D g(\hat{\mathcal{U}}_1)$, since obviously $g(\mathcal{U}_1) \approx_D g(\hat{\mathcal{U}}_1)$. □

We have seen that g^* is not an 1D-embedding, i.e., the function does not embed 1MTSs into DMTSs such that the identification of fully determined systems and refinement orders are respected. However, g is an embedding with respect to the alternative implementation-based refinement, in which one system refines another, if and only if the set of implementations of the first is a subset of the set of implementations of the second. This is expressed in the following proposition:

Proposition 4.19. *For all $\mathcal{T} \in \mathbb{TS}$ and $\mathcal{U}_1 \in \mathbb{1MTS}$, we have*

$$\mathcal{T} \prec_1 \mathcal{U}_1 \Leftrightarrow \mathcal{T} \prec_D g(\mathcal{U}_1).$$

Proof. We start with the implication "⇒": Let $\mathcal{T} \in \mathbb{TS}$ and $\mathcal{U}_1 \in \mathbb{1MTS}$ such that $\mathcal{T} \prec_1 \mathcal{U}_1$. Then, by Proposition 3.8(ii), $\mathsf{1MTS}(\mathcal{T}) \lhd_1 \mathcal{U}_1$, which implies by Lemma 4.14 $g(\mathsf{1MTS}(\mathcal{T})) \lhd_{\mathrm{D}} g(\mathcal{U}_1)$. $g(\mathsf{1MTS}(\mathcal{T}))$ is fully determined, because $\mathsf{1MTS}(\mathcal{T})$ is fully determined and g only renames the states. Then Proposition 3.8(i) yields

$$\mathsf{TS}_{\mathrm{D}}(g(\mathsf{1MTS}(\mathcal{T}))) \prec_{\mathrm{D}} g(\mathcal{U}_1). \tag{4.9}$$

By Lemma 4.15, we have $\mathsf{DMTS}(\mathcal{T}) \approx_{\mathrm{D}} g(\mathsf{1MTS}(\mathcal{T}))$ and since $g(\mathsf{1MTS}(\mathcal{T}))$ is fully determined, Proposition 3.14 implies $\mathcal{T} \sim \mathsf{TS}_{\mathrm{D}}(g(\mathsf{1MTS}(\mathcal{T})))$. This, together with (4.9), implies $\mathcal{T} \prec_{\mathrm{D}} g(\mathcal{U}_1)$ by Proposition 2.13, as required.

It remains to prove the implication "⇐". Let $\mathcal{T} = (S, L, \longrightarrow, s^0) \in \mathbb{TS}$ and $\mathcal{U}_1 = (U_1, L, \longmapsto_1, \dashrightarrow_1, U_1^0) \in \mathbb{1MTS}$ such that $\mathcal{T} \prec_{\mathrm{D}} \mathcal{U}_{\mathrm{D}}$. Define $\mathcal{U}_{\mathrm{D}} = (U_{\mathrm{D}}, L, \longmapsto_{\mathrm{D}}, \dashrightarrow_{\mathrm{D}}, U_{\mathrm{D}}^0) \stackrel{\text{def}}{=} g(\mathcal{U}_1)$. There is a disjunctive simulation $R_{\mathrm{D}} \subseteq S \times U_{\mathrm{D}}$ between $\mathcal{T}$ and $\mathcal{U}_{\mathrm{D}}$. We prove that R_1, defined by

$$sR_1u \stackrel{\text{def}}{\Leftrightarrow} \exists \gamma \in \mathsf{choice}(u \dashrightarrow_1) : sR_{\mathrm{D}}(u, \gamma),$$

is an 1-selecting refinement between $\mathcal{T}$ and $\mathcal{U}_1$.

(i) Since R_{D} is a disjunctive refinement, we can choose $(u, \gamma) \in U_{\mathrm{D}}^0$ such that $s^0 R_{\mathrm{D}}(u, \gamma)$. Hence $u \in U_1^0$ and $s^0 R_1 u$, as required.

(ii) Let $(s, u) \in R_1$. By definition of R_1, we can choose $\gamma \in \mathsf{choice}(u \dashrightarrow_1)$ such that $sR_{\mathrm{D}}(u, \gamma)$.

(a) Let $(a, s') \in (s \longrightarrow)$. Since R_{D} is a disjunctive simulation, we can choose $\{(\hat{a}, (u', \gamma'))\} \in ((u, \gamma) \dashrightarrow_{\mathrm{D}})$ such that $a = \hat{a}$ and $s' R_{\mathrm{D}}(u', \gamma')$. The latter implies $s' R_1 u'$ by definition of R_1. Furthermore, by definition of $\dashrightarrow_{\mathrm{D}}$, $(u, \gamma) \dashrightarrow_{\mathrm{D}} \{(\hat{a}, (u', \gamma'))\}$ implies $(\hat{a}, u') \in \gamma(u \dashrightarrow_1)$. Since $a = \hat{a}$ and $s' R_1 u'$, we get $(a, s') R_1 (\hat{a}, u')$, as required.

(b) Let $(\hat{a}, u') \in \gamma(u \longmapsto_1)$. Then, by definition of $\longmapsto_{\mathrm{D}}$,

$$(u, \gamma) \longmapsto_{\mathrm{D}} \{(\hat{a}, (u', \gamma')) \mid \gamma' \in \mathsf{choice}(u' \dashrightarrow_1)\}.$$

Since R_{D} is a disjunctive simulation, we can choose $(a, s') \in (s \longrightarrow)$ and $\gamma' \in \mathsf{choice}(u' \dashrightarrow_1)$ such that $a = \hat{a}$ and $s' R_{\mathrm{D}}(u', \gamma')$. The latter implies $s' R_1 u'$ by definition of R_1. Consequently $(a, s') R_1 (\hat{a}, u')$, as required. □

4.4 No 1D-Embedding

In Section 4.2, we have seen that there is a D1-embedding, and in Section 4.3, a straightforward 1D-homomorphism was presented, that is not an 1D-embedding. Now, we will see that there exists no 1D-embedding at all, and consequently no 1D-isomorphism either. Thus the two formalisms are not equally expressive; 1MTSs have strictly more expressive power than DMTSs.

We consider the 1MTS $\hat{\mathcal{U}}_1$ that was examined in Section 3.7, and prove that its Hasse structure (Figure 3.4) cannot be embedded in the DMTS-formalism. With $\hat{\mathcal{U}}_1$ being the 1MTS introduced at the beginning of Section 3.7, we show that there is no possible $g^*([\hat{\mathcal{U}}_1]_{\approx_1})$ for an 1D-embedding g^*. As a consequence there is no 1D-embedding.

Theorem 4.20. *There is no 1D-embedding.*

Proof. Assume there is an 1D-embedding $g^* : \underline{\mathbb{1MT\$}} \to \underline{\mathbb{DMT\$}}$. We define $\bar{\mathcal{K}}_1 \stackrel{\text{def}}{=} [\hat{\mathcal{U}}_1]_{\approx_1}$, where $\hat{\mathcal{U}}_1$ is the 1MTS defined at the beginning of Section 3.7, and $\bar{\mathcal{K}}_\text{D} \stackrel{\text{def}}{=} g^*(\bar{\mathcal{K}}_1)$. g^* satisfies the following properties:

(i) For any fully determined (1-selecting) refinement $\mathcal{K}_1$ of $\bar{\mathcal{K}}_1$, we have

$$\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1)) = g^*(\mathcal{K}_1) \trianglelefteq_\text{D} g^*(\bar{\mathcal{K}}_1) = \bar{\mathcal{K}}_\text{D}.$$

Thus $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1))$ is a fully determined (disjunctive) refinement of $\bar{\mathcal{K}}_\text{D}$. We have

$$\{\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1)) \mid \mathcal{K}_1 \in \underline{\mathbb{DMT\$}}^\text{det} \wedge \mathcal{K}_1 \trianglelefteq_1 \bar{\mathcal{K}}_1\} \subseteq \{\mathcal{K}_\text{D} \in \underline{\mathbb{DMT\$}}^\text{det} \mid \mathcal{K}_\text{D} \trianglelefteq_\text{D} \bar{\mathcal{K}}_\text{D}\}.$$

(ii) For any fully determined (disjunctive) refinement $\mathcal{K}_\text{D}$ of $\bar{\mathcal{K}}_\text{D}$, we have

$$\mathcal{K}_\text{D} = \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D})))).$$

Then $\mathcal{K}_\text{D} = g^*(\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D})))$ and consequently $g^*(\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D})))$ $\trianglelefteq_\text{D} g^*(\bar{\mathcal{K}}_1)$, which implies $\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D})) \trianglelefteq_1 \bar{\mathcal{K}}_1$. Thus $\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D}))$ is a fully determined (1-selecting) refinement of $\bar{\mathcal{K}}_1$. We have

$$\{\mathcal{K}_1 \in \underline{\mathbb{DMT\$}}^\text{det} \mid \mathcal{K}_1 \trianglelefteq_1 \bar{\mathcal{K}}_1\} \supseteq \{\underline{\mathsf{1MTS}}(\underline{\mathsf{TS}}_\text{D}(\mathcal{K}_\text{D})) \mid \mathcal{K}_\text{D} \in \underline{\mathbb{DMT\$}}^\text{det} \wedge \mathcal{K}_\text{D} \trianglelefteq_\text{D} \bar{\mathcal{K}}_\text{D}\},$$

which is equivalent to

$$\{\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1)) \mid \mathcal{K}_1 \in \underline{\mathbb{DMT\$}}^\text{det} \wedge \mathcal{K}_1 \trianglelefteq_1 \bar{\mathcal{K}}_1\} \supseteq \{\mathcal{K}_\text{D} \in \underline{\mathbb{DMT\$}}^\text{det} \mid \mathcal{K}_\text{D} \trianglelefteq_\text{D} \bar{\mathcal{K}}_\text{D}\}.$$

(iii) We define $\tilde{\mathcal{K}}_\text{D} \stackrel{\text{def}}{=} g^*([\mathcal{U}_1^3]_{\approx_\text{D}})$, where $[\mathcal{U}_1^3]_{\approx_\text{D}}$ is the equivalence class represented by $\mathcal{U}_1^3$ in Figure 3.4, and get

- $\tilde{\mathcal{K}}_\text{D} \trianglelefteq_\text{D} \bar{\mathcal{K}}_\text{D}$,
- $\bar{\mathcal{K}}_\text{D} \ntrianglelefteq_\text{D} \tilde{\mathcal{K}}_\text{D}$,
- $\tilde{\mathcal{K}}_\text{D} \ntrianglelefteq_\text{D} g^*([\mathcal{U}_1^1]_{\approx_1}) = \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1}))$, and
- $\tilde{\mathcal{K}}_\text{D} \ntrianglelefteq_\text{D} g^*([\mathcal{U}_1^2]_{\approx_1}) = \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))$.

Since by Theorem 3.34 we have

$$\{\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1(\mathcal{K}_1)) \mid \mathcal{K}_1 \in \underline{\mathbb{DMTS}}^{\det} \wedge \mathcal{K}_1 \trianglelefteq_1 \bar{\mathcal{K}}_1\} = \{\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1})), \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))\},$$

properties (i) and (ii) imply

$$\{\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1})), \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))\} = \{\mathcal{K}_\mathrm{D} \in \underline{\mathbb{DMTS}}^{\det} \mid \mathcal{K}_\mathrm{D} \trianglelefteq_\mathrm{D} \bar{\mathcal{K}}_\mathrm{D}\}. \quad (4.10)$$

Furthermore, (iii) implies that

$$\tilde{\mathcal{K}}_\mathrm{D} \trianglelefteq_\mathrm{D} \bar{\mathcal{K}}_\mathrm{D} \;\wedge\; \tilde{\mathcal{K}}_\mathrm{D} \notin \{\bar{\mathcal{K}}_\mathrm{D}, \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1})), \underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))\}. \quad (4.11)$$

The rest of the proof is dedicated to showing that there are no $\bar{\mathcal{K}}_\mathrm{D}, \tilde{\mathcal{K}}_\mathrm{D} \in \underline{\mathbb{DMTS}}$ satisfying properties (4.10) and (4.11), i.e., there are no elements that $\bar{\mathcal{K}}_1 = [\hat{\mathcal{U}}_1]_{\approx_\mathrm{D}}$ and $[\mathcal{U}_1^3]_{\approx_\mathrm{D}}$ could be mapped to by an 1D-embedding. Consequently there is no 1D-embedding.

Let $\bar{\mathcal{U}}_\mathrm{D} \in \bar{\mathcal{K}}_\mathrm{D}$. Then $\bar{\mathcal{U}}_\mathrm{D}$ satisfies the following properties:

(i) $\bar{\mathcal{U}}_\mathrm{D}$ has no components with labels different from a and b. Otherwise, there were fully determined refinements that are not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1}))$ and not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))$, which would be a contradiction to (4.10).

(ii) $\bar{\mathcal{U}}_\mathrm{D}$ has no components with an outgoing transition from a state that is not the root state of the component. Otherwise, there were fully determined refinements not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1}))$ and not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))$, which would be a contradiction to (4.10).

This implies that each component of $\bar{\mathcal{U}}_\mathrm{D}$ is DR-equivalent to a component that has only two states, where one of them is the root state of the component and the other has no outgoing transitions.

(iii) $\bar{\mathcal{U}}_\mathrm{D}$ has no components that have two outgoing may transitions starting in their root state, where one of them is labelled with a and the other is labelled with b. Otherwise, a fully determined refinement with one transition labelled with a and another transition labelled with b would be in an equivalence class different from $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1}))$ and $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))$, which would be a contradiction to (4.10).

This implies that $\bar{\mathcal{U}}_\mathrm{D}$ has no components that have two outgoing *must* transitions starting in their root state, where one of them is labelled with a and the other is labelled with b.

Furthermore, this implies that $\bar{\mathcal{U}}_\mathrm{D}$ has no components that have a must hypertransition with label a and b starting in their root state (if there was such a component with such a hypertransition, condition (2.1) of DMTSs would imply may transitions with labels a and b). As other labels than a

and b are not possible (property (i)), we cannot have hypertransitions with different labels. However, if all labels in a hypertransition are equal, then property (ii) implies that each component is DR-equivalent to a component with no hypertransitions at all.

(iv) $\bar{\mathcal{U}}_{\mathrm{D}}$ has a component with a root state that has an outgoing may transition with label a. Otherwise, $\mathsf{DMTS}(\mathsf{TS}_1(\mathcal{U}_1^1))$ would not be a fully determined refinement of $\bar{\mathcal{U}}_{\mathrm{D}}$, which would be a contradiction to (4.10).

Furthermore, $\bar{\mathcal{U}}_{\mathrm{D}}$ has a component with a root state with outgoing may transition with label b. Otherwise, $\mathsf{DMTS}(\mathsf{TS}_1(\mathcal{U}_1^2))$ would not be a fully determined refinement of $\bar{\mathcal{U}}_{\mathrm{D}}$, which would be a contradiction to (4.10).

(v) The root state of each component must have an outgoing must transition. Otherwise, the DMTS with no transitions would be a fully determined refinement that is not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^1]_{\approx_1}))$ and not in $\underline{\mathsf{DMTS}}(\underline{\mathsf{TS}}_1([\mathcal{U}_1^2]_{\approx_1}))$, which would be a contradiction to (4.10). By property (i), the must transition is either labelled with a or with b.

By property (v), each component has a must transition labelled with a or b. By property (iii), it is not a hypertransition and there are no further may or must transitions in the component. By property (iv), we have one component with a must transition labelled with a and one component with a must transition labelled with b. Possibly existing further components must be DR-equivalent to one of these two components. Thus $\bar{\mathcal{U}}_{\mathrm{D}}$ is DR-equivalent to $\mathcal{U}_{\mathrm{D}}^5$ from Figure 3.2. Then, in contradiction to (4.11), Theorem 3.24 implies that there is no refinement of $\bar{\mathcal{U}}_{\mathrm{D}}$ that is not DR-equivalent to $\bar{\mathcal{U}}_{\mathrm{D}}$, not DR-equivalent to $\mathsf{DMTS}(\mathsf{TS}(\mathcal{U}_1^1))$ and not DR-equivalent to $\mathsf{DMTS}(\mathsf{TS}(\mathcal{U}_1^2))$: By transitivity of disjunctive refinement, such a refinement would also be a refinement of $\hat{\mathcal{U}}_{\mathrm{D}}$ from Figure 3.2, which would be a contradiction to Theorem 3.24. □

Chapter 5

Conclusion and Related Work

We developed 1-selecting modal transition systems (1MTS), a new abstraction/ refinement framework, by modifying the established disjunctive modal transition systems (DMTS). The key difference lies in the interpretation of hypertransitions: DMTSs require *at least* one alternative to be taken in an implementation, whereas 1MTSs require *exactly* one alternative to be taken. The two resulting refinement notions were illustrated by two diagrams showing all refinements of a simple DMTS example and a simple 1MTS example. We introduced a general notion of relative expressiveness in abstraction/refinement frameworks that is defined via the existence of special order embeddings. It takes the refinement ordering structure into account and requires the preservation of fully determined systems. We used this notion to compare DMTSs with 1MTSs and got the result that 1MTSs are not only as expressive as DMTS, but even strictly more expressive. Thus 1MTSs might turn out to be a useful formalism for refinement and abstraction in the context of common refinement:

For refinement in software development, the additional power of 1MTSs (over DMTSs) shows quite directly: It becomes easily possible to specify a system where in some part of the system (e.g., a function or a class) *exactly one* of several alternatives should be implemented. It is hard to express underspecification of that kind in DMTSs. Consequently, 1MTSs might be a suitable semantics for specification languages.

For the abstraction approach, 1MTSs might be useful in the field of model checking. Since 1MTSs allow more alternatives than DMTSs, there is a chance to find 1MTS abstractions, for which satisfaction can be checked more efficiently than for any DMTS. Therefore, a satisfaction relation on 1MTSs should be defined and examined, which is out of the scope of this thesis.

However, it should also be noted that the result of additional expressive power does not hold when regarding an alternative refinement notion: If we use implementation-based refinement, i.e., if we define a (disjunctive or 1-selecting) modal transition system $\mathcal{U}$ to be a refinement of another (disjunctive or 1-selecting) modal transition system $\hat{\mathcal{U}}$, if and only if the set of implementations of $\mathcal{U}$ is a subset of the sets of implementations of $\hat{\mathcal{U}}$, the question of relative expressiveness is simply answered by comparing the sets of implementations expressible in the two formalisms. It was shown that they can express the same implementations by giving translations in both directions and consequently we have equal expressive powers. These embeddings in the context of implementation-based refinement, as well as the embedding given from 1MTSs to DMTSs in the context of common refinement, allow to translate algorithms (e.g., for refinement checks) from one formalism to another. In this context, it is interesting to determine the complexity of the translations, since it is desirable not to introduce too much inefficiency.

Related Work

The approach of extending common transition systems by a second transition relation expressing which steps *may* appear in an implementation was followed by Larsen and Thomsen, who introduced *modal transition systems* [15], and by Dams, who called his extension *mixed transition systems* [4, 5]. Modal transition systems require that every must transition has to be contained in the may transition relation, whereas mixed transition systems do not have this restriction. Both modal and mixed transition systems come with a refinement notion: For modal transition systems it is simply called *refinement*, whereas in the context of mixed transition system, the term *mixed simulation* is used.

Larsen and Xinxin were the first to extend the must and may transition approach by hypertransitions, which resulted in their definition of *disjunctive modal transition systems* [16], which we call DMTSs. The refinement notion on DMTSs, that is also considered in this work, is a straightforward extension of the refinement notion on modal transition systems. In [16], Larsen and Xinxin also showed how a DMTS can be used to express the solution set of an equation system formulated in process algebra.

Kripke modal transition systems (KMTS) [10, 12], state-based versions of modal transition systems, are used as a model for abstraction in order to investigate more efficient model checking. KMTSs do not have action labels on transitions. Instead, every state is labelled with the set of propositions holding there. Then validity of properties expressed in modal logics like the μ-calculus [14] can be checked with respect to all possible implementations. Obviously, besides the two outcomes *true* (all implementations satisfy the formula) and *false* (all implementations do not satisfy the formula), underspecification introduces a third truth value *unknown* (e.g., when some implementations satisfy the formula and others do not), resulting in three-valued approaches for program analysis [11].

Generalized Kripke modal transition systems are KMTSs that also feature hypertransitions, thus are the state-based version of DMTSs. These were used by Shoham and Grumberg in [23].

Alfaro et al. used in [8] a DMTS-like approach for the underspecification of turn-based games [2], extending these structures by must and may transitions and hypertransitions. This resulted in their definition of *abstract game structures*.

Validity of formulas in the alternating-time μ-calculus [3] can be checked, which again raises the need for three-valued semantics.

In [6], Dams and Namjoshi have presented yet another transition system variant called *focused transition systems (FTS)*. The corresponding abstraction framework is *complete* in the sense that for every system one can find a finite abstraction such that a given correctness property can be shown. The authors extend mixed transition systems by fairness constraints and two types of hypertransitions, so called *focus* and *de-focus* steps that model disjunction, respectively conjunction in some sense. Satisfaction of formulas is defined via a game on the focused transition system and an alternating tree automaton [19] for the given formula. The notion of refinement is also defined via a game.

Dams and Namjoshi also considered *μ-automata* [13], that have OR-states which introduce disjunction similar to hypertransitions, and demonstrated in [7], how they yield complete models for abstraction with respect to the modal μ-calculus. In that paper, they also defined *modal automata* as 3-valued variants of μ-automata.

Future Work

We proved that 1MTSs have strictly more expressive power than DMTSs in the context of the common refinement notion. However, this is not the case in the context of implementation-based refinement, where we get the result of equal expressive powers. It is future work to determine the complexity of the translations given from DMTSs to 1MTSs and vice versa. A further interesting question still to be answered is, what might be the reason for the discrepancy between the two refinement notions and how this result can be interpreted. Why does the common refinement notion introduce the additional expressive power of 1MTSs over DMTSs?

The common refinement notion has the substantial advantage over the implementation-based approach that satisfaction of formulas can be more efficiently determined. Therefore, the result of additional expressive power in this context is of special interest and it should be examined, how 1MTSs could be used in the various applications of underspecification. In these applications, it might be interesting to combine DMTSs with 1MTSs, getting transition systems with both disjunctive- and 1-selecting-style hypertransitions. Also, a generalised variant called *n-selecting modal transition systems* is worth considering, in which every hypertransition carries a number n indicating how many targets should be implemented (instead of exactly one in 1MTSs). Another variant has hypertransitions with only a single label, but several successor states. The variants remain to get (partially) ordered by relative expressiveness, using the notion proposed in this thesis. Fields of applications, in which 1MTSs or their variants might be useful, include the following:

For refinement in software development, we remarked that the additional power of 1MTSs (over DMTSs) enables us to easily specify a system where in some part of the system *exactly one* of several alternatives should be implemented. As a consequence, 1MTSs might be a suitable semantics for specification languages. It is future work to check, how 1MTSs could be integrated in specification formalisms and tools.

For the abstraction approach, the additional power of 1MTSs possibly allows more flexible abstractions and smaller representations that allow more efficient model-checking. As remarked before, a satisfaction relation on 1MTSs should be developed and examined. Furthermore, the question arises, whether it is possible to even get a complete abstraction formalism (i.e., one, such that for every system a finite abstraction can be found), if 1MTSs are extended with fairness constraints similar as in [6].

Bibliography

[1] *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*. IEEE Computer Society Press, 2004.

[2] S. Abramsky. Semantics of interaction. In H. Kirchner, editor, *Trees in Algebra and Programming – CAAP'96*, volume 1059 of *Lecture Notes in Computer Science*, page 1. Springer-Verlag, 1996.

[3] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49(5):672–713, 2002.

[4] D. Dams. *Abstract interpretation and partition refinement for model checking*. PhD thesis, Technische Universiteit Eindhoven, The Netherlands, 1996.

[5] D. Dams, R. Gerth, and O. Grumberg. Abstract interpretation of reactive systems. *ACM Transactions on Programming Languages and Systems*, 19(2):253–291, 1997.

[6] D. Dams and K. S. Namjoshi. The existence of finite abstractions for branching time model checking. In *LICS '04* [1].

[7] D. Dams and K. S. Namjoshi. Automata as abstractions. In R. Cousot, editor, *Verification, Model Checking, and Abstract Interpretation*, volume 3385 of *Lecture Notes in Computer Science*, pages 216–232. Springer-Verlag, 2005.

[8] L. de Alfaro, P. Godefroid, and R. Jagadeesan. Three-valued abstractions of games: Uncertainty, but with precision. In *LICS '04* [1].

[9] C. A. R. Hoare. A model for communicating sequential processes. In *On the Construction of Programs*, pages 229–254. Cambridge University Press, 1980.

[10] M. Huth. Model checking modal transition systems using Kripke structures. In A. Cortesi, editor, *VMCAI 2002*, volume 2294 of *Lecture Notes in Computer Science*, pages 302+. Springer-Verlag, 2002.

[11] M. Huth, R. Jagadeesan, and D. Schmidt. Modal transition systems: A foundation for three-valued program analysis. In D. Sands, editor, *ESOP*

2001, volume 2028 of *Lecture Notes in Computer Science*, pages 155–169. Springer-Verlag, 2001.

[12] M. Huth, R. Jagadeesan, and D. A. Schmidt. A domain equation for refinement of partial systems. *Mathematical Structures in Computer Science*, 2003.

[13] D. Janin and I. Walukiewicz. Automata for the modal μ-calculus and related results. In J. Wiedermann and P. Hájek, editors, *Mathematical Foundations of Computer Science*, volume 969 of *Lecture Notes in Computer Science*, pages 552–562. Springer-Verlag, 1995.

[14] D. Kozen. Results on the propositional μ-calculus. In M. Nielsen and E. M. Schmidt, editors, *Proceedings of the 9th Colloquium on Automata, Languages and Programming*, volume 140 of *Lecture Notes in Computer Science*, pages 348–359. Springer-Verlag, 1982.

[15] K. G. Larsen and B. Thomsen. A modal process logic. In *LICS '88*, pages 203–210. IEEE Computer Society Press, 1988.

[16] K. G. Larsen and L. Xinxin. Equation solving using modal transition systems. In *LICS '90*, pages 108–117. IEEE Computer Society Press, 1990.

[17] R. Milner. An algebraic definition of simulation between programs. In *IJCAI*, pages 481–489, 1971.

[18] R. Milner. *A Calculus for Communicating Systems*, volume 92 of *LNCS*. Springer-Verlag, 1980.

[19] D. Niwiński. Fixed point characterization of infinite behavior of finite-state systems. *Theoretical Computer Science*, 189(1-2):1–69, 1997.

[20] Object Management Group. *UML 2.0 Infrastructure Specification*, 2003. `http://www.omg.org/cgi-bin/doc?ptc/03-09-15`.

[21] Object Management Group. *UML 2.0 Superstructure Specification*, 2003. `http://www.omg.org/cgi-bin/doc?ptc/03-08-02`.

[22] H. Schmidt. The one-selecting variant of disjunctive modal transition systems, 2005. Studienarbeit. Available at `http://www.informatik.uni-kiel.de/~hsc/studienarbeit.pdf`.

[23] S. Shoham and O. Grumberg. Monotonic abstraction-refinement for CTL. In K. Jensen and A. Podelski, editors, *Tools and Algorithms for the Construction and Analysis of Systems*, volume 2988 of *Lecture Notes in Computer Science*, pages 546–560. Springer-Verlag, 2004.

[24] R. J. van Glabbeek. The linear time – branching time spectrum I; the semantics of concrete, sequential processes. In J. Bergstra, A. Ponse, and S. Smolka, editors, *Handbook of Process Algebra*, chapter 1, pages 3–99. Elsevier, 2001. Available at `http://boole.stanford.edu/pub/spectrum1.ps.gz`.

[25] N. Wirth. Program development by stepwise refinement. *Communications of the ACM*, 14(4):221–227, 1971.

[26] L. Xinxin. *Specification and Decomposition in Concurrency*. PhD thesis, Aalborg University, 1992.

Index